Three for the Money: An Hour in the Classroom *

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I am in a crowded, windowless, cubicle of a classroom, winging it somewhat, as always. The chairs are arranged a circle, although at the very most, a quarter of the students are sitting down. On the floor in front of us is the largest known 3-regular planar graph of diameter 3. (See Figure 1.) It is drawn with masking tape and its 12 vertices are large enough for a person to stand in. A jumble of conversations fills the room. Students interrupt one another; occasionally someone asks me a question. Mostly I am watching. What I see is so exciting that it is hard for me not to demand that they all sit down and be quiet so that I can give a lecture about what I see. For the first time since the class period began, I relax. Teaching, learning, the classroom—it is all going the way it is supposed to.

The 20 or so young people in the classroom are high school students participating in the Idaho Science Camp [1]. This is our fourth one-hour session together; we are not quite the strangers that we were to one another a few days earlier. I have been trying to expand their understanding of what it means to do mathematics, and also put into practice some things that I understand theoretically—that students can learn much more in a rich and stimulating environment, that hierarchical exposition isn't always the best way to convey information, and that students learn a lot by talking to each other about their ideas. I am trying to understand what it means to have a learning-centered classroom instead of a teacher-centered one. It is the uncertainty of giving up tight control of the program that has made me tense; seeing what good things can happen when I do is what makes me relax.

Before the students arrived, I had drawn the graph on the floor with masking tape. I know a lot about this particular graph and about graphs in general. Trying to organize that information into a coherent outline seemed impossible. The structure of the information in

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[1]The Idaho Science Camp is sponsored by the University of Idaho College of Engineering and the U.S. Department of Energy

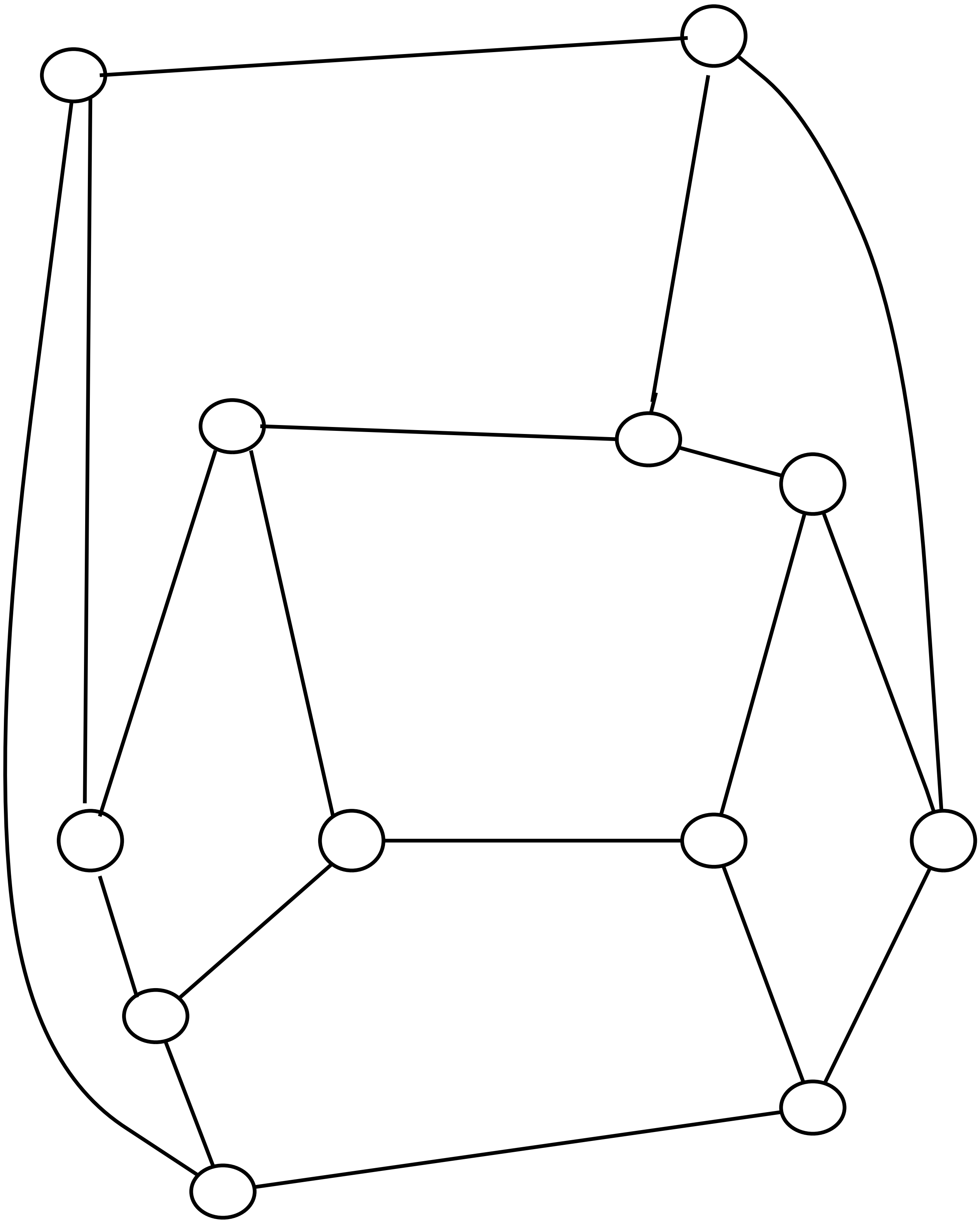

Figure 1: *This is the graph that was drawn on the classroom floor with masking tape. Every vertex of this planar graph has degree 3. The maximum diameter in the graph is 3. It is not known whether it is possible to draw a graph with more vertices which preserves these properties.*

my own mind more closely resembled the drawing on the floor than an ordered list with indentations. I decided to take a risk and come to the classroom equipped primarily with my understanding of graphs. I had in mind a brief introduction with which I would begin. Otherwise, my only plans were to draw the graph in Figure 1 on the floor ahead of time, to take my cues from the students, to be alert, and to try to think on my feet. I was more than a little bit nervous when it was time to get started.

> I asked: *Do you remember I told you about my friend Mike*[2]*—the Computer Scientist?*

"The guy from San Diego who invented the Orange Game[3]!" someone called out.

> *Yes, he's the one. Well, I have learned a lot from him—a lot of mathematics, and also a lot about what mathematicians actually* do. *One thing that drives him crazy is that people have the mistaken idea that mathematics is primarily about numbers. Really, there is much more to mathematics than that. One day when he was particularly excited, he plucked a drawing from a jumble of papers on the table and exclaimed, 'An object like this is every bit as important as a number—and just as useful, too. Why,* this *is a number! This is a* psychedelic *number!'*
>
> *Well, this drawing is the one he was referring to. What he called a psychedelic number that day is what most mathematicians call a graph. It is made of dots (or circles) and lines. You'll notice right away, I hope, that even though it's called a graph, it doesn't have anything to do with the kind of graph you've probably studied in school—you know, bar graphs line graphs and such. This a a whole different kind of a graph, simply made of dots and lines. The thing that is particularly interesting about this graph is that it figures into a contest that anyone in this room could perhaps win. A famous mathematician has offered a cash prize for any person who can draw a graph that is larger than this one, yet preserves three critical properties.*

There was a flurry of excitement—how much money, how soon do we get it, and quickly someone called out, "I can draw one bigger, just gimme the masking tape!"

> I clarify: *Bigger means having more nodes or circles, In graph theory they are called vertices. Also, you still need to know what the three critical properties*

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[3]The Orange Game illustrates routing and deadlock in networks. Any number of people can play. All players label two oranges with their names, then the oranges are mixed up, a single orange is removed, and the players each pick up two oranges which are not their own. (One person will have only one.) Players then stand in a circle. A player may place his or her orange into the *empty* hand of an *adjacent* player only. The game ends when all players have their own oranges back.

> *are that you have to preserve. Size is a property. The size of this graph is 12 because it has 12 vertices. Obviously that's not the property that you have to preserve.*

"Well, what are the properties then?"

At that moment the idea occurred to me to turn the problem inside-out and make them guess what the properties were. I also decided to use precise mathematical terms whenever the opportunity arose, and not to define them unless someone asked me what they meant. In retrospect, I would see that these were two wonderful ideas. Scrutinizing and manipulating a mathematical object so as to understand its properties is so much more like what mathematicians do than being presented with a list to memorize of the properties that some object has. Because I used the special terminology of graph theory right off, the students heard the words and began using them themselves, long before they were absolutely sure what they meant. It was as though the words were with us from the beginning, and they grew meanings as the class period progressed.

> I answered: *You have to guess what the properties are. Name all the properties that you can think of and each time you name one of the three critical ones, I will tell you that you have guessed one.*

Students began talking all at once, one of the many chaotic eruptions that this class period would have. I doubted that the students had more than a foggy notion of what I meant by the word *property*, but since this didn't dampen their enthusiasm, I wasn't going to interrupt and tell them that. I also didn't think that they would be able to guess the three properties without help. I waited until the excited conversation died down before I continued.

> *The prize-winning graph will be of size greater than 12 and also be* planar, 3-regular, *and have a* diameter *of 3. That's a hint. All you have to do is figure out what that means and you will know what the three properties are.*

A few people wrote those words own. Quite a few students were copying the graph. Others were excited about winning the money.

Next time I had everyone's attention, I told them about Paul Erdos, an elderly Hungarian mathematician who spends most of his time on the road, finding it far more interesting to travel and visit mathematicians all over the world, to work with them and get them excited about problems that interest him, than to stay home and be famous. I held up a photocopied article about him[4].

[4] Erdos citation

He has a great mathematical mind, and would be famous and world-renowned by simply working alone. But he prefers working with other people, planting seeds, sharing ideas and questions. As a result, university mathematicians love to have him visit and give talks. These visits often result in a collaboration between Paul Erdos and a professor or student, and together they share the credit for some breakthrough.

He lives simply. His hosts pay a fee for his visits and pay his travel expenses. He's not exactly rich, but he has plenty of money. There are so many problems that fascinate him that he can't possibly work on all of them, so he offers prizes to anyone who can solve them. This is one of his prizewinning questions.

Now here is an interesting thing that Mike told me: He thinks that the reason why no one has found the solution for this problem because no one has fiddled with it enough.

So what you have to do here, is either draw a larger graph, or prove that it can't be done.

Think about it. Imagine someone offered you a prize if you could find two odd numbers that made an odd number when you added them together ...

Immediately a voice called out, "You can't do that!" Other voices agreed.

Right. You wouldn't never say that someone who couldn't add two odd numbers and get an odd number needed to try harder. You would be sure they couldn't do it. With some careful thinking, you can come up with an explanation why no one will be able to add two odds and get an odd .

Some students began explaining to one another why this is so, but I continued.

But this problem is different. No one has been able to come up with a good logical argument for why it would be impossible to draw a graph larger than this one that preserves the three properties. So if there's no reason that anyone knows of why there can't be a bigger one, it's likely enough that there is one, only no one has discovered it. If that's the case, it doesn't take any special advanced mathematics to just draw it. It just takes a pencil, and a lot of time, patience, and inclination to fiddle with it. Mike shows this problem to grade school kids whenever he can because he thinks one of them might just stumble on the solution.

"Tell us what the properties are, then!" The room became quiet.

OK, more hints. I'll give you some hints: Planar, 3-regular, and diameter 3. That's a lot of 3's. That's a hint. The number three. What's with the number 3?

Someone noticed that there are three edges (lines) touching every vertex. This was explained to others. They checked all the vertices. Indeed it was true. I told them that the number of edges that touch a vertex is called the *degree* of that vertex. In this graph, every vertex is of degree three. When every vertex of a graph has the same degree, then the graph is *regular*. So this graph is *3-regular*. When you draw the graph with more vertices, it has to be 3-regular.

Immediately some students began trying to draw 3-regular graphs of size larger than 12. Soon the chaos was back. A short while passed before enough students realized that they didn't have all the information to solve the problem. A few more moments passed before they could shush the others so I could tell them more.

> *I'll give you hints by showing you some games that you can play on this graph. The games will help you understand the structure of the graph better. No doubt that the person who actually solves this problem will know many more things about the structure of this graph than just the three properties that are preserved. We'll play some games that will help you see lots of properties. When someone guesses a property that is one of the three critical ones, I will tell you.*

I asked a couple of students to label the vertices with little strips of masking tape with the numbers from 1–12, reminding everyone that the numbers weren't part of the graph. The numbers would serve as names for the vertices and make it easier to talk about them.

Two boys had begun to play cards. I asked them to loan me the clubs. Surprised, they shurgged and obliged. I removed the King, handed the remaining clubs to a student, and asked him to go stand in any circle. Then I explained a game.

> *Here's the rule: You can travel along the lines and walk from vertex to vertex. Every time you come to a vertex, you must lay a card down in it before you leave. You cannot return to a vertex that already has a card in it. Can you deal out all the cards?*

He set out. It is not very hard to do. Soon all the cards were laid in circles on the floor. "It's traversible!" cried a voice from the crowd. "We had that last year."

> *Right.* Traversible *is a word you can use to describe this graph. You can find a path through the graph that allows you to travel to all of the vertices without touching any of them twice. When you can do that, you can say that the graph is traversible, or that the graph has a Hamiltonian Path. That is one of the properties of this graph. It's not one of the critical properties that has to be preserved, but it is a property. Do you think it will be preserved in the larger one?*

Some yes's, some no's. They wanted more properties. I announced another game, walked to the center of the room, picked up a card from the graph, and called for volunteers.

We need 11 volunteers. One person for each vertex where there is a card.

Students came eagerly forward and scrambled for places. When they were set, I gave them directions.

> *OK, pick up the cards. Look at the number that you have. Jack is 11, Queen is 12, Ace is 1.*

"Oh no!" groaned a voice. "The Orange Game!" We had played the Orange Game a few days earlier. This was indeed a variation.

> *You'absolutely right! Routing and deadlock in networks. Let's give it a try.*
> *Everyone has to get back to the vertex with their number on it, but the only way anyone can move is from the place where they are standing and into an empty vertex.*

The fun begins. There is only one empty vertex. This requires a lot of maneuvering, planning and working together. It is much harder than the Orange Game. Once I realize this, I am not even sure it is possible, and wonder if I haven't made an awful mistake by asking them to do it.

Several leaders emerge, they argue. Some students are confused, but follow orders willingly. Some refuse to do what others tell them. No one gives up. A few students in the audience add their suggestions, too. Others continue to pour over graphs as they have for the last 20 minutes. A girl has gone to the board to demonstrate to two friends why you can't draw a 3-regular graph with 13 vertices. ("With 39 ends-of-lines, one line would just have to hang there loose, with no vertex on its other end," she says.) When a student struggling with the puzzle on the floor is too frustrated and suggests they quit, someone jumps up and offers to take his place. The student who quits stands aside, continues to watch, and is soon making suggestions.

With the students moving about the graph in front of me, once again the series of questions related to those that leaped to my mind when I watched them play the Orange Game comes back. What is the fewest number of moves it will take to get everyone back to their places when there is just one empty slot? How is it different when there are 2, 3, or 4 empty slots? If you chart these numbers, will you see a pattern? Won't that depend on how "mixed up" everyone was in the first place? What does "mixed up" mean? How did we know the numbers were "good and mixed up" when we started? Is it possible to start out so mixed up that everyone can't get back to their places? How would the game be different if you didn't know what numbers other people had? What if you couldn't know the number of a vertex until you stood in it?

Another observer might find this classroom disorderly and unfocused. It *is* disorderly and unfocused. This, however, is the moment at which I relax and feel that I have done my job as teacher well. The room is filled with excitement and enthusiasm about mathematics. As the students explore this problem, they are participating in a large collaboration set up by one of the greatest mathematicians of our time. Even though there is a monetary prize for a "right answer", this is not the sole focus of the room. In fact, none of the students understands the question yet! They have lost themselves in the preliminary exploration. I pat myself on the back for the lucky on-the-spot inspiration that had me use the technical terms for the hints. Those big, unfamiliar words have indeed grown meanings as they practiced saying them and tried to figure out what they mean.

I regret that I will say goodbye to these students for the last time tomorrow. When will be the next time that I will have 20 enthused and clever people to think about problems like this with me so that we can act them out together? Next week, my learning of graph theory will slow to its usual solitary, plodding pace as I sit at a table and move labeled bottle caps over graphs drawn on sheets of paper, trying to explore some of the questions raised for me during these sessions.

At last the students have sorted themselves on the network. They stand triumphantly in the vertices whose numbers correspond with their cards. A student who is in his seat looks up from his notebook with an expression that is both confused and intense. He asks, "What are those properties again?"

> *You are trying to find a planar graph of size larger than 12 that is 3-regular, and has a diameter of 3.*

I count to 3 on my fingers as I say the words *planar*, *3-regular*, and *diameter of 3.* Voices join and say the words along with me. I am ready with another proposal.

> *Do you want to play a game that is easier than that last one? When you figure the game out, you will know what the diameter business is.*

Of course they do. I held up an illustrated storybook[5] I had made.

> *This is a game that I invented to teach this problem to a group of 3-year-olds...*

[5] The text of this and other stories with Gertrude, Superperson, and the Monster that illustrate properties and problems on graphs through games are available on the World Wide Web `http://www.cs.uidaho.edu/~casey931/mega-math/workbk/graph.html` and in Chapter 3, Games on Graphs in *This is Mega-Mathematics* by Nancy Casey and Mike Fellows, available via anonymous ftp from `ftp.cs.uidaho.edu` in the directory `pub/mega-math/workbk`

Now, with children that age, it takes a lot of effort just to get them to stay on the edges when they walk around the graph. So you can imagine how teaching them about properties takes a lot of doing. I invented this story and the game to go with it after watching them play dress-up one day. The characters were ones that they had made up for their game: Gertrude (she's a goose), a Monster and Superperson. We'll need three volunteers to play.

When the three-year-olds play, they spend the first 10 minutes dressing up in their costumes and capes. But you know that we're doing this so that you can find out what diameter *means, so we can get right to the point.*

In this game, the circles of the graph are ponds. The lines that connect them are the flyways that you can use to fly from one pond to the next. Superperson has the most flying power, Gertrude has the least, and the Monster is in the middle. The Monster is frightening and ugly, but harmless. He wants to play with Gertrude, but Gertrude is afraid of him. Since Gertrude is going to try to escape from the Monster, and since the Monster has more traveling power, it's more fair to let the Monster choose where he is going to start first. Then Gertrude can pick a good place to hide from him. Superperson will pick her place last.

The students who have volunteered to play the roles of each of the characters choose places to stand on the graph, and the game is ready to start.

The game begins because Gertrude is bored, this pond is boring and she wants to be somewhere else. So she flies off to another pond. She's not in very good shape, though, so by the time she gets to the next pond, she crash lands in it and has to rest.

The person playing Gertrude acts out her part.

Now the monster wants to play with Gertrude. He is so ridiculously large that it takes a huge amount of effort just for him to get off the ground. He is still going up when he passes the first pond. But soon he is tired and he crashes into the second one with an enormous splash.

The good-natured Monster acts out his ungainly flight. This time, anyway, he doesn't reach the pond where Gertrude is hiding from him.

Now Superperson flies. She has the most power of all. She can fly three ponds before she gets tired, zip-zip-zoom! She flies over two ponds and lands in the third.

Superperson does this.

> *Now it's Gertrude's turn. All three of them will keep flying around—first Gertrude one, then the Monster two, and Superperson three—until the Monster catches up with Gertrude.*

It doesn't take long before the Monster lands in the same pond as Gertrude, and the two students perform the ensuing action.

> *Now the Monster isn't going to hurt Gertrude, but Gertrude doesn't know that, so she attacks the Monster who has to cry to Superperson for help. C'mon Gertrude! Attack! Monster, yell for help!*
>
> *Can Superperson come and save the Monster before Gertrude drowns him in mis-directed self-defense?*

Of course she can. The diameter of the graph is three. But it takes a few times before the students figure this out. Soon they complain that the the game is boring and kind of silly. The Monster always gains a step on Gertrude and will always catch up with her, and Superperson always gets there in time to save him.

> *Yes! The game is horribly dull because of the properties of the graph that is being used for a game board. Superperson always saves the day, and Superperson always saves the day because the* diameter of the graph is 3. *So what does that mean? Think about that statement: the graph has a diameter of 3?*

It takes a little bit of arguing and some experimentation, but soon everyone realizes that you can get from one vertex (no matter which) to any other in three or fewer steps.

Now the students are divided roughly into two groups: those who want to keep playing the Gertrude game and invent more rules so it is less predictable, harder, and more interesting, and those who want to work on the prizewinning problem. I insist that everyone sit down and listen one last time.

> *Planar. That's the other property. What could that mean?*

The students are silent. There is an uncomfortable shifting of feet. They avoid my gaze.

> *It has something to do with a plane. Like points, and lines and planes that you may have learned about in geometry.*

There are still no guesses. They need a hint. What hint can I give that's not tantamount to telling them? On the spur of the moment I can only come up with one idea.

OK, here's a hint. Imagine that this is the map of a city. The edges are roads, the verticess are intersections. What is something that is expensive to build that this city doesn't have any of?

What a stupid, misleading hint, I think, and am surprised when a chorus of voices shouts, "Bridges!" Soon everyone knows that *planar* means you can draw the graph on a plane or flat surface in such a way that the edges only touch at vertices. I tell them that planarity is especially important when people are trying to design electronic circuits that can fit on one side of a silicon chip.

About ten minutes remain in the class period. On the one hand, it has taken over 45 minutes to simply to state a problem. On the other hand, it is mind-boggling to consider the amount of things we have done in that short time. This group is homogeneous, not "gifted". In two days the Science Camp will be over. They are winding down, have been staying up too late, far more interested in getting in the last days of socializing than the last moments of mathematics. I did not anticipate this level of enthusiasm and tenacity. I never could have planned this. Had I not been there, I would argue that it is not possible to "cover" so much in one class period. And yet, it has happened.

Alone, in pairs or in groups some students are thoughtfully drawing graphs. Others want to keep playing games and I turn my attention to them. I rip up five sheets of paper that are different colors, and randomly place strips of all five colors on the vertices.

You want to make it so that when you walk along the edges and leave from a vertex of one color, the next vertex you come to will not be the same color. You might need these.

I give a handful of the colored strips to each student. It's not hard to do a vertex-coloring of this graph with five colors. When they are done, I ask them what is the fewest number of colors they can use to do it. They begin experimenting.

I turn to the students who have queued up to show me graphs that they think might be prizewinners. I systematically check out the properties, showing everyone my thought-processes out loud. Soon they are checking their own and each others. It is more efficient to help each other than to wait for my undivided attention.

When the game-playing students have found a vertex-coloring that works for three colors, I ask them why they can't do it with two colors, and they don't have a hard time showing me why two colors just can't work. I scoop up the colored strips from the vertices and place them on the edges and ask them what they think they are supposed to do now. They don't seem to hear me. They are already working, trying to find a way to arrange the colors so that no two edges of the same color touch.

I interrupt and ask them how many colors they think it will take. Three, someone says. Agreement is muttered; they all seem sure. I ask how they know that, but no one answers. When they finish the edge-coloring, there is still time to color the regions.

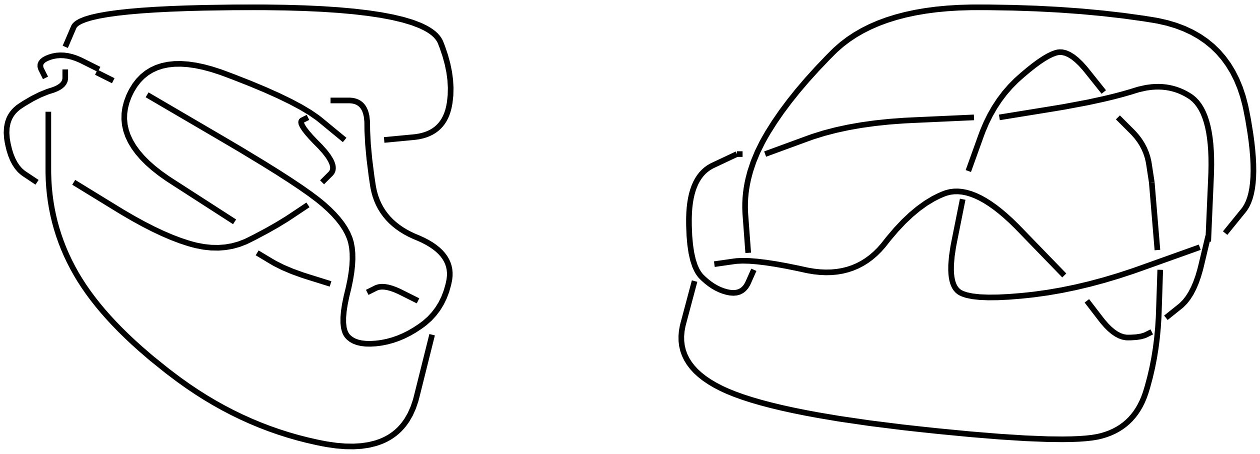

Figure 2: *The story of mathematics is much more the story of long and fruitless searches for solutions to problems, than it is the story of a discipline full of quick answers that are easily verified as right and wrong—even though the latter is the more common experience of many students. For 75 years mathematicians did not think that it was possible to twist and pull either of these two knots (without cutting) so that they were identical. Then one day, someone with tenacity and determination did it. Make the knots out of rope and try for yourself. Chances are it won't take you 75 years.*

A few students drawing graphs are convinced that it is impossible.

"Tell that famous guy you can't do it," someone says. I explain that if they can demonstrate why it is not possible, they can probably win the prize. They are excited—until I insist that a valid demonstration is not simply saying that you tried and tried and couldn't find it. How do you know if you have tried long and hard enough? I happen to have a book[6] with me that has a picture of two knots which were thought to be different for 75 years. (See Figure 2.) Then after all that time, someone showed that they were really the same.

> I warn them: *Even if you try for a long time and can't do it, that doesn't mean that someone else won't get lucky and figure it out.*

We review the reason why we are sure that you can't draw a graph with 13 (or any odd number) vertices that has these properties. Maybe it won't work for 14, but how can you be sure it doesn't work for 50, or 246? And how do you *know* it doesn't work for 14? How can you be sure you have tried every possible combination? Is there a systematic way to list the different things you try to that you can tell if you have left any possibility out? They want that prize. They are talking about how they will spend the money. They keep working.

At the end of the class period, I shoo the students away so they will be on time for their next class. Later in the afternoon and the next day, the final day of classes, students will

[6]Peterson, Ivars. (1990). *Islands of Truth: A Mathematical Mystery Cruise.* New York: W.H. Freeman & Co., p.55

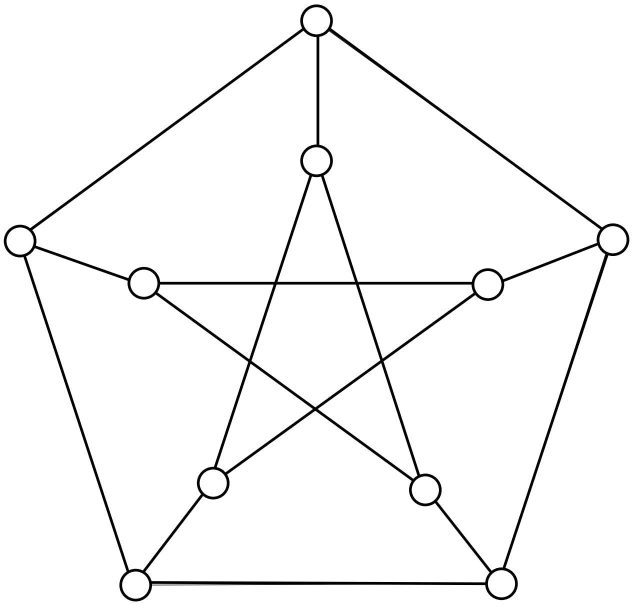

Figure 3: *The Peterson Graph is one of the most famous graphs in all of graph theory. Its renown stems from the unusual list of properties it both possesses and fails to possess. It is an interesting graph for students to examine to see what properties it has and does not have.*

bring me graphs that they have drawn and we will go over them carefully. No one solves the problem, but it is hard to find the flaw in some of the graphs they have drawn.

"What will you do if I find it?" someone asks.

"We will get on the phone right away and call Mike. It will be a great day in the history of mathematics." I mean it.

I ask the students to write down what they have learned about mathematics this week. One girl writes, "My teachers try to make me think abstractly, and I refuse. You tricked me into doing it."

How I wish I had been able to teach for *both* weeks of the Science Camp. This session would have been in the first week, and on the weekend I could have accompanied them on their Saturday trip to picnic and swim at the Snake River. They would show me even more graphs, we could look more closely at the games and talk about other things that came up during the week, such as logic puzzles, and what it takes, in mathematics, to say that something is true. Perhaps there would be a student or two tenacious enough to pour over Mike's paper on dense planar networks[7] with me and decipher it. Certain elements of his methods for drawing graphs with minimal degree and diameter are accessible and might help them in their efforts to find a prizewinning graph.

[7]Fellows, M., Hell, P., and Seyffarth, K., (1993). Constructions of Dense Planar Networks, available from Mike Fellows, Department of Computer Science, University of Victoria, Victoria, BC, Canada V8W 2Y2 or `mfellows@csr.uvic.ca`

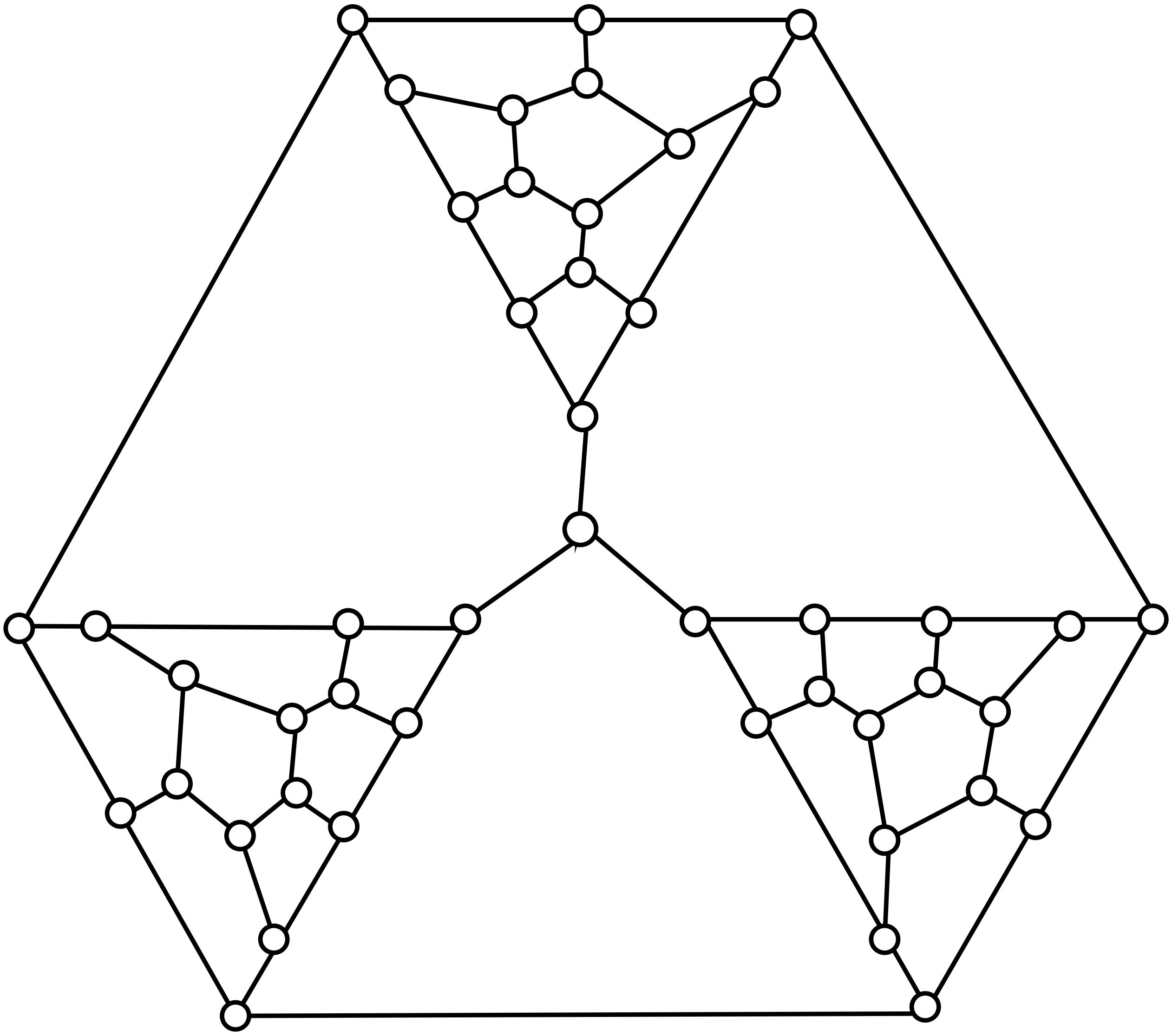

Figure 4: *When Canadian mathematician R.L. Tutte drew this graph in 1946, a question which had been open for over 60 years was solved. The Tutte graph is 3-regular and has the property of being* 3-connected*, which means that in order to break the graph into two disconnected components, you must delete at least 3 vertices. In 1880, the English mathematician P.G. Tait conjectured that every 3-connected, planar, 3-regular graph is hamiltonian. (A graph is hamiltonian if it is possible to find a path that allows you to walk from vertex to vertex, touching each vertex exactly once and return to the vertex where you began.) The Tutte graph is not hamiltonian, so it provides the counterexample which disproved Tait's conjecture—even though Tait's conjecture seemed to be true for all those years.*

If we were together a second week, we would surely gather with a different graph drawn on the floor in front of us. The Tutte Graph perhaps, or the Peterson Graph—they are pretty and symmetric and have different properties to discover and talk about. (See Figures 3 & 4.) We could even draw it in colored chalk on the sidewalk in front of the library or somewhere downtown; or we could use lawn paint to spray paint it on the grass.

If we had more time, we could return to Mike's idea of a graph as a psychedelic number, and try to figure out what he meant. I would be able to talk about mathematical objects, their properties and operations on them. I could watch these terms acquire meaning for the students as we looked for properties of objects such as graphs and numbers, knots and maps and then discussed what it meant to do operations—such as add, subtract, multiply and divide—on them.

No matter how engaging my presentations, I would expect them to stop listening to my teacherly lectures and begin to experiment and talk about properties that they suspect or discover. They may adapt versions of the games we have played to the mathematical object that is at hand. They will get in each others' way, interrupt one another and make each other angry. Some students may try to take charge. Others will ignore them and work furiously in their seats on problems they may or may not understand. Some students will stand aside and appear to be doing nothing at all. This, I am learning, is often a symptom of careful reflection.

I will remind myself how hoarse I get whenever I raise my voice in the classroom. As I (and my agenda) slip from the center of their attention, I can be confident that their attention will come back. Perhaps someone on the other side of the room will call out a question to me. I will pantomime my inability to talk loud enough to be heard. Perhaps I will adopt a mock professorial attitude and say, *"Ahem, I thought I was asked to impart some valuable information here."*

Eventually someone who likes to give orders will want to hear what I have to say and shout, "Hey! Psssh! Hey you guys, *shut up!* Can't you see she's trying to *tell* you something?!"

I will begin talking in a quiet voice, but before I have finished a sentence, all the other conversations in the room will resume. I will stop talking, and the students who have accepted the responsibility of making the others be quiet and listen will again take the floor and make it quiet so that I can talk. Eventually, they will be ready to listen, and I will draw on what has happened to pick up the thread of the story that wove throughout all of our class sessions—the story of what it means to do mathematics.

I will point out that whenever I try to tell them about a good math problem, if it *is* a good problem, it is so rich and many-tentacled that immediately, when I begin talking about it, their minds dart to interesting and compelling places. They notice things, they want to experiment, they make guesses, argue, and they want to tell someone else what they are thinking. This happens because they are doing mathematics: churning up ideas, filling the room with creative thoughts and guesses that eventually they can pin down, refine, and either keep or reject by marching carefully through their thoughts with rigor, logic and proof. They are *doing* mathematics, and that is far more stimulating than sitting still and listening to a teacher talk about it. I hope I will have to shoo them out the door to be on time for their next class, and I will leave them, as I always did, excited about learning, about mathematics and about questions of my own, because their enthusiasm and excitement are so very, very contagious.