FORMULAS FOR BLOCKING PROBABILITY[1]

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The assumptions and notation involved in the various formulas for blocking probability that we shall present and discuss are illustrated in Figure 1. M users are modelled as having the combined average rate of generating calls denoted by ν, and the arrival times of these first attempts to place a call are assumed to be independent; these assumptions lead [1, ch. 2, 10] to a Poisson distribution for the number of calls generated in a given time interval, T:

$$\Pr\{n \text{ calls generated in } T\} = \frac{(\nu T)^n}{n!}\, \mathrm{e}^{-\nu T},\ n \geq 0; \tag{1a}$$

for an incremental interval dt, the probability of a call arrival in that interval is

$$\Pr\{\text{one arrival in } dt\} \doteq \nu\, dt. \tag{1b}$$

Also, the probability density function for the time between call arrivals is exponential:

$$\mathrm{p}_a(t) = \nu\, \mathrm{e}^{-\nu t},\ t \geq 0. \tag{2}$$

The users seek to place these calls through a system that has N channels (also called servers) that, on the average, are each occupied with a call for the mean call duration or service time $h = 1/\mu$. For telephone traffic the probability density function of the service time for a given channel is well-modelled as exponential:

$$\mathrm{p}_s(t) = \mu\, \mathrm{e}^{-\mu t},\ t \geq 0. \tag{3}$$

For this assumption, and if a given channel is constantly in use, the number of call completions in time T is a Poisson random variable:

$$\Pr\{n \text{ calls completed in } T\} = \frac{(\mu T)^n}{n!}\, \mathrm{e}^{-\mu T},\ n \geq 0, \tag{4a}$$

and the probability that the given channel, if occupied, will complete a call in the interval dt is

$$\Pr\{\text{service completion in one channel in } dt\} \doteq \mu\, dt. \tag{4b}$$

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[1] Excerpted and adapted from the following technical report: L. E. Miller, "Models for MSE Traffic and Blocking Under Stress," J. S. Lee Associates, Inc. Report JC-2092-2-FF (91 pages) under contract DAAL02-89-C-0040 (Army Survivability Management Office), July 1992. (DTIC accession number AD-B166477).

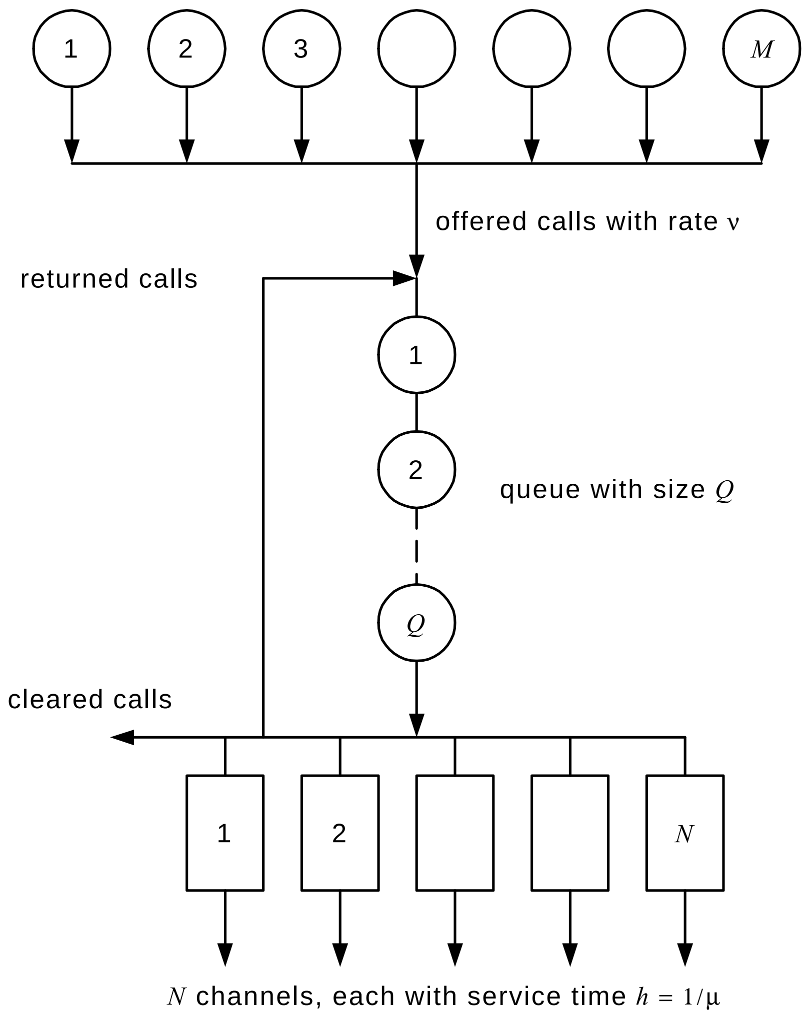

Figure 1. Model used in blocking analyses

The amount of offered call traffic $A = \nu/\mu$ in terms of its loading on the system is often expressed in terms of an occupancy or channel utilization factor

$$\rho \triangleq \frac{\nu}{N\mu} = \frac{\nu\, h}{N} = \frac{A}{N} = \frac{M}{N}\,\rho_0\,. \tag{5}$$

This measure of traffic is the expected M users' rate of arriving calls normalized by the system's rate of servicing calls (call arrival rate for M users $\div$ call departure rate for N channels). Since it is required for stable operation that $\rho < 1.0$, we observe from (5) that, when $M > N$, there must be a restriction on the callers' utilization rates; that is, $\rho_0 < N/M < 1$ for $M > N$.

If all N channels are occupied when a call arrives, the arriving call cannot be completed and is handled in one of several ways: (a) the call is “cleared," in effect forgotten; (b) the call “returns" after a random delay as the user tries to reinitiate the same call; or (c) the call is put on hold (“held") in a queue of a certain size, Q, while waiting to be serviced.

A. Blocking Analyses for an Infinite Number of Sources

When there is an infinite number of users or sources ($M \to \infty$), the rate of calls being offered to the system is not affected by how many calls are currently in the system. Below, in Section B, the case of a finite number of sources will be treated.

Cleared Calls. For blocked calls cleared, the analysis of blocking probability proceeds by noting the relationships between the probabilities of possible system states (numbers of calls in the system, occupying one or more channels) at different times. Let $\Pr\{k,\ t\}$ denote the probability at time t that there are k calls in the system. Using (1b) and (4b), we have

$$\Pr\{0,\, t{+}dt\} = (1 - \nu\, dt)\Pr\{0,\, t\} + \mu\, dt\Pr\{1,\, t\} \tag{6a}$$

$$\Pr\{1,\, t{+}dt\} = \nu\, dt\Pr\{0,\, t\} + (1 - \nu\, dt - \mu\, dt)\Pr\{1,\, t\} + 2\mu\, dt\Pr\{2,\, t\} \tag{6b}$$

$$\vdots$$

$$\Pr\{k,\, t{+}dt\} = \nu\, dt\Pr\{k{-}1,\, t\} + (1 - \nu\, dt - k\mu\, dt)\Pr\{k,\, t\} + (k{+}1)\mu\, dt\Pr\{k{+}1,\, t\} \tag{6c}$$

$$\vdots$$

$$\Pr\{N,\, t{+}dt\} = \nu\, dt\Pr\{N{-}1,\, t\} + (1 - N\mu\, dt)\Pr\{N,\, t\}. \tag{6d}$$

As a steady state or equilibrium is reached, the time derivative of these probabilities is assumed to vanish, giving, for example,

$$\Pr\{k,\, t{+}dt\} = \Pr\{k,\, t\} + dt\,\tfrac{\partial}{\partial t}\Pr\{k,\, t\} = \Pr\{k,\, t\} \equiv \mathrm{P}(k) \tag{7}$$

and resulting in

$$\nu\,\mathrm{P}(0) = \mu\,\mathrm{P}(1) \tag{8a}$$

$$(\nu + \mu)\,\mathrm{P}(1) = \nu\,\mathrm{P}(0) + 2\mu\,\mathrm{P}(2) \tag{8b}$$

$$\vdots$$

$$(\nu + k\mu)\,\mathrm{P}(k) = \nu\,\mathrm{P}(k{-}1) + (k{+}1)\mu\,\mathrm{Pr}(k{+}1) \tag{8c}$$

$$\vdots$$

$$N\mu\,\mathrm{P}(N) = \nu\,\mathrm{P}(N{-}1), \tag{8d}$$

together with the normalization requirement

$$\sum_{k=0}^{N}\mathrm{P}(k) = 1. \tag{8e}$$

Since $A = \dfrac{\nu}{\mu}$, the system of equations (8) is solved by the probability expression

$$\mathrm{P}(k) = \begin{cases} \dfrac{A^k}{k!}\left(\displaystyle\sum_{n=0}^{N}\frac{A^n}{n!}\right)^{-1}, & 0 \le k \le N \\ 0, & k > N. \end{cases} \tag{9}$$

The blocking probability for cleared calls and an infinite number of sources then is

$$B = \mathrm{P}(N) = \frac{A^N}{N!}\left(\sum_{n=0}^{N}\frac{A^n}{n!}\right)^{-1} \triangleq B_{c,\infty}(A, N). \tag{10}$$

This is *Erlang's Loss Formula* or the *Erlang B* formula for blocking probability [2].

Returning Calls. When blocked calls are not cleared but are offered again after random delays, approximately the analysis proceeds as for cleared calls with the exception that the effective incoming rate of calls ν' in terms of the originally offered rate ν is [3]

$$\nu' = \nu + \nu B + \nu B^2 + \nu B^3 + \cdots = \frac{\nu}{1 - B}\,. \tag{11}$$

Thus the argument of the blocking probability is $A' = A/(1 - B)$, a function of the probability itself. This situation calls for an iterative solution for B or, more simply, a parametric solution in which B is expressed by (10) with A' as the argument, and the offered traffic calculated also as a function of A' using

$$A = A'(1 - B). \tag{12}$$

Held Calls. When up to Q blocked calls are held (delayed) in a queue, with the remainder cleared, the analysis of the transition of probabilities leads to

$$\Pr\{0, t{+}dt\} = (1 - \nu\, dt)\Pr\{0, t\} + \mu\, dt\Pr\{1, t\}; \tag{13a}$$

$$\begin{aligned}\Pr\{k, t{+}dt\} = \nu\, dt\Pr\{k{-}1, t\} + (1 - \nu\, dt - k\mu\, dt)\Pr\{k, t\}&\\ + (k{+}1)\mu\, dt\Pr\{k{+}1, t\},\ 1 \le k < N;&\end{aligned} \tag{13b}$$

$$\begin{aligned}\Pr\{k, t{+}dt\} = \nu\, dt\Pr\{k{-}1, t\} + (1 - \nu\, dt - N\mu\, dt)\Pr\{k, t\}&\\ + N\mu\, dt\Pr\{k{+}1, t\},\ N \le k < N{+}Q;&\end{aligned} \tag{13c}$$

$$\Pr\{N{+}Q, t{+}dt\} = \nu\, dt\Pr\{N{+}Q{-}1, t\} + (1 - N\mu\, dt)\Pr\{N{+}Q, t\}. \tag{13d}$$

Note that, for $k \ge N$, the probability of a channel's becoming unoccupied is $N\mu\, dt$, rather than $k\mu\, dt$, because there are N calls being processed, even though there are more than N calls in the system. At steady state, (13) results in

$$\nu\,\mathrm{P}(0) = \mu\,\mathrm{P}(1, t\}; \tag{14a}$$

$$(\nu + \mu)\mathrm{P}(k) = \nu\,\mathrm{P}(k{-}1) + (k{+}1)\mu\,\mathrm{P}(k{+}1),\ 1 \le k < N; \tag{14b}$$

$$(\nu + N\mu)\mathrm{P}(k) = \nu\,\mathrm{P}(k{-}1) + N\mu\,\mathrm{P}(k{+}1),\ N \le k < N{+}Q; \tag{14c}$$

$$N\mu\,\mathrm{P}(N{+}Q) = \nu\,\mathrm{P}(N{+}Q{-}1),\ k = N{+}Q; \tag{14d}$$

with the normalization requirement

$$\sum_{k=0}^{N+Q}\mathrm{P}(k) = 1. \tag{14e}$$

Since $A = \dfrac{\nu}{\mu}$, the system of equations (14) is solved by the probability expression

$$\mathrm{P}(k) = \begin{cases} \dfrac{A^k}{k!}\, G^{-1}, & 0 \le k < N \\ \dfrac{A^N}{N!} \left(\dfrac{A}{N}\right)^{k-N} G^{-1}, & N \le k \le N{+}Q \\ 0, & k > N{+}Q \end{cases} \tag{15a}$$

where

$$G = \sum_{n=0}^{N-1} \frac{A^n}{n!} + \frac{A^N}{N!} \sum_{n=N}^{N+Q} \left(\frac{A}{N}\right)^{n-N} = \sum_{n=0}^{N-1} \frac{A^n}{n!} + \frac{A^N}{N!} \cdot \frac{1-(A/N)^{Q+1}}{1-(A/N)} \tag{15b}$$

$$= \sum_{n=0}^{N} \frac{A^n}{n!} + \frac{A^N}{N!} \sum_{n=N+1}^{N+Q} \left(\frac{A}{N}\right)^{n-N} = \sum_{n=0}^{N} \frac{A^n}{n!} + \frac{A^N}{N!} \cdot \frac{(A/N)[1-(A/N)^Q]}{1-(A/N)}\,. \tag{15c}$$

In this case of infinite sources and held calls, using (15b) the blocking probability is

$$B = \sum_{k=N}^{N+Q} \mathrm{P}(k) = \frac{\frac{A^N}{N!} \cdot \frac{1-(A/N)^{Q+1}}{1-(A/N)}}{\sum\limits_{n=0}^{N-1} \frac{A^n}{n!} + \frac{A^N}{N!} \cdot \frac{1-(A/N)^{Q+1}}{1-(A/N)}} \triangleq B_{h,\infty}(A,\ N;\ Q). \tag{16a}$$

This formula (for $Q \to \infty$) is known [2] as the *Erlang C formula*. Using (10) and (15c), we can also write

$$B_{h,\infty}(A,\ N;\ Q) = \frac{B_{c,\infty}(A,\ N) \cdot \frac{1-(A/N)^{Q+1}}{1-(A/N)}}{1 + B_{c,\infty}(A,\ N) \cdot \frac{(A/N)[1-(A/N)^Q]}{1-(A/N)}}\,. \tag{16b}$$

It is customary for telephone traffic analyses to regard the queue size as being infinite ($Q \to \infty$), since a delayed call, even if the user hangs up and tries again, is assumedly not cleared. For infinite queue size and $\rho = A/N < 1$, (16b) becomes

$$B_{h,\infty}(A,\ N) = \frac{B_{c,\infty}(A,\ N)}{1 - \frac{A}{N} + B_{c,\infty}(A,\ N) \cdot \frac{A}{N}}\,. \tag{16c}$$

Another formula ([3], [4]) for the case of infinite sources, held calls, and infinite queue size is due to Molina:

$$B = 1 - \mathrm{e}^{-A} \cdot \sum_{n=0}^{N-1} \frac{A^n}{n!} = \mathrm{e}^{-A} \cdot \sum_{n=N}^{\infty} \frac{A^n}{n!} \triangleq B'_{h,\infty}(A,\ N)\,, \tag{17}$$

which can be recognized as the blocking probability that would result if (13c) and (13d) were modified to become

$$\begin{aligned}\Pr\{k,\,t{+}dt\} &= \nu\,dt\,\Pr\{k{-}1,\,t\} + (1-\nu\,dt - k\mu\,dt)\Pr\{k,\,t\} \\ &\qquad + (k{+}1)\mu\,dt\,\Pr\{k{+}1,\,t\},\; N \le k < N{+}Q; \end{aligned} \tag{18a}$$

$$\Pr\{N{+}Q,\,t{+}dt\} = \nu\,dt\,\Pr\{N{+}Q{-}1,\,t\} + [1-(N{+}Q)\mu\,dt]\Pr\{N{+}Q,\,t\}. \tag{18b}$$

The changes from (13c) and (13d) consist in making the probability that a call is completed in the interval dt proportional to the total number of calls in the system, including the queue, rather than just the number of calls currently being serviced by the N channels. The significance of these modified equations is that (17) reflects an assumption that a queued call is held for the original duration of that call, whether or not during that time a channel becomes free to service the call, with queued calls being regarded as blocked (even though some of them will eventually be serviced).

B. **Blocking Probability Analyses for a Finite Number of Sources**

When there is a finite number, M, of users or sources, the amount of offered traffic is affected by the number of users whose calls are already in the system, and it is customary to consider the offered traffic as M times the average source activity. Using ν' to denote the average call arrival rate for an individual source that is free to make a call, then when k users' calls are in the system the state-dependent call arrival rate is

$$\nu_k = (M-k)\,\nu'. \tag{19}$$

Cleared Calls. Taking into account the dependence of the call arrival rate on the number of calls in the system, the probability equations (6) become

$$\begin{aligned}\Pr\{0,\,t{+}dt\} &= (1-\nu_0\,dt)\Pr\{0,\,t\} + \mu\,dt\,\Pr\{1,\,t\} \\ &= (1-M\nu' dt)\Pr\{0,\,t\} + \mu\,dt\,\Pr\{1,\,t\},\; k=0; \end{aligned} \tag{20a}$$

$$\begin{aligned}\Pr\{k,\,t{+}dt\} &= \nu_{k-1}\,dt\,\Pr\{k{-}1,\,t\} + (1-\nu_k\,dt - k\mu\,dt)\Pr\{k,\,t\} \\ &\qquad + (k{+}1)\mu\,dt\,\Pr\{k{+}1,\,t\} \\ &= (M{-}k{+}1)\,\nu' dt\,\Pr\{k{-}1,\,t\} + [1-(M{-}k)\,\nu' dt - k\mu\,dt]\Pr\{k,\,t\} \\ &\qquad + (k{+}1)\mu\,dt\,\Pr\{k{+}1,\,t\},\; 0<k<N; \end{aligned} \tag{20b}$$

$$\begin{aligned}\Pr\{N,\,t{+}dt\} &= \nu_{N-1}\,dt\,\Pr\{N{-}1,\,t\} + (1-N\mu\,dt)\Pr\{N,\,t\} \\ &= (M{-}N{+}1)\,\nu' dt\,\Pr\{N{-}1,\,t\} + (1-N\mu\,dt)\Pr\{N,\,t\},\; k=N. \end{aligned} \tag{20c}$$

At steady state these equations lead to

$$M\nu'\,\mathrm{P}(0) = \mu\,\mathrm{P}(1),\; k=0; \tag{21a}$$

$$[(M{-}k)\,\nu' {+} k\mu]\,\mathrm{P}(k) = (M{-}k{+}1)\,\nu'\,\mathrm{P}(k{-}1) + (k{+}1)\mu\,\mathrm{P}(k{+}1),\; 0<k<N; \tag{21b}$$

$$N\mu\,\mathrm{P}(N) = (M{-}N{+}1)\,\nu'\,\mathrm{P}(N{-}1),\; k=N; \tag{21c}$$

together with the normalization requirement

$$\sum_{k=0}^{N} \mathrm{P}(k) = 1. \tag{21d}$$

Using $\rho_0' = \dfrac{\nu'}{\mu}$ the system of equations (21) is solved by the probability expression

$$\mathrm{P}(k) = \begin{cases} \dbinom{M}{k} (\rho_0')^k \left(\displaystyle\sum_{n=0}^{N} \dbinom{M}{n} (\rho_0')^n \right)^{-1}, \; 0 \le k \le N & M > N \\ 0, & k > N. \end{cases} \tag{22}$$

These probabilities form the *Engset* or *Truncated Bernoulli* probability distribution [2].

Using $A' \triangleq M\rho_0'$ to relate traffic in Erlangs to the free source activity, in terms of ρ_0' the probability that the system is blocked, called the “time congestion" ([2], [3]), is

$$\begin{aligned} B &= \Pr\{N \text{ channels are busy}\} = \mathrm{P}(N) \\ &= \frac{\binom{M}{N} (\rho_0')^N}{\sum_{n=0}^{N} \binom{M}{n} (\rho_0')^n} \triangleq B_{c,M}(\rho_0', N). \end{aligned} \tag{23a}$$

The probability that a user's call is lost given that the call arrives, called the “call congestion," is the probability that N channels are being used by the other $M-1$ users:

$$\mathrm{P}_L = \frac{\binom{M-1}{N} (\rho_0')^N}{\sum_{n=0}^{N} \binom{M-1}{n} (\rho_0')^n}. \tag{23b}$$

Note that for $M \to \infty$, the blocking and loss probabilities become the same for a system in which calls are cleared.

The offered traffic $A \triangleq M\rho_0$ for a finite number of sources is defined in terms of the average source activity $\rho_0 = \mathrm{E}\{\nu_k / M\mu\}$, where

$$\begin{aligned} \rho_0 = \mathrm{E}\left\{ \frac{\nu'}{\mu} \cdot \frac{M-k}{M} \right\} &= \rho_0' \sum_{k=0}^{N} \frac{M-k}{M} \mathrm{P}(k) = \rho_0' \, G_M^{-1} \sum_{k=0}^{N} \frac{M-k}{M} \binom{M}{k} (\rho_0')^k \\ &= \rho_0' \, G_M^{-1} \sum_{k=0}^{N} \binom{M-1}{k} (\rho_0')^k = \rho_0' G_{M-1} / G_M \,, \end{aligned} \tag{24a}$$

where

$$G_M \triangleq \sum_{k=0}^{N} \binom{M}{k} (\rho_0')^k. \tag{24b}$$

The quantity G_M can be expanded to yield the expression

$$G_M = \sum_{k=0}^{N}\left[\binom{M-1}{k} + \binom{M-1}{k-1}\right](\rho_0')^k = G_{M-1} + \sum_{k=1}^{N}\binom{M-1}{k-1}(\rho_0')^k$$
$$= G_{M-1} + \sum_{k=0}^{N-1}\binom{M-1}{k}(\rho_0')^{k+1} = (1+\rho_0')G_{M-1} - \binom{M-1}{N}(\rho_0')^{N+1}$$
$$= \{1 + \rho_0'[1 - B_{c,M-1}(\rho_0', N)]\}G_{M-1}\,, \tag{25}$$

which, when substituted in (24a), leads to the expression

$$\rho_0 = \frac{\rho_0'}{1+\rho_0'[1-B_{c,M-1}(\rho_0', N)]} < \frac{\rho_0'}{1+\rho_0'[1-B_{c,M}(\rho_0', N)]}\,; \tag{26}$$

the upper bound is commonly used ([3], [4]), and its accuracy improves as $\frac{M}{N}$ increases. Rather than solve (26) for ρ_0' to substitute in (23a), the quantities in both of these equations are calculated parametrically, and $B_{c,M}(\rho_0', N)$ may be plotted as a function of $A = M\rho_0(\rho_0')$.

Held Calls. Assuming the total queue size is $Q = M - N$, the probability transition equations for a finite number of sources and held calls are

$$\Pr\{0, t{+}dt\} = (1 - M\nu' dt)\Pr\{0, t\} + \mu\, dt\Pr\{1, t\},\ k = 0; \tag{27a}$$
$$\Pr\{k, t{+}dt\} = (M{-}k{+}1)\,\nu' dt\Pr\{k{-}1, t\} + [1 - (M{-}k)\,\nu' dt - k\mu\, dt]\Pr\{k, t\} + (k{+}1)\mu\, dt\Pr\{k{+}1, t\},\ 0 < k < N; \tag{27b}$$
$$\Pr\{k, t{+}dt\} = (M{-}k{+}1)\,\nu' dt\Pr\{k{-}1, t\} + [1 - (M{-}k)\,\nu' dt - N\mu\, dt]\Pr\{k, t\} + N\mu\, dt\Pr\{k{+}1, t\},\ N \le k < M; \tag{27c}$$
$$\Pr\{M, t{+}dt\} = \nu' dt\Pr\{M{-}1, t\} + (1 - N\mu\, dt)\Pr\{M, t\},\ k = M. \tag{27d}$$

At steady state these equations lead to

$$M\nu'\,\mathrm{P}(0) = \mu\,\mathrm{P}(1),\ k = 0; \tag{28a}$$
$$[(M{-}k)\nu' {+} k\mu]\,\mathrm{P}(k) = (M{-}k{+}1)\,\nu'\,\mathrm{P}(k{-}1) + (k{+}1)\mu\,\mathrm{P}(k{+}1),\ 0 < k < N; \tag{28b}$$
$$[(M{-}k)\nu' {+} N\mu]\,\mathrm{P}(k) = (M{-}k{+}1)\,\nu'\mathrm{P}(k{-}1) + N\mu\,\mathrm{P}(k{+}1),\ N \le k < M; \tag{28c}$$
$$N\mu\,\mathrm{P}(M) = \nu'\,\mathrm{P}(M{-}1),\ k = M; \tag{28d}$$

together with the normalization requirement

$$\sum_{k=0}^{M}\mathrm{P}(k) = 1. \tag{28e}$$

Using $\rho_0' = \dfrac{\nu'}{\mu}$, the system of equations (28) is solved by the expression

$$\mathrm{P}(k) = \begin{cases} \binom{M}{k} (\rho_0')^k F_M^{-1}, & 0 \le k < N \\ \binom{M}{k} \left(\frac{\rho_0'}{N}\right)^k \frac{k!\, N^N}{N!}\, F_M^{-1}, & N \le k \le M \\ 0, & k > M \end{cases} \tag{29a}$$

where

$$\begin{aligned} F_M &= \sum_{n=0}^{N-1} \binom{M}{n} (\rho_0')^n + \frac{N^N}{N!} \sum_{n=N}^{M} n! \binom{M}{n} \left(\frac{\rho_0'}{N}\right)^n && \text{(29b)} \\ &= \sum_{n=0}^{N} \binom{M}{n} (\rho_0')^n + \frac{N^N}{N!} \sum_{n=N+1}^{M} n! \binom{M}{n} \left(\frac{\rho_0'}{N}\right)^n \\ &= \sum_{n=0}^{N} \binom{M}{n} (\rho_0')^n + \binom{M}{N} (\rho_0')^N \sum_{n=1}^{M-N} n! \binom{M-N}{n} \left(\frac{\rho_0'}{N}\right)^n . && \text{(29c)} \end{aligned}$$

In this case of a finite number of sources and held calls, using (29b) the blocking probability is

$$B = \sum_{k=N}^{M} \mathrm{P}(k) = \frac{\frac{N^N}{N!} \sum\limits_{n=N}^{M} n! \binom{M}{n} \left(\frac{\rho_0'}{N}\right)^n}{\sum\limits_{n=0}^{N-1} \binom{M}{n} (\rho_0')^n + \frac{N^N}{N!} \sum\limits_{n=N}^{M} n! \binom{M}{n} \left(\frac{\rho_0'}{N}\right)^n} \triangleq B_{h,M}(\rho_0', N). \tag{30a}$$

Using the factoring shown in (29c), we can also write

$$B_{h,M}(\rho_0', N) = \frac{B_{c,M}(\rho_0', N) \cdot \sum\limits_{n=0}^{M-N} n! \binom{M-N}{n} \left(\frac{\rho_0'}{N}\right)^n}{1 + B_{c,M}(\rho_0', N) \cdot \sum\limits_{n=1}^{M-N} n! \binom{M-N}{n} \left(\frac{\rho_0'}{N}\right)^n} . \tag{30b}$$

The average source activity ρ_0 is

$$\begin{aligned} \rho_0 &= \rho_0' \sum_{k=0}^{M} \frac{M-k}{M} \mathrm{P}(k) = \rho_0' \sum_{k=0}^{M-1} \frac{M-k}{M} \mathrm{P}(k) \\ &= \rho_0' F_M^{-1} \left\{ \sum_{k=0}^{N-1} \binom{M-1}{k} (\rho_0')^k + \frac{N^N}{N!} \sum_{k=N}^{M-1} k! \binom{M-1}{k} \left(\frac{\rho_0'}{N}\right)^k \right\} \\ &= \rho_0' F_{M-1} / F_M , && \text{(31a)} \end{aligned}$$

where the normalization factor F_M can be expanded as follows:

$$\begin{aligned}
F_M &= \sum_{n=0}^{N}\left[\binom{M-1}{n}+\binom{M-1}{n-1}\right](\rho_0')^n + \frac{N^N}{N!}\sum_{n=N+1}^{M} n!\left[\binom{M-1}{n}+\binom{M-1}{n-1}\right]\left(\frac{\rho_0'}{N}\right)^n \\
&= F_{M-1} + \sum_{n=1}^{N}\binom{M-1}{n-1}(\rho_0')^n + \frac{N^N}{N!}\sum_{n=N+1}^{M} n!\binom{M-1}{n-1}\left(\frac{\rho_0'}{N}\right)^n \\
&= F_{M-1} + \sum_{n=0}^{N-1}\binom{M-1}{n}(\rho_0')^{n+1} + \frac{N^N}{M\cdot N!}\sum_{n=N+1}^{M} n\,(n!)\binom{M}{n}\left(\frac{\rho_0'}{N}\right)^n \\
&> F_{M-1}\{1+\rho_0'[1-B_{h,M-1}(\rho_0',N)]\} + \frac{N}{M}F_M B_{h,M}(\rho_0',N) - \frac{N}{M}\binom{M}{N}(\rho_0')^N \\
&< F_{M-1}\{1+\rho_0'[1-B_{h,M-1}(\rho_0',N)]\} + F_M B_{h,M}(\rho_0',N) - \frac{N}{M}\binom{M}{N}(\rho_0')^N;
\end{aligned} \tag{31b}$$

solving for F_{M-1}/F_M leads to the inequality

$$\frac{\rho_0'[1-B_{h,M}(\rho_0',N)]}{1+\rho_0'[1-B_{h,M-1}(\rho_0',N)] - \frac{\binom{M}{N}(\rho_0')^N}{\frac{M}{N}F_{M-1}}} < \rho_0 < \frac{\rho_0'[1-\frac{N}{M}B_{h,M}(\rho_0',N)]}{1+\rho_0'[1-B_{h,M-1}(\rho_0',N)] - \frac{\binom{M}{N}(\rho_0')^N}{\frac{M}{N}F_{M-1}}}\,. \tag{31c}$$

For $M \gg N$ the upper bound in (31c) approaches the expression for ρ_0 in (26)—with $B_{h,M}$ replacing $B_{c,M}$—and both the offered traffic and the blocking probability for held calls can be calculated parametrically as functions of ρ_0'. Therefore, the calculations use

$$A = M\rho_0 \lesssim \frac{M\rho_0'}{1+\rho_0'[1-B_{h,M}(\rho_0',N)]}\,. \tag{31d}$$

Analogous to Molina's approach that leads to (17) above, an alternate expression for the blocking probability with calls held and an infinite number of sources, there is an alternate expression for calls held and a finite number of sources [3]. According to this approach, the appearance and disappearance of held calls on the queue is treated in the same manner as carried calls, as if there were M channels, although the system is said to be blocked if there are N or more channels occupied (time congestion) and a call is said to be held (not lost) if N or more channels are occupied when a call arrives from a free source (call congestion). The transition equations are

$$\Pr\{0,t+dt\} = (1-M\nu'dt)\Pr\{0,t\} + \mu\,dt\Pr\{1,t\},\ k=0; \tag{32a}$$

$$\begin{aligned}\Pr\{k,t+dt\} = (M-k+1)\,\nu'dt\Pr\{k-1,t\} + [1-(M-k)\,\nu'dt - k\mu\,dt]\Pr\{k,t\} \\ + (k+1)\mu\,dt\Pr\{k+1,t\},\ 0<k<M;\end{aligned} \tag{32b}$$

$$\Pr\{M,t+dt\} = \nu'dt\Pr\{M-1,t\} + (1-M\mu\,dt)\Pr\{M,t\},\ k=M. \tag{32c}$$

At steady state these equations lead to

$$M\nu'\,\mathrm{P}(0) = \mu\,\mathrm{P}(1),\ k=0; \tag{33a}$$

$$[(M-k)\nu'+k\mu]\,\mathrm{P}(k) = (M-k+1)\,\nu'\,\mathrm{P}(k-1) + (k+1)\mu\,\mathrm{P}(k+1),\ 0<k<M; \tag{33b}$$

$$M\mu\,\mathrm{P}(M) = \nu'\,\mathrm{P}(M-1),\ k=M; \tag{33c}$$

together with the normalization requirement

$$\sum_{k=0}^{M}\mathrm{P}(k) = 1. \tag{33d}$$

The system of equations is solved by

$$\mathrm{P}(k) = \begin{cases}\binom{M}{k}(\rho_0')^k(1+\rho_0')^{-M}, & 0\le k\le M\\ 0, & k>M.\end{cases} \tag{34a}$$

With no calls cleared, the average source activity ρ_0 is

$$\rho_0 = \frac{\rho_0'}{(1+\rho_0')^M}\sum_{k=0}^{M}\frac{M-k}{M}\binom{M}{k}(\rho_0')^k = \frac{\rho_0'}{(1+\rho_0')^M}(1+\rho_0')^{M-1} = \frac{\rho_0'}{1+\rho_0'}. \tag{34b}$$

Substitution in (34a) gives the blocking probability (time congestion)

$$\begin{aligned}B = \sum_{n=N}^{M}\mathrm{P}(n) &= \frac{\sum\limits_{n=N}^{M}\binom{M}{n}(\rho_0')^n}{(1+\rho_0')^M} = \frac{\sum\limits_{n=N}^{M}\binom{M}{n}\left(\frac{\rho_0}{1-\rho_0}\right)^n}{\left(1+\frac{\rho_0}{1-\rho_0}\right)^M}\\ &= \sum_{n=N}^{M}\binom{M}{n}\rho_0^n(1-\rho_0)^{M-n} \triangleq B'_{h,M}(\rho_0,N).\end{aligned} \tag{35}$$

The call congestion (holding probability) is the probability that N channels are busy with $M-1$ users:

$$\mathrm{P}_h = \sum_{n=N}^{M-1}\binom{M-1}{n}\rho_0^n(1-\rho_0)^{M-1-n}. \tag{36}$$

C. **Numerical Comparisons Of Blocking Formulas**

In this section we present numerical comparisons of the various formulas shown above for blocking probability. The formulas used in the numerical comparisons are the following:

Formula	Eq. #	#Sources	Treatment of Lost Calls
$B_{c,\infty}$	(10)	Infinite	Cleared (Erlang B)
$B_{r,\infty}$	(10), (12)	Infinite	Returning
$B_{h,\infty}$	(16c)	Infinite	Held (Erlang C)
$B'_{h,\infty}$	(17)	Infinite	Held (Molina approach)
$B_{c,M}$	(23a), (26)	Finite	Cleared (Engset)
$B_{h,M}$	(30b), (31d)	Finite	Held
$B'_{h,M}$	(35)	Finite	Held (Bernoulli: Molina approach)

In the numerical results that follow, these formulas are plotted together as functions of the amount of offered traffic, A, in Erlangs, for a given number of channels, N. The calculation of the formulas was done using the BASIC program listed in the Appendix, which generated a file for input to a plotting program.

In Figures 2 through 8, respectively, the seven formulas for blocking probability are plotted, one formula per figure, as functions of the amount of offered traffic per channel, A/N. The values $N =$ 8, 10, 13, 30, and 62 are used in each figure to produce a family of curves. From this set of figures we are able to observe the following comparative behavior of the formulas: all of the held calls formulas produce values that are independent or relatively independent of N in the neighborhood of $A = N$, whereas the cleared calls formulas and calls returning formulas do not. For the values of N used in these figures, the blocking probability for held calls is roughly 50% when the amount of offered traffic equals the number of channels, except for the Erlang C formula, which gives $B = 1$ when $A = N$.

The significance of these observations is that the selection of a formula for blocking probability is not as critical for small levels of traffic, but becomes more important as the level of traffic approaches the number of channels; the Erlang B formula then gives an answer that can be an order of magnitude different from that provided by the Erlang C formula. Traditionally, formulas based on the infinite-sources, held calls model—that is, the Erlang C and the formula based on Molina's approximate analysis—have been preferred for network performance estimates because they provide a conservative estimate. Intuitively the Molina formula is more satisfying as well, since if the *average* amount of offered traffic in Erlangs equals the number of channels, then—assuming that the average value is close to the median value—it is very likely that about 50% of the the time the traffic will be greater than N, and arriving calls will be blocked.

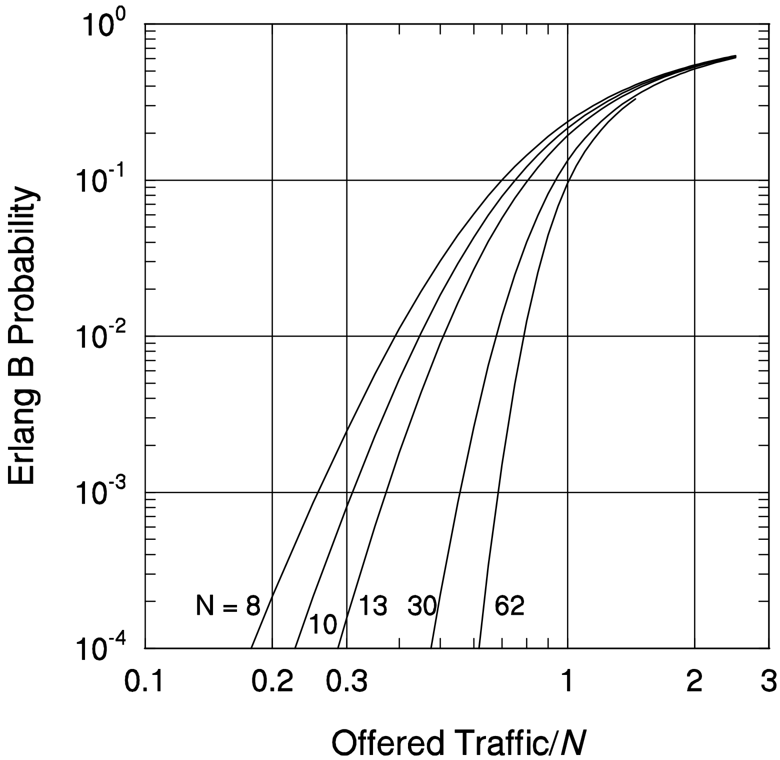

Figure 2. Erlang B Formula with N Varied

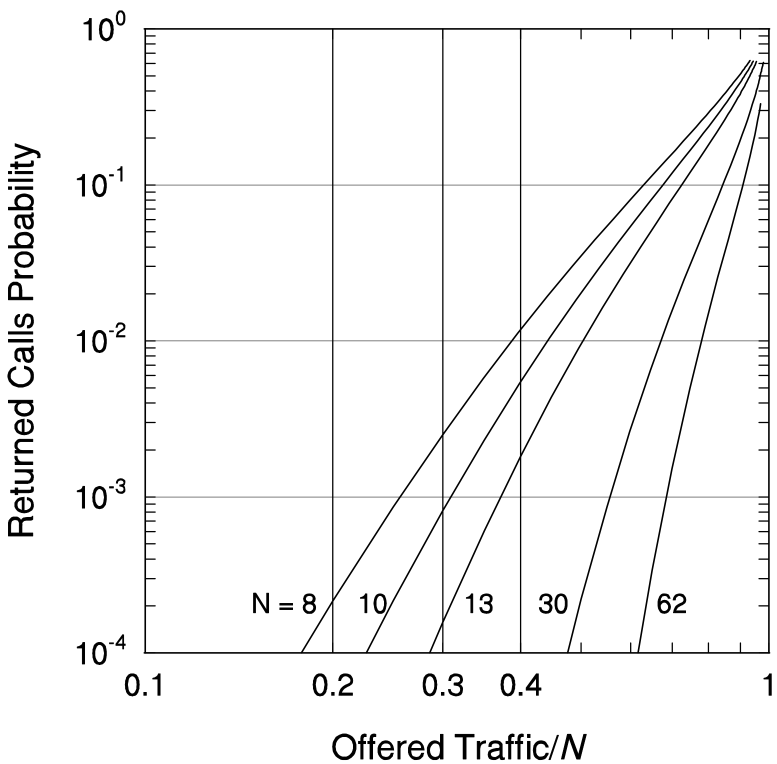

Figure 3. Returned Calls Formula With N Varied

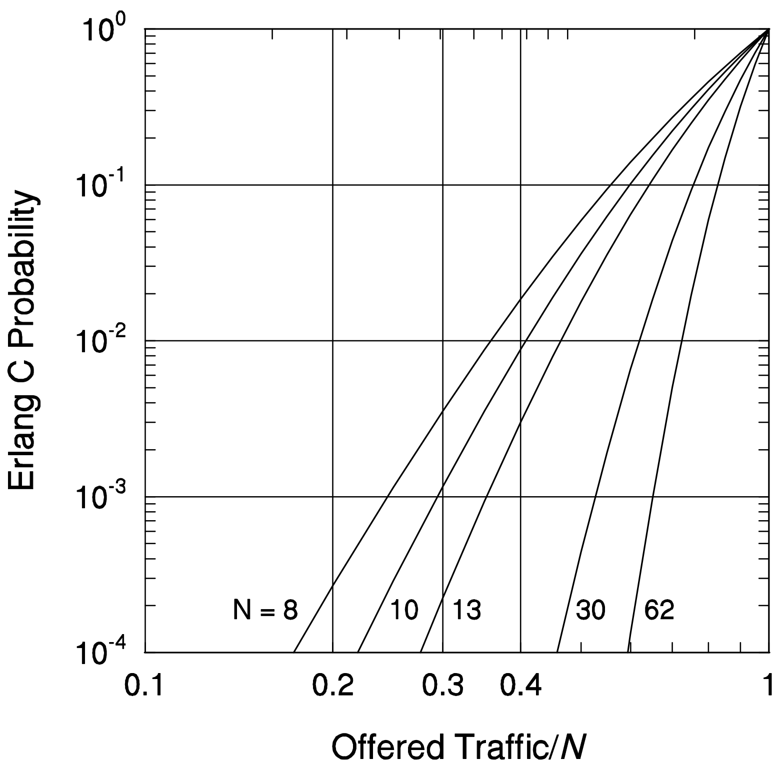

Figure 4. Erlang C Formula with N Varied

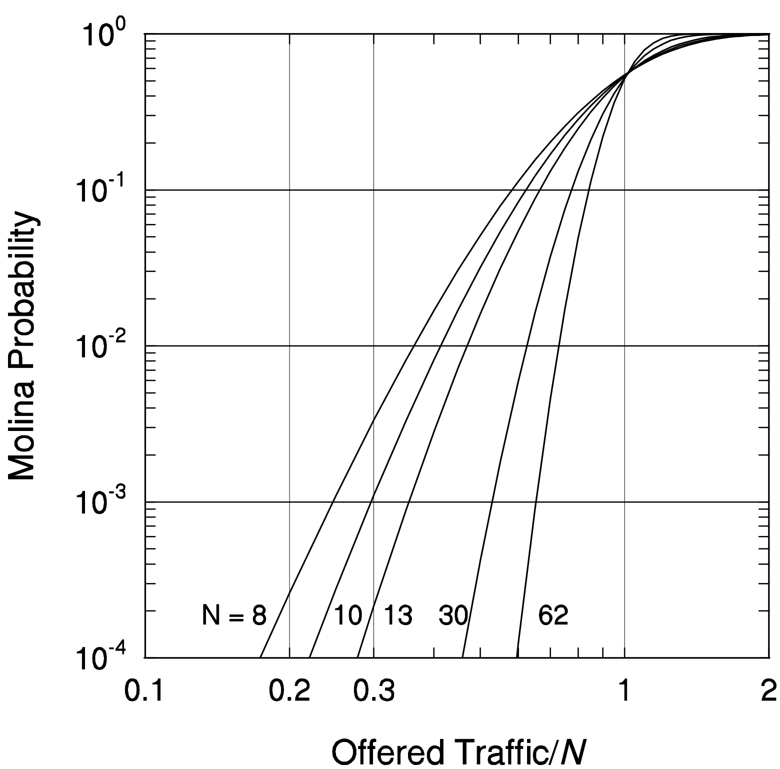

Figure 5. Molina Formula With N Varied

N	8	10	13	30	62
Number of sources	23	26	41	88	219

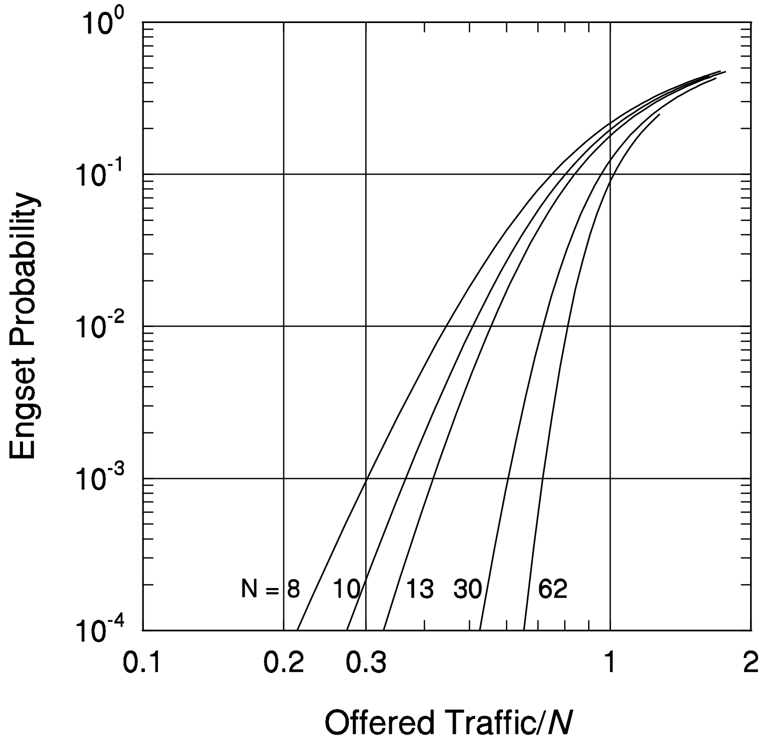

Figure 6. Engset-Cleared Formula with N Varied

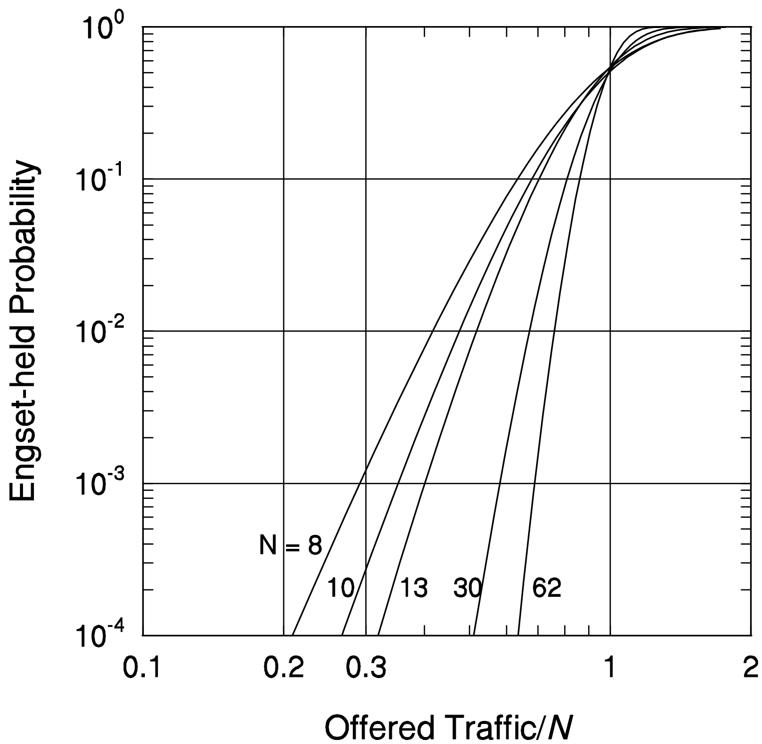

Figure 7 Engset-Held Calls Formula with N Varied

N	8	10	13	30	62
Number of sources	23	26	41	88	219

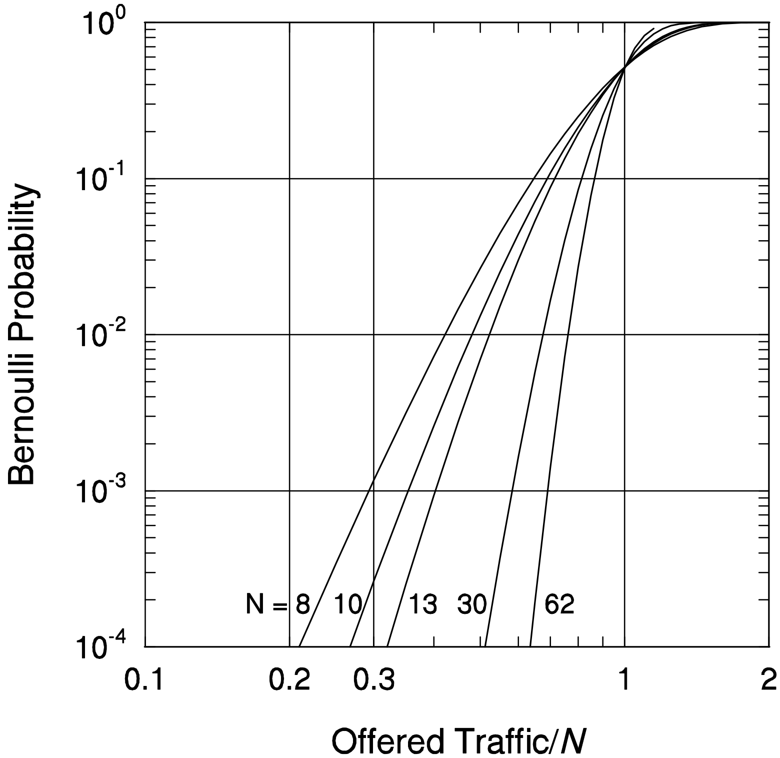

Figure 8. Bernoulli Formula with N Varied

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[2] P. Beckmann, *Elementary Queuing Theory and Telephone Traffic*. A volume in a series on telephone traffic published by Lee's ABC of the Telephone, Geneva, IL. 1977.

[3] J. Bellamy, *Digital Telephony*. Wiley, New York. 1982.

[4] D. Bear, *Principles of Telecommunication Traffic Engineering* (revised third edition). Peter Peregrinus Ltd on behalf of the Institution of Electrical Engineers, London. 1988.

APPENDIX: COMPUTER PROGRAM FOR CALCULATING BLOCKING PROBABILITIES

The following program, written in GW-BASIC©, calculates the seven formulas for blocking probability for plotting and comparison, as discussed above.

Key to program variables:

A	offered traffic
PB1	Erlang B formula
AR	traffic modified to account for returning calls
PB2	Erlang C formula
PB3	Molina formula
PB4	Engset-cleared formula
AE	traffic modified for Engset formula
PB5	Engset-held formula
PB6	Bernoulli formula

```
10 REM BLOCKING PROBABILITIES--7/92
20 INPUT "Enter output filename"; F$
30 OPEN "O", #1, F$
40 LPRINT "Filename = ";F$
50 INPUT "Enter number of channels"; N
60 INPUT "Enter number of users";M
70 LPRINT N;" channels and ";M;" users"
80 DA=N/20
130 LPRINT
140 LPRINT USING"\     \"; "Load";"LRet";"LE";"LQ";"EB";"EC";"Mol";"ES";
"ESH";"MESH"
150 PRINT #1, "Load", "PB1","LRet","PB2","PB3","PB4","LE","PB5","LQ","PB6"
160 FOR I=1 TO 50
170 A=I*DA
180 GOSUB 1000
200 GOSUB 1300
220 LPRINT USING "###.## ";A,AR,AE,AQ,
225 LPRINT USING "#.#### ";PB1,PB2,PB3,PB4,PB5,PB6
230 PRINT #1, A,PB1,AR,PB2,PB3,PB4,AE,PB5,AQ,PB6
240 NEXT I
300 STOP
1000 REM Infinite-source formulas
1010 SUM=1
1020 TERM=1
1030 FOR K=1 TO N
1040 TERM=TERM*A/K
1050 SUM=SUM+TERM
1060 NEXT K
1070 PB1=TERM/SUM
1080 AR=A*(1-PB1)
1090 PB2=PB1/(1-A/N+A*PB1/N)
1100 PB3=1-EXP(-A)*(SUM-TERM)
1110 RETURN
1300 REM Finite-source formulas
1305 RHOP=A/M
1310 TERM=1
1320 SUM=1
1330 FOR K=1 TO N
1340 TERM=TERM*RHOP*(M+1-K)/K
```

```
1350 SUM=SUM+TERM
1360 NEXT K
1370 PB4=TERM/SUM
1380 AE=A/(1+(1-PB4)*RHOP)
1390 RON=RHOP/N
1400 SUM=1
1410 TERM=1
1420 FOR K=1 TO M-N
1430 TERM=TERM*(M-N-K+1)*RON
1440 SUM=SUM+TERM
1450 NEXT K
1460 Y=1/SUM
1470 PB5=PB4/(Y+(1-Y)*PB4)
1480 AQ=A/(1+RHOP)
1490 TERM=(1-RHOP)^M
1500 FOR K=1 TO N
1510 TERM=TERM*(M-K+1)*RHOP/K/(1-RHOP)
1520 NEXT K
1530 SUM=TERM
1540 FOR K=N+1 TO M
1550 TERM=TERM*(M-K+1)*RHOP/K/(1-RHOP)
1560 SUM=SUM+TERM
1570 NEXT K
1580 PB6=SUM
1590 RETURN
2000 END
```