Loop calculus in statistical physics and information theory

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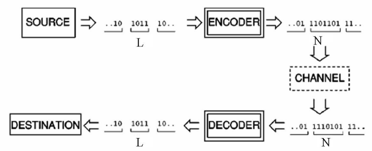
Thanks to M. Stepanov (UofA, Tucson)

Outline

Error Correction

Cell phone Hard disk Optical disk Fiber

Scheme:



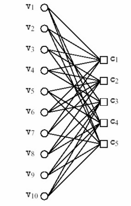
Example of Additive White Gaussian Channel:

$$P(\mathbf{x}_{out}|\mathbf{x}_{in}) = \prod_{i=bits} p(x_{out;i}|x_{in;i})$$

$$p(x|y) \sim \exp(-s^2(x-y)^2/2)$$

- Channel
  is noisy "black box" with only statistical information available
- Encoding:
  use redundancy to redistribute damaging effect of the noise
- Decoding:
  reconstruct most probable codeword by noisy (polluted) channel

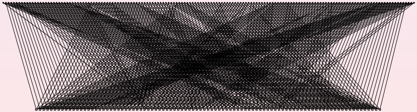
Low Density Parity Check Codes

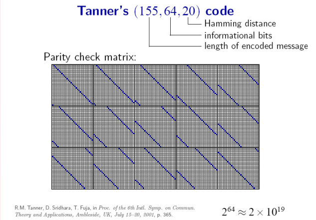


- N bits, M checks, $L = N - M$ information bits
  example: $N = 10$, $M = 5$, $L = 5$
- 2^L codewords of 2^N possible patterns
- Parity check: $\hat{H}\mathbf{v} = \mathbf{c} = \mathbf{0}$
  example:

$$\hat{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

- LDPC = graph (parity check matrix) is sparse





Maximum Likelihood/Maximum-a-Posteriori

Exhaustive search for pre-image = the best one can possibly do

$$\text{ML} = \arg \max_{\boldsymbol{\sigma}=\text{codewords}} P(\mathbf{x}_{\text{out}}|\boldsymbol{\sigma}); \quad \text{MAP} = \text{sign}\left(\frac{\sum_{\boldsymbol{\sigma}} \boldsymbol{\sigma} P(\mathbf{x}_{\text{out}}|\boldsymbol{\sigma})}{\sum_{\boldsymbol{\sigma}} P(\mathbf{x}_{\text{out}}\boldsymbol{\sigma})}\right)$$

MAP≈BP=Belief-Propagation (Bethe-Pieirls) Gallager '61

- Exact on a tree [Derivation Sketch]
- Trading optimality for reduction in complexity: $\sim 2^L \rightarrow \sim L$
- BP = solving equations on the graph:

$$\eta_{j\alpha} = h_j + \sum_{\beta\neq\alpha}^{j\in\beta} \tanh^{-1}\left(\prod_{i\neq j}^{i\in\beta} \tanh \eta_{i\beta}\right)$$

- Message Passing = iterative BP
- Applies to a general inference problem on a (sparse) graph

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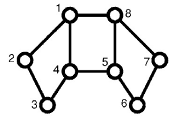
Graphical models of Statistical Inference

Factorization (Forney '01, Loeliger '01)

$$P\{\boldsymbol{\sigma}\} = Z^{-1}\prod_{a\in X} f_a(\boldsymbol{\sigma}_a)$$

$$Z = \sum_{\{\boldsymbol{\sigma}\}} P\{\boldsymbol{\sigma}\}$$

X = edges



$f_a \geq 0$

$\sigma_{ab} = \sigma_{ba} = \pm 1$

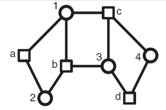
$\boldsymbol{\sigma}_1 = (\sigma_{12}, \sigma_{14}, \sigma_{18})$

$\boldsymbol{\sigma}_2 = (\sigma_{12}, \sigma_{13})$

Error-Correction (bipartite)

$$f_i(\boldsymbol{\sigma}_i) = \begin{cases} 1, & \sigma_{i\alpha} = \sigma_{i\beta} \\ 0, & \text{otherwise} \end{cases}$$

$$f_\alpha(\boldsymbol{\sigma}_\alpha) = \delta\left(\prod_{i\in\alpha} \sigma_i, +1\right) \exp\left(\sum_{i\in\alpha} \sigma_i h_i / q_i\right)$$



h_i - log-likelihoods

q_i-connectivity degrees

Bethe free energy: variational approach
(Yedidia,Freeman,Weiss '01 - inspired by Bethe '35, Peierls '36)

$$F = -\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln f_a(\boldsymbol{\sigma}_a) + \sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln b_a(\boldsymbol{\sigma}_a) - \sum_{(a,c)} b_{ac}(\sigma_{ac}) \ln b_{ac}(\sigma_{ac})$$

constraints:

$$\forall\ a,c;\ c \in a: \quad 0 \le b_a(\boldsymbol{\sigma}_a), b_{ac}(\sigma_{a,c}) \le 1$$

$$\forall\ a,c;\ c \in a: \quad \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) = \sum_{\sigma_{a,c}} b_{ac}(\sigma_{a,c}) = 1$$

$$\forall\ a;\ c \in a: \quad b_{ac}(\sigma_{ac}) = \sum_{\boldsymbol{\sigma}_a \setminus \sigma_{ac}} b_a(\boldsymbol{\sigma}_a)$$

Belief-Propagation Equations:

$$\left.\frac{\delta F}{\delta b}\right|_{\text{constr.}} = 0$$

▸ Variational Method in Stat Mech

- Relaxation to minimum of the Bethe Free energy enforces convergence of iterative BP (Stepanov, Chertkov '06)

LP decoding
Feldman, Wainwright, Karger '03

- LP decoding = minimization of a linear function over a bounded domain described by linear constraints
- Fast and Discrete
- "Large SNR" limit of BP: $F \approx E = -\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln f_a(\boldsymbol{\sigma}_a)$

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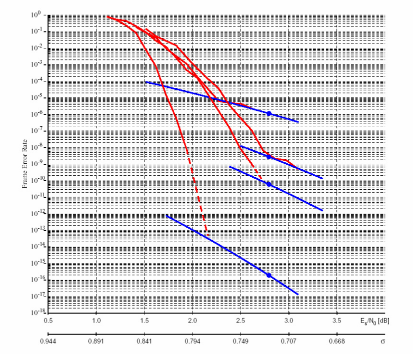
Error-Floor



T. Richardson, Allerton '03

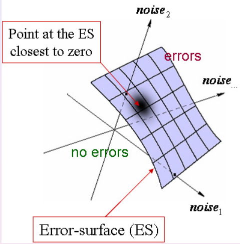
- BER vs SNR = measure of performance
- Waterfall $\leftrightarrow$ Error-floor
- Suboptimal decoding causes error-floor: at $E_s/N_0 \to \infty$, $FER_{ML} \sim \exp(-d_{ML} E_s/N_0)$ vs $FER_{sub} \sim \exp(-d_{sub} E_s/N_0)$ where $d_{ML} > d_{sub}$
- Monte-Carlo is useless at BER $\lesssim 10^{-8}$

Pseudo-codewords and Instantons

Error-floor is caused by (rare) troublemakers

instantons → pseudo-codewords

log-likelihoods → a-posteriori log-likelihoods



Bibliography:

- Pseudo-codewords & Error-floor:
  Wiberg '96; Forney et.al'99; Frey et.al '01; Richardson '04; Vontobel, Koetter '04–'06
- Instantons:
  Stepanov et.al '04–'06 ▸ Pseudo-Codeword Search

Shopping list

- Analyze the troublemakers
- Improve decoding to reduce the error-floor

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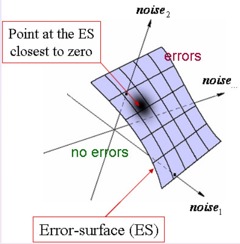

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(Chertkov,Chernyak '06)

Exact expression (for partition function, etc) in terms of BP

$$Z=\sum_{\boldsymbol{\sigma}_\sigma}\prod_a f_a(\boldsymbol{\sigma}_a)=Z_0\left(1+\sum_C r(C)\right)$$

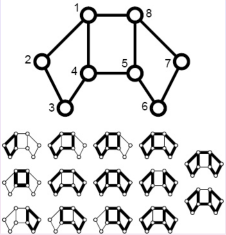
$$r(C)=\frac{\prod\limits_{a\in C}\mu_a}{\prod\limits_{(ab)\in C}(1-m_{ab}^2)}$$

$C\in$ Generalized Loops $=$ Loops without loose ends

$$m_{ab}=\int d\boldsymbol{\sigma}_a b_a^{(bp)}(\boldsymbol{\sigma}_a)\sigma_{ab}$$

$$\mu_a=\int d\boldsymbol{\sigma}_a b_a^{(bp)}(\boldsymbol{\sigma}_a)\prod_{b\in a,C}(\sigma_{ab}-m_{ab})$$

Derivation Sketch



- The Loop Series is finite
- $b_{ab}^{(bp)}, b_a^{(bp)}, Z_0$ and all terms in the series are calculated within BP
- BP is exact on a tree
- BP is a Gauge fixing condition. Other choices of Gauges would lead to different representation.

Synthesis of Pseudo-Codeword Search Algorithm & Loop Calculus

- Consider pseudo-codewords one after other
- For an individual pseudo-codeword/instanton identify a critical loop giving major contribution to the loop series: $Z = Z_0(1+\sum_C r_C) \approx Z_0(1+r(\Gamma))$
- Hint: look for single connected loops and use local "triad" contributions as a tester:

$$r(\Gamma) = \prod_{\alpha\in\Gamma} \tilde{\mu}_\alpha^{(bp)}, \quad \tilde{\mu}_\alpha^{(bp)} = \frac{\mu_\alpha^{(bp)}}{\sqrt{(1-(m_i^{(bp)})^2)(1-(m_j^{(bp)})^2)}}$$

Proof-of-Concept test [(155, 64, 20) code over AWGN]

- $\forall$ pseudo-codewords with $16.4037 < d < 20$ (~ 200 found) there always exists a simple single-connected critical loop(s) with $r(\Gamma) \sim 1$.
- Pseudo-codewords with the lowest d show $r(\Gamma) = 1$
- Invariant with respect to other choices of the original codeword
- Correction to log-likelihood at a bit of a critical loop brings the cumulative result to zero. Correction to an a-posteriori log-likelihood always aligns with correction to respective log-likelihood.

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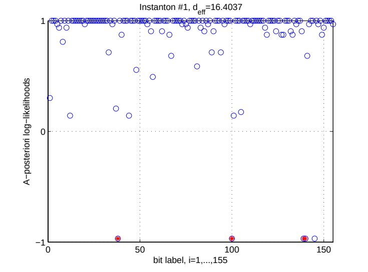
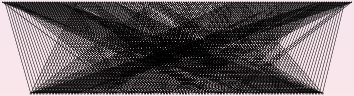


Bigger Set



Extended Variational Principle & Loop-Corrected BP

Bare BP Variational Principle:

$$\left.\frac{\delta Z_0}{\delta \eta_{ab}}\right|_{\eta^{(bp)}} = 0, \qquad Z_0 = \left(\prod_{bc} 2\cosh(\eta_{bc} + \eta_{cb})\right)^{-1} \sum_{\boldsymbol{\sigma}} \prod_a P_a(\boldsymbol{\sigma}_a)\Big|_{\eta^{(bp)}}$$

New choice of Gauges guided by the knowledge of the critical loop Γ

$$\left.\frac{\delta \exp(-\mathcal{F})}{\delta \eta_{ab}}\right|_{\eta_{\rm eff}} = 0, \quad \mathcal{F} \equiv -\ln(Z_0 + Z_\Gamma)$$

BP-equations are modified along the critical loop Γ

$$\left.\frac{\sum_{\sigma_a}(\tanh(\eta_{ab}+\eta_{ba})-\sigma_{ab})P_a(\sigma_a)}{\sum_{\sigma_a} P_a(\sigma_a)}\right|_{\eta_{\rm eff}} = \left.\frac{\prod_{d\in\Gamma}\mu_{d;\Gamma}}{\prod_{(a'b')\in\Gamma}(1-(m^{(*)}_{a'b'})^2)}\,\delta m_{a\to b;\Gamma}\right|_{\eta_{\rm eff}} \neq 0 \quad [\text{along } \Gamma]$$

Loop-Corrected BP Algorithm

- **1.** Run bare BP algorithm. Terminate if BP succeeds (i.e. a valid code word is found).
- **2.** If BP fails find the most relevant loop Γ that corresponds to the maximal $|r_\Gamma|$. Triad search is helping.
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LP-erasure (even simpler) algorithm

- **1.** Run LP algorithm. Terminate if LP succeeds (i.e. a valid code word is found).
- **2.** If LP fails, find the most relevant loop Γ that corresponds to the maximal amplitude $r(\Gamma)$.
- **3.** Modify the log-likelihoods along the loop Γ introducing a shift towards zero, i.e. introduce a complete or partial erasure of the log-likelihoods at the bits. Run LP with modified log-likelihoods. Terminate if the modified LP succeeds.
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$(155, 64, 20)$ Test

- **IT WORKS!**
  All troublemakers ($\sim$ 200 of them) previously found by LP-based Pseudo-Codeword-Search Algorithm method were successfully corrected by the loop-improved LP algorithm.
- Method is invariant with respect the choice of the codeword (used to generate pseudo-codewords).

General Conjecture:

Loop-erasure algorithm is capable of reducing the error-floor.

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Results

- Loop Calculus
  = Generic Tool for calculating marginal probabilities in terms of loops on underlying graphical structure
- Pseudo-Codeword-Search Algorithm
  = Efficient way of describing pseudo-codeword spectrum for LP-decoding
- Pseudo-Codewords are all associated with respective Critical Loops
- Effective Free Energy Approach
  = Variational Principle improving BP with a Critical Loop information
- Loop-corrected BP and Loop-erasure
  = Algorithms improving BP/LP with a Critical Loop information

Future Efforts

- Improve and continue testing the simple LP-erasure algorithm.
- The major improvement required is in automatization of the critical loop identification scheme. (Towards Monte Carlo test.)
- Testing other (longer) codes.
- Testing other (e.g. correlated) channels.
- All of the above for improving Loop-corrected BP.
- Stat Mech & Inf. Theory (inter-symbol interference, network capacity) on a d-dimensional lattice/graph with/without disorder.

Other complementary developments, e.g. on

- Improving BP [Survey Propagation = Mezard et.al '02; Generalized BP = Yedidia et.al '01]
- Correcting for Loops in BP [Montanarri, Rizzo '05; Parisi, Slanina '05]
- Accelerating convergence of bare BP [Stepanov, Chertkov '06]
- Reducing LP complexity [Taghavi, Siegel '06; Vontobel, Koetter '06]
- Improving LP [Dimakis, Wainwright '06]

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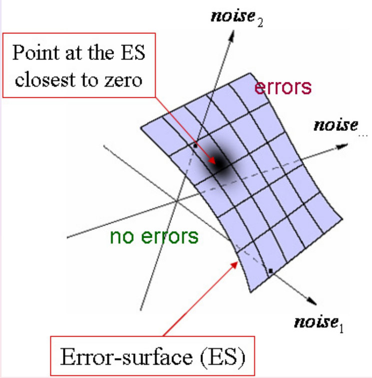
- V. Chernyak, M. Chertkov, M. Stepanov, B. Vasic, *Error correction on a tree: An instanton approach*, Phys. Rev. Lett. **93**, 198702-1 (2004).
- M.G. Stepanov, M. Chertkov, *Instanton analysis of Low-Density-Parity-Check codes in the error-floor regime*, arXiv:cs.IT/0601070, ISIT 2006 (July 2006, Seattle, WA)
- M.G. Stepanov, M. Chertkov, *Improving convergence of belief propagation decoding*, arXiv:cs.IT/0607112, 44th Allerton Conference (September 27-29, 2006, Allerton, IL)

All papers are available at http://cnls.lanl.gov/∼chertkov/pub.htm

Optimal Fluctuation

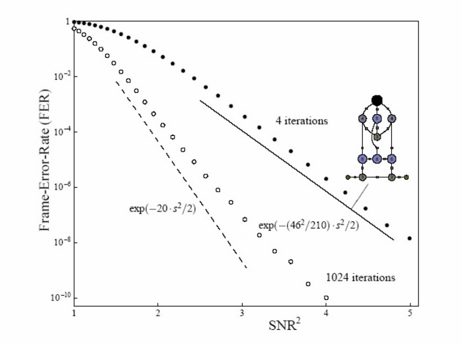
- Laplace method = Large deviation = Steepest descent = instanton = optimal fluctuation method
- BER $= \int d$noise Weight(noise)
- BER $\sim$ Weight $\begin{pmatrix} \text{optimal configuration} \\ \text{of the noise} \end{pmatrix}$
- $\begin{pmatrix} \text{optimal configuration} \\ \text{of the noise} \end{pmatrix} = \begin{pmatrix} \text{point at the ES} \\ \text{closest to zero} \end{pmatrix}$

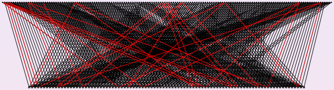


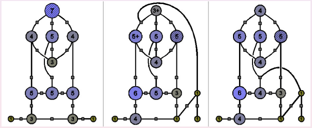
Chernyak, Chertkov, Stepanov, Vasic Phys.Rev.Lett **93**, 198702 (2004)

(155,64,20) Tanner code. Gaussian Channel.

- Instanton-amoeba (numerical minimization) implemented for 4-iterations of iterative BP



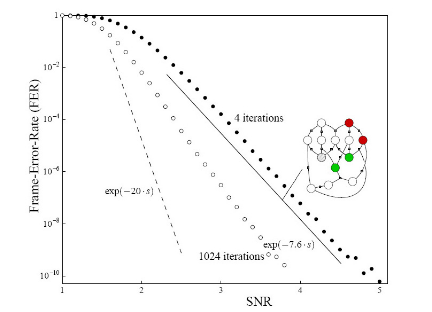


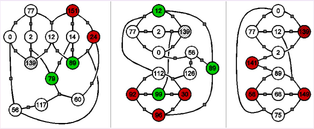


$l_{eff}^2 = \frac{46^2}{210} \approx 10.076$ $\quad l_{eff}^2 = \frac{806}{79} \approx 10.203$ $\quad l_{eff}^2 = \frac{44^2}{188} \approx 10.298$

Stepanov, Chertkov, Chernyak, Vasic, Phys Rev Lett **95**, *228701 (2005) + cond-mat/0506037 + ISIT 2006*

(155,64,20) Tanner code. Laplacian Channel.





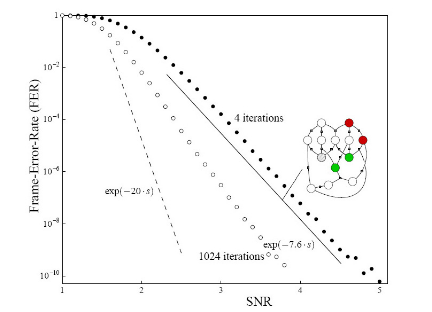
$l_{eff}^2 = 7.6$ $l_{eff}^2 = 8$ $l_{eff}^2 = 8$

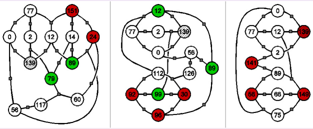
Lesson:

Strong dependence of the error-floor performance on the channel

Stepanov, Chertkov, 43rd Allerton conference (2005), arXiv:cs.IT/0507031

(155,64,20) Tanner code. Laplacian Channel.





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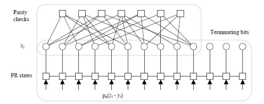
Stepanov, Chertkov, 43rd Allerton conference (2005), arXiv:cs.IT/0507031

Example: Inter-Symbol interference (Partial Response) channel

Di-code formulation

- output $= y_i = \sum_j J_{ij}\sigma_j + \xi_i$
  $\langle \xi_i \rangle = 0, \quad \langle \xi_i \xi_j \rangle = \delta_{ij}/s^2$
- $\boldsymbol{\sigma}$ is encoded by an LDPC code
- di-code=Inter-Symbol Interference + LDPC code
- Decoding of the di-code = solving inference problem on an extended Tanner graph



Our approach

- Formulate di-BP = minimum of the joint (di-) Bethe free energy
- Test iterative version of the di-BP against Monte-Carlo simulations
- Apply instanton (and instanton-LP) approach to analysis of di-BP (di-code) error-floor
- Develop loop-improved di-BP/LP

Anguita, Chertkov – in progress

Dilute Gas of Loops

If d (dimensionality) is large for a lattice problem or
If N is large for a random graph of size N:

$$Z = Z_0 \left(1 + \sum_C r_C\right) \approx Z_0 \prod_{C_{sc}} (1 + r_{sc})$$

C_{sc} are single connected loops.

- The approximation allows an easy multi-scale re-summation.

Ising model

General Graphical Model

$$W(\boldsymbol{\sigma}) = Z^{-1}\prod_{a\in X} f_a(\boldsymbol{\sigma}_a), \quad Z=\sum_{\boldsymbol{\sigma}}\prod_{a\in X} f_a(\boldsymbol{\sigma}_a),$$

Ising model

$$W(\boldsymbol{\sigma}) = Z_I^{-1}\exp\left(\beta\sum_{\alpha=(i,j)\in X}\sigma_i\sigma_j+\sum_{i\in X}h_i\sigma_i\right)$$

$$Z=\sum_{\boldsymbol{\sigma}}\prod_{\alpha=(i,j)\in X}\exp\left(\beta\sigma_i\sigma_j\right)=\sum_{\boldsymbol{\sigma}}\prod_{a\in X} f_a(\boldsymbol{\sigma}_a); \quad \{a\}=\{i\}\cup\{\alpha\}$$

$$f_i(\boldsymbol{\sigma}_i)=\begin{cases}\exp(h_i\sigma_i), & \sigma_{i\alpha}=\sigma_{i\beta}=\sigma_i \quad \forall\alpha,\beta\ni i\\ 0, & \text{otherwise;}\end{cases}$$

$$f_\alpha\left(\boldsymbol{\sigma}_\alpha=(\sigma_{\alpha i},\sigma_{\alpha j})\right)=\exp\left(\beta\sigma_{\alpha i}\sigma_{\alpha j}\right)$$

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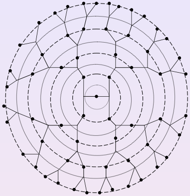
Paramagnetic (high-temperature) State at $\mathbf{h} = 0$

- The only solution of the BP equations is the trivial one: $\eta = 0$
- The Loop Series reduces to the High-Temperature Expansion



$$Z(\mathbf{h}) = \sum_{\boldsymbol{\sigma}} \prod_{\alpha=1}^{M} \delta\left(\prod_{i\in\alpha} \sigma_i, 1\right) \exp\left(\sum_{i=1}^{N} h_i \sigma_i\right)$$

h_i is a log-likelihood at a bit (outcome of the channel)

$$Z^{\pm}_{j\alpha}(\mathbf{h}^{>}) \equiv \sum_{\boldsymbol{\sigma}^{>}}^{\sigma_j=\pm 1} \prod_{\beta^{>}} \delta\left(\prod_{i\in\beta} \sigma_i, 1\right) \exp\left(\sum_{i^{>}} h_i \sigma_i\right)$$

$$Z^{\pm}_{j\alpha} = \exp(\pm h_j) \prod_{\beta\neq\alpha}^{j\in\beta} \frac{1}{2}\left(\prod_{i\neq j}^{i\in\beta}(Z^{+}_{i\beta} + Z^{-}_{i\beta}) \pm \prod_{i\neq j}^{i\in\beta}(Z^{+}_{i\beta} - Z^{-}_{i\beta})\right)$$

$$\eta_{j\alpha} \equiv \frac{1}{2}\ln\left(\frac{Z^{+}_{j\alpha}}{Z^{-}_{j\alpha}}\right), \quad \eta_{j\alpha} = h_j + \sum_{\beta\neq\alpha}^{j\in\beta} \tanh^{-1}\left(\prod_{i\neq j}^{i\in\beta} \tanh \eta_{i\beta}\right)$$

◂ Statistical Inference

Gibbs measure: $P(\mathbf{x}) = \frac{\exp(-E(\mathbf{x}))}{Z}, \quad Z \equiv \sum_{\mathbf{x}} \exp(-E(\mathbf{x}))$

Exact Variational Principe

$$F\{b(\mathbf{x})\} = \sum_{\mathbf{x}} b(\mathbf{x}) E(\mathbf{x}) - \sum_{\mathbf{x}} b(\mathbf{x}) \ln b(\mathbf{x})$$

$$\left.\frac{\delta F}{\delta b(\mathbf{x})}\right|_{b(\mathbf{x})=p(\mathbf{x})} = 0 \quad \text{under} \quad \sum_{\mathbf{x}} b(\mathbf{x}) = 1$$

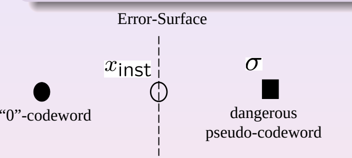
Factorized form of $E(\mathbf{x})$ or other considerations may suggest an approximate variational ansatz (mean-field, $b(\mathbf{x}) = \prod_i b_i(x_i)$, is the famous example)

◂ Bethe Free Energy

LP decoding ($\sigma_i = 0, 1$ AWGN channel)

Minimize, $E = \sum_\alpha \sum_{\sigma_\alpha} b_\alpha(\sigma_\alpha) \sum_{i\in\alpha} \sigma_i(1-2x_i)/q_i$, under $0 \le b_i(\sigma_i), b_\alpha(\sigma_\alpha) \le 1$

$\forall\ \alpha :\ \sum_{\sigma_\alpha} b_\alpha(\sigma_\alpha) = 1,\quad \&\quad \forall\ i\ \forall\ \alpha \ni i :\ b_i(\sigma_i) = \sum_{\sigma_\alpha \setminus \sigma_i} b_\alpha(\sigma_\alpha)$



Weighted Median:

$$\mathbf{x}_{\text{inst}} = \frac{\boldsymbol{\sigma}}{2}\frac{\sum_i \sigma_i}{\sum_i \sigma_i^2},\quad d = \frac{(\sum_i \sigma_i)^2}{\sum_i \sigma_i^2}$$

$$\text{FER} \sim \exp(-d \cdot s^2/2)$$

Wiberg '96; Forney et.al '01
Vontobel, Koetter '03,'05

Pseudo-Codeword-Search Algorithm

Chertkov, Stepanov '06

- **Start:** Initiate $\mathbf{x}^{(0)}$.
- **Step 1:** $\mathbf{x}^{(k)}$ is decoded to $\boldsymbol{\sigma}^{(k)}$.
- **Step 2:** Find $\mathbf{y}^{(k)}$ - weighted median between $\boldsymbol{\sigma}^{(k)}$, and "0"
- **Step 3:** If $\mathbf{y}^{(k)} = \mathbf{y}^{(k-1)}$, $k_* = k$ End. Otherwise go to **Step 2** with $\mathbf{x}^{(k+1)} = \mathbf{y}^{(k)} + 0$.

$(155, 64, 20)$, AWGN test:

- Fast Convergence

Enlarge

~ 200 pseudo-codewords within $16.4037 < d < 20$

Pseudo-Codeword General

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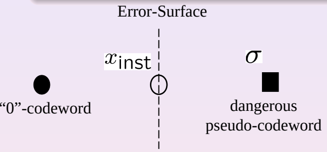

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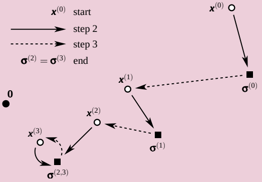

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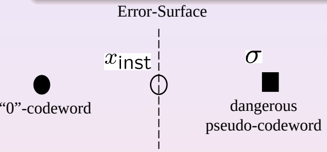

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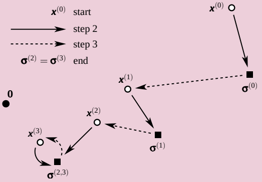

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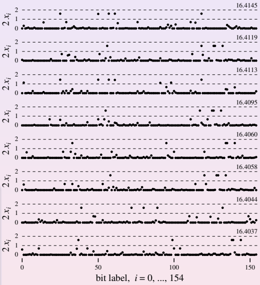

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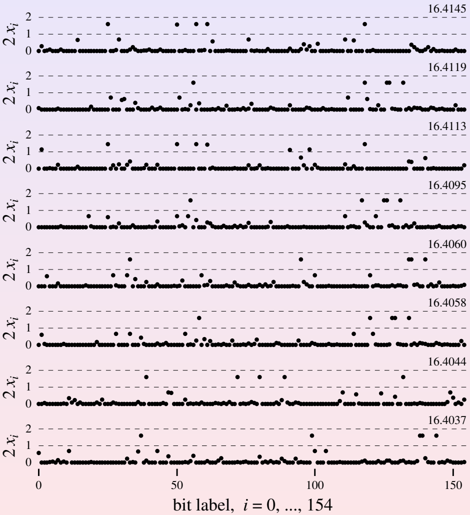
16.4145
16.4119
16.4113
16.4095
16.4060
16.4058
16.4044
16.4037
2 x_i
0
1
2
0
50
100
150
bit label, $i = 0, ..., 154$

Back

- Replication:

$$Z=\sum_{\boldsymbol\sigma}\prod_a f_a(\boldsymbol\sigma_a)=\sum_{\boldsymbol\sigma'}\prod_a f_a(\boldsymbol\sigma_a)\prod_{bc}\frac{1+\sigma_{bc}\sigma_{cb}}{2},\quad \sigma_{bc}\neq\sigma_{cb}$$

- Local Gauge Freedom:

$$1+\pi\sigma=\frac{\exp(\sigma\eta+\pi\chi)}{\cosh(\eta+\chi)}\Big(1+(\tanh(\eta+\chi)-\sigma)(\tanh(\eta+\chi)-\pi)\cosh^2(\eta+\chi)\Big)$$

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$$V_{bc}(\sigma_{bc},\sigma_{cb})=1+(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{bc})(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{cb})\cosh^2(\eta_{bc}+\eta_{cb})$$

- Fixing the Gauge (η-fields on the graph)

BP equations

$$\sum_{\boldsymbol\sigma_a}\Big(\tanh(\eta_{ab}^{(bp)}+\eta_{ba}^{(bp)})-\sigma_{ab}\Big)P_a(\boldsymbol\sigma_a)=0\quad\Rightarrow\quad \eta_{j\alpha}^{bp}=h_j+\sum_{\beta\neq\alpha}^{j\in\beta}\tanh^{-1}(\prod_{i\neq j}^{i\in\beta}\tanh\eta_{i\beta}^{bp})$$

Geometrical Principle: no loose ends

$$\prod_{(bc)}V_{bc}=1+\sum_{\text{colored edges}}*\cdots*\cdots*\cdots$$

Variational Principle:

$$\prod_{(bc)}V_{bc}\to 1,\quad Z\to Z_0,\quad \left.\frac{\delta Z_0}{\delta\eta_{ab}}\right|_{\eta^{(bp)}}=0$$

$$Z_0=(\prod_{bc}2\cosh(\eta_{bc}+\eta_{cb}))^{-1}\sum_{\boldsymbol\sigma}\prod_a P_a(\boldsymbol\sigma_a)$$

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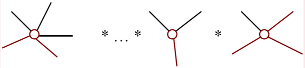

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$$\prod_{(bc)} V_{bc} = 1 + \sum_{\text{colored edges}} * \cdots * \cdots * \cdots$$

Variational Principle:

$$\prod_{(bc)} V_{bc} \to 1, \quad Z \to Z_0, \quad \left.\frac{\delta Z_0}{\delta \eta_{ab}}\right|_{\eta^{(bp)}} = 0$$

$$Z_0 = \left(\prod_{bc} 2\cosh(\eta_{bc}+\eta_{cb})\right)^{-1} \sum_{\boldsymbol\sigma} \prod_a P_a(\boldsymbol\sigma_a)$$

◂ Loop Series

- Replication:

$$Z=\sum_{\boldsymbol\sigma}\prod_a f_a(\boldsymbol\sigma_a)=\sum_{\boldsymbol\sigma'}\prod_a f_a(\boldsymbol\sigma_a)\prod_{bc}\frac{1+\sigma_{bc}\sigma_{cb}}{2},\quad \sigma_{bc}\neq\sigma_{cb}$$

- Local Gauge Freedom:

$$1+\pi\sigma=\frac{\exp(\sigma\eta+\pi\chi)}{\cosh(\eta+\chi)}\Big(1+(\tanh(\eta+\chi)-\sigma)(\tanh(\eta+\chi)-\pi)\cosh^2(\eta+\chi)\Big)$$

$$Z=\Big(\prod_{bc}2\cosh(\eta_{bc}+\eta_{cb})\Big)^{-1}\sum_{\boldsymbol\sigma}\prod_a P_a\prod_{bc}V_{bc},\quad P_a(\boldsymbol\sigma_a)=f_a(\boldsymbol\sigma_a)\prod_{b\in a}\exp(\eta_{ab}\sigma_{ab})$$

$$V_{bc}(\sigma_{bc},\sigma_{cb})=1+(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{bc})(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{cb})\cosh^2(\eta_{bc}+\eta_{cb})$$

- Fixing the Gauge (η-fields on the graph)

BP equations

$$\sum_{\boldsymbol\sigma_a}\Big(\tanh(\eta^{(bp)}_{ab}+\eta^{(bp)}_{ba})-\sigma_{ab}\Big)P_a(\boldsymbol\sigma_a)=0\quad\Rightarrow\quad \eta^{bp}_{j\alpha}=h_j+\sum^{j\in\beta}_{\beta\neq\alpha}\tanh^{-1}\Big(\prod^{i\in\beta}_{i\neq j}\tanh\eta^{bp}_{i\beta}\Big)$$

Geometrical Principle: no loose ends

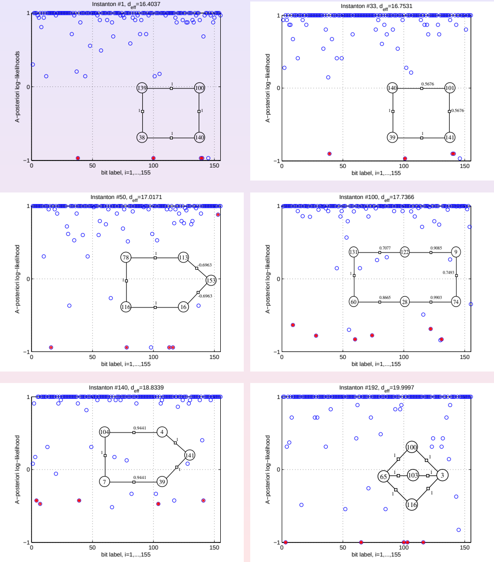
$$\prod_{(bc)}V_{bc}=1+\sum_{\text{colored edges}}*\cdots*\cdots*\cdots$$

Variational Principle:

$$\prod_{(bc)}V_{bc}\to 1,\quad Z\to Z_0,\quad \frac{\delta Z_0}{\delta\eta_{ab}}\bigg|_{\eta^{(bp)}}=0$$

$$Z_0=\Big(\prod_{bc}2\cosh(\eta_{bc}+\eta_{cb})\Big)^{-1}\sum_{\boldsymbol\sigma}\prod_a P_a(\boldsymbol\sigma_a)$$

Loop Series


Instanton #1, d_eff=16.4037
Instanton #33, d_eff=16.7531
Instanton #50, d_eff=17.0171
Instanton #100, d_eff=17.7366
Instanton #140, d_eff=18.8339
Instanton #192, d_eff=19.9997
A−posteriori log−likelihood
bit label, i=1,...,155

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