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STRUCTURAL DESIGN CONCEPTS FOR FUTURE SPACE MISSIONS

Progress Report on

A CONCEPTUAL INVESTIGATION OF ROTATION-TRANSLATION TRANSFORMATION OF PLATONIC POLYHEDRA

THE SCHOOL OF TECHNOLOGY
of
SOUTHERN ILLINOIS UNIVERSITY

STRUCTURAL DESIGN CONCEPTS FOR FUTURE SPACE MISSIONS

A CONCEPTUAL INVESTIGATION OF ROTATION-TRANSLATION TRANSFORMATION OF PLATONIC POLYHEDRA

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NASA Contract
NsG-607

Joseph D. Clinton

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FOREWARD

This report was prepared in the School of Technology at Southern Illinois University under National Aeronautics and Space Administration Contract No. NsG - 607.

This report covers the technical data gathered on the subject of "Rotation-Translation Transformation of Platonic Polyhedra."

This research was conducted by Joseph D. Clinton under the direction of Julian H. Lauchner, principal investigator NASA Contract NsG - 607; John Pollock, thesis advisor; Charles Bunten, John Erickson, and Wayne Ramp, thesis advisory committee.

ABSTRACT

From the *"Jitterbug"* concept of R. Buckminster Fuller a new concept of expandables was derived. The *Rotation-Translation Transformation of Platonic Polyhedra*. The investigation within this report illustrates the transformation concept and presents filling concepts stemming from the transformation.

Three concepts for filling the void areas created by face rotation-translation transformation of Platonic Polyhedra are presented. One concept utilized the transformation concept itself in conjunction with the transformation concepts: *Double Rotation-Translation Transformation*. The other two concepts depend upon long recognized expandable concepts: *Folding Rotation-Translation Transformation* and *Fanning Rotation-Translation Transformation*.

ACKNOWLEDGEMENTS

The writer wishes to express his thanks to all people who helped make this study possible.

The helpful criticism of Dean Julian Lauchner, principal investigator for NASA contract NsG-607, and Professor John M. Pollock, director of this thesis is certainly appreciated. Thanks are indeed due the National Aeronautics Space Administration for allowing the writer to conduct the research done under NASA contract NsG-607. The writer also thanks Dane Hildebrecht, student assistant, for his help in construction of the models, and above all the writer expresses his gratitude to his wife, Elberta, for her patience, understanding and assistance in the preparation of this thesis

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INTRODUCTORY NOTE

In order to make the transformation more meaningful and to aid in visualizing the series, stereo photographs have been used. With the use of stereo photographs a three dimensional effect can be obtained, which will allow the reader to grasp the transformations more quickly. With the aid of the stereo viewer supplied with the report and by following the simple steps outlined below, little difficulty will be incurred in obtaining the stereo effect.

How to Use the Stereo Viewer

1. Place the report directly in front of you, so that the page lies flat.

2. Place viewer and picture toward a light source. Both stereo pictures must be illuminated equally without any shadow.

3. Make sure the bottom edge of the stereo pictures is parallel to the line between the lenses. (Some defects of vision may necessitate a slight twist off the parallel.) Both pictures must be level--one must not be higher than the other.

4. If you wear glasses, use them while viewing.

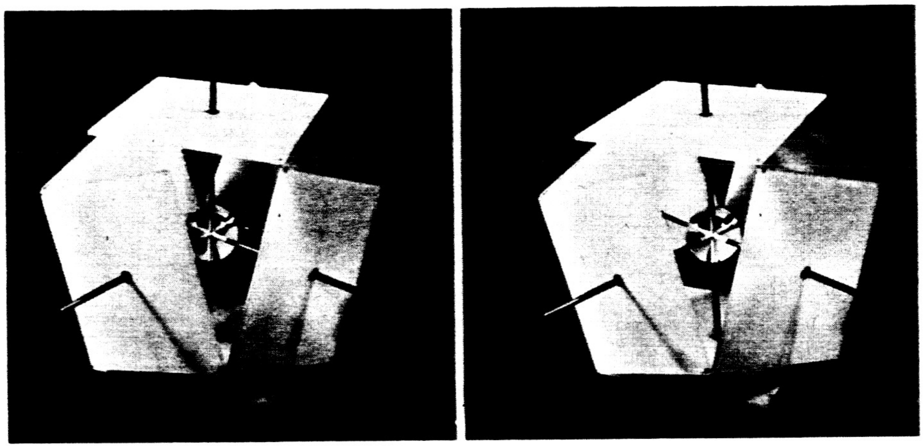
5. In some cases, focusing may be improved by raising or lowering the viewer.

6. If you don't see stereo immediately, relax your eyes as if gazing into a distance; this will put your eyes in parallel so that the pictures fuse together and combine into one.

7. Give your eyes a fair amount of time to fuse the stereo pictures; only 8% of the people cannot see stereo.

8. By following the above procedure and after a little practice all other viewings will come with little effort.

9. If it gets dirty, the stereo viewer should be cleaned only with a soft white cloth.



Try out the stereo viewer on the stereo pair shown here,following the procedure suggested on the previous page.

I. INTRODUCTION & BACKGROUND

With the increase in technological change which is manifest around us, we never cease using materials and knowledge congruently in everything that is created or evolved today. We no longer use one area as an entity within itself but co-ordinate and unite it with other areas to advance ideas, concepts, and designs. The conventional boundaries between sciences are found to be neither fixed nor absolute. An increasing tendency has been to explore in every science those areas that lie near the boundaries, and this has inevitably led to the crossing of these boundaries. Most researchers, however, who are in a border province have to specialize in their province almost as narrowly as those who work within a single discipline. They concentrate on a particular problem, straddling the border, just as the researcher within a more conventional field concentrates on a particular problem lying within that field. Yet, in our industrial-technological culture today, there is a definite need for researchers who can cross the lines and make use of the language arts, mathematics, sciences, and social sciences and who can become involved in study, experimentation, and the application among these areas. With the complexity of our technological environment utilizing the specialized sciences in relationship to each other, it is most desirable to include researchers who can coordinate research done in many areas.

One area of concern here, that draws upon many fields, and crosses over many boundaries, is that of structural design concepts for aerospace application. The use of structures to house and protect the variety of objects man builds will be needed to aid in his conquest and exploration of space. Of the variety of structural design concepts, considerable attention has been brought to bear on the potential of expandable structures for aerospace applications. The possibility of fabricating a structural configuration on earth, packaging it in a small container for launch, and expanding it upon command to a predetermined size and shape has distinct and obvious advantages.

According to Webster's Dictionary the word *expand* indicates *increase in extent*. Therefore, the use of the term *expandable structures* has a broad connotation. It includes any structure that geometrically expands--occurring along one axis only; it may also mean an increase in two directions; or it may mean expansion in any direction utilizing three or more axes.

There are generally three classifications in which expandable structures for aerospace applications fall:*

Inflated, pressure-stabilized structures;

Inflated, rigidized structures;

Mechanically expanded, framework stabilized structures.

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*Aerospace Expandable Structures 1., p. 7.

The mechanically expanded or rigid system "is made up of rigid components compacted into a small package which, upon signal, rearrange themselves to provide for greater surface area and enclosed volume."* Five basic concepts have been considered as applicable to rigid expandable structures--one is relatively new and little in regards to application has been done with it.† A sixth concept shall be introduced later in this report.

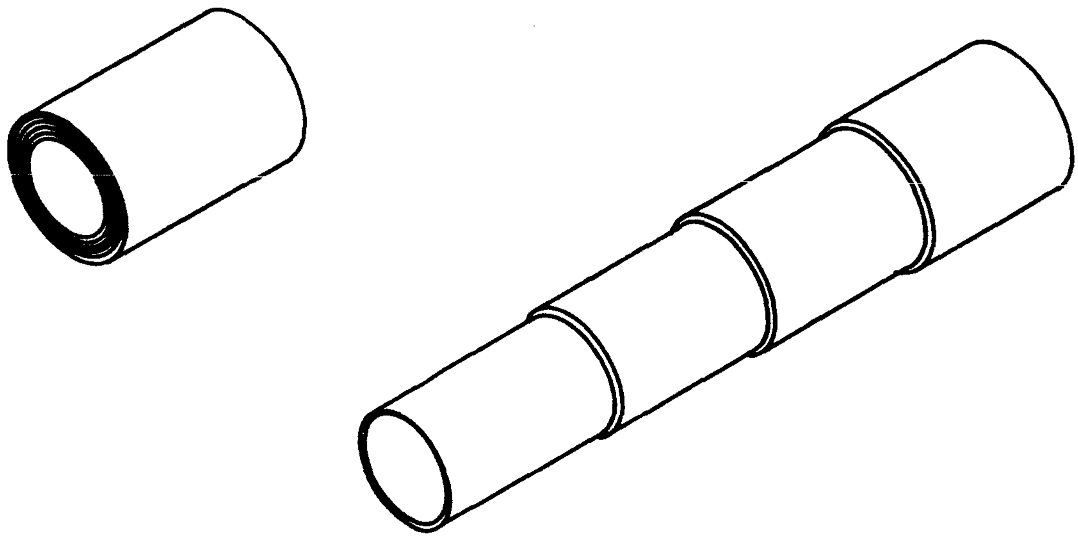

Figure 1

Telescoping Concept:

The telescoping mechanisms consist of a series of rigid components that translate on a common axis sliding in and out of each other, thus permitting unidirectional expansion in packing.

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*Wright, F. N. 1., p. 11.

†Lebovits, M. et al; Wright, F. N. 1., pp. 11-16.

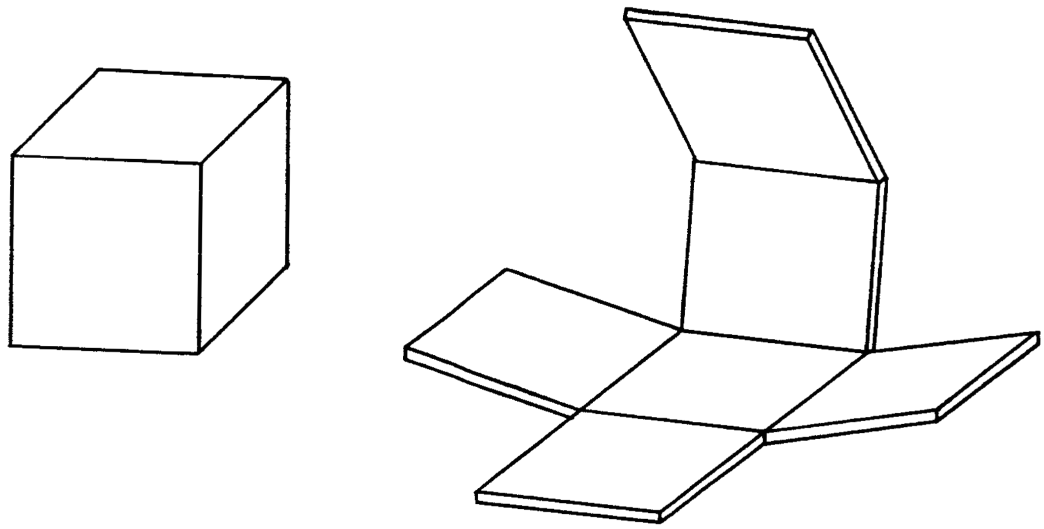

Figure 2

Folding Concept:

The folding mechanisms consist of a series of rigid bars and/or panels which are hinged together at the ends or sides. They are expanded or compacted about their hinged joints until they lie in a predetermined position.

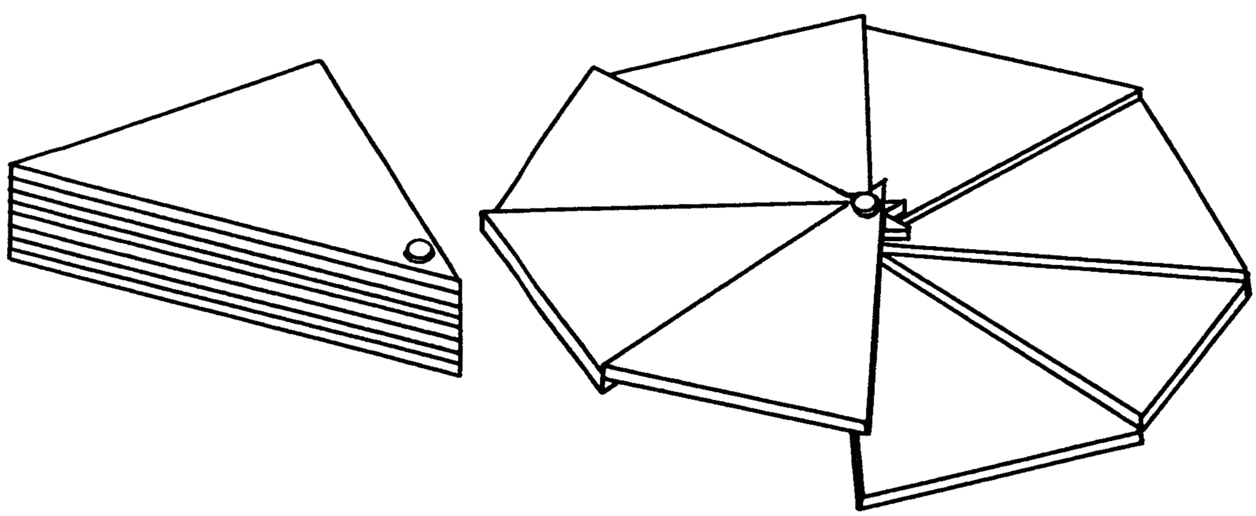

Figure 3

Fan Concept:

The fan concept entails rotation of rigid components in common planes about a central axis. By this means, an expanded or compacted structure results.

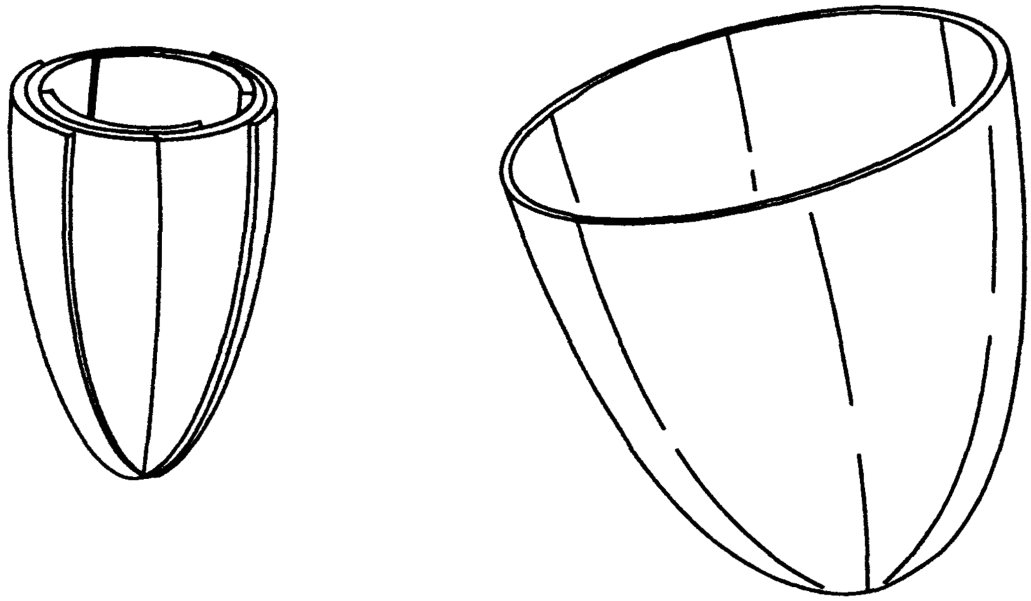

Figure 4

Umbrella Concept:

the umbrella concept utilizes rigid component rotation in mutually perpendicular planes about a common point.

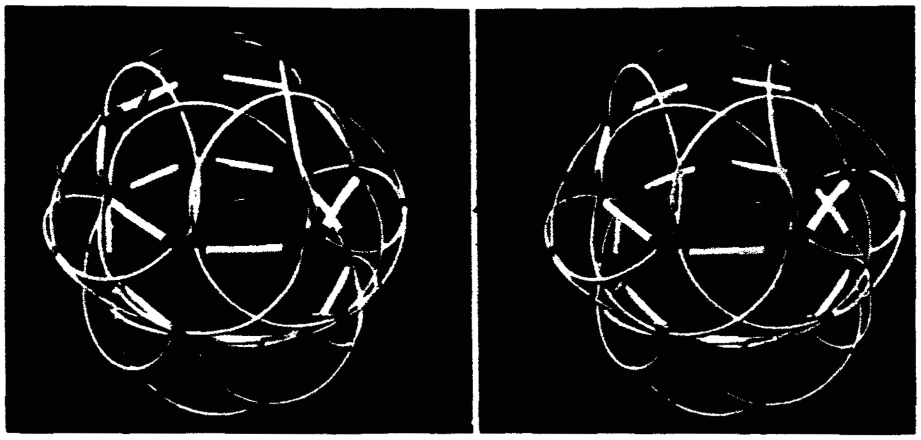

Figure 5

Variable Geometry Concept:

The variable geometry concept utilizes frame arches and a base ring. The arches are attached about the base ring and by means of hinging and actuating, the arches can be rotated to assume the desired configuration. The arches lie in one plane in the compressed state; in the deployed state they are rotated to assume a three-dimensional structural framework.

II THE TRANSFORMATION OF POLYHEDRA

It has long been known that spheres closely packed around a central sphere do not form a larger sphere. A polyhedron, however, bounded by 14 faces is formed. This polyhedron consists of six squares and eight triangles. R. Buckminster Fuller, in his discussion of Energetic-Synergetic Geometry, refers to this 14-faced geometric configuration as the *Vector Equilibrium.*

The polyhedron, commonly known as the *Hexoctahedron* or *Cuboctahedron,* is literally an equilibrium of vectors. The value of its radial vectors is exactly the same as that of its circumferential vectors.* The length of the distance from any of the polyhedron's center to its vertices is equal to the length of any of its elements. For this reason an equilibrium exists where the lines of force radiate from its center, and bind inward around its periphery.

Fuller indicated that the removal of the center sphere would cause a significant change in the close packing of spheres, a 20-sided polyhedron would result--an *icosahedron.* This change suggested that a Vector Equilibrium could be translated into an icosahedron and vice versa. The same number of surface-defining spheres exist between the two polyhedra and they both have 12 vertices. Each has symmetrical similarities. A family of relationships which is

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*Marks, R. W. 1., p. 41.

capable of cycling through a sequence of phases existed which Fuller called *"Regenerative"*. This sequence of phases can be visually illustrated with the construction of (as Fuller calls it) the *"Jitterbug"*.* The sequence starts with the Vector Equilibrium and when released it compresses symmetrically into an icosahedron and then into an octahedron.

Through investigation of Fuller's *"Regenerative"* and coordination with research conducted on the *Platonic* and *Archimedean Polyhedra* a very unique concept of polyhedral transformation was derived.† This sixth‡ concept of expandable structures using rigid expandable components came to light only recently through the efforts of Duncan Stuart of the School of Design, North Carolina State University. His article, "Polyhedra and Mosiac Transformations," 1963, reports his findings.

The concept of polyhedral transformation incorporates the use of the familiar regular‖ and semi-regular polyhedra. It introduces a new dimension to the old "classical" concept of polyhedra transformation** by introducing a new concept of *Rotation-Translation Transformation* of polyhedra.

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*Marks, R. W. 1., p. 42; See figure 6.

†An interview held with R. Buckminster Fuller on 3 May 1965.

‡The other five expandable concepts are discussed in the preceding chapter.

‖See Figure 7.

**See Figure 8.

In contrast to the "classical" method of transformation of polyhedra this new concept maintains dimensionality during transformation and generates a polyhedral form without recourse to any special knowledge other than the rules of transformation described in Table I.

This concept of Rotation-Translation Transformation is characteristic of all regular and semi-regular polyhedra (Table II). By allowing each surface to rotate about its axis, translate along its axis, and maintain connection with one of its paired vertices; the surfaces enclosing the polyhedron will transform into another polyhedral form.*

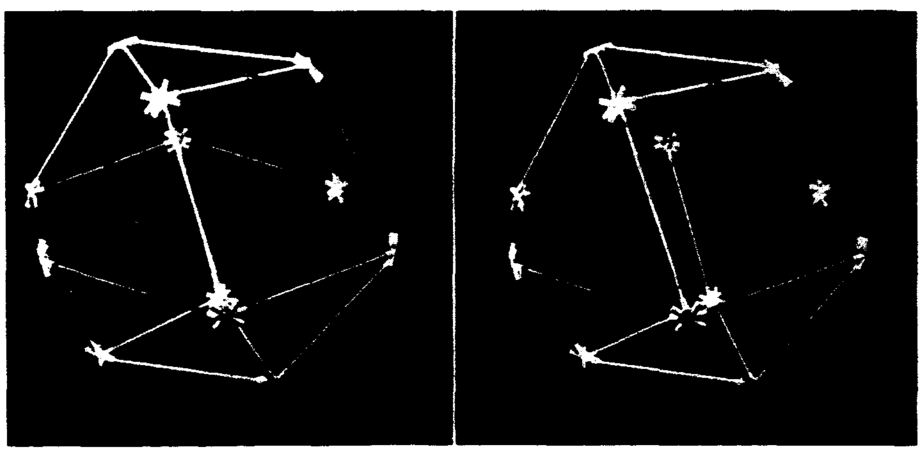

Figure 6

"Jitterbug"

The Vector Equilibrium Phase

The Vector Equilibrium is constructed with circumferential vectors only and with flexible joints.

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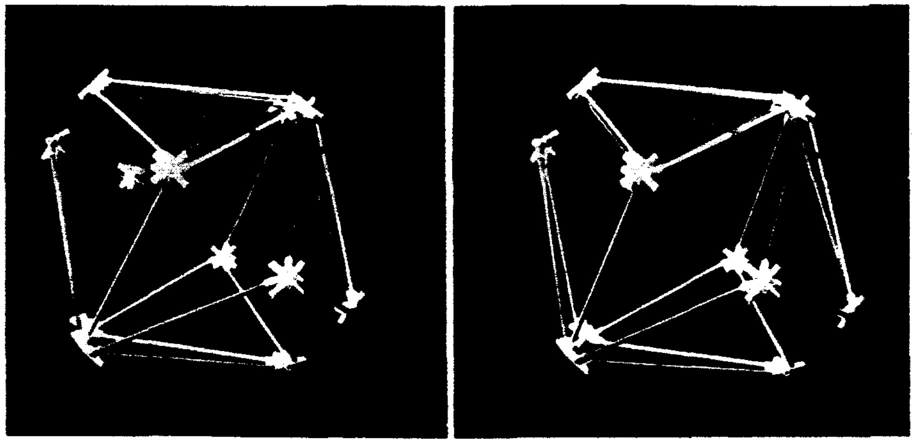
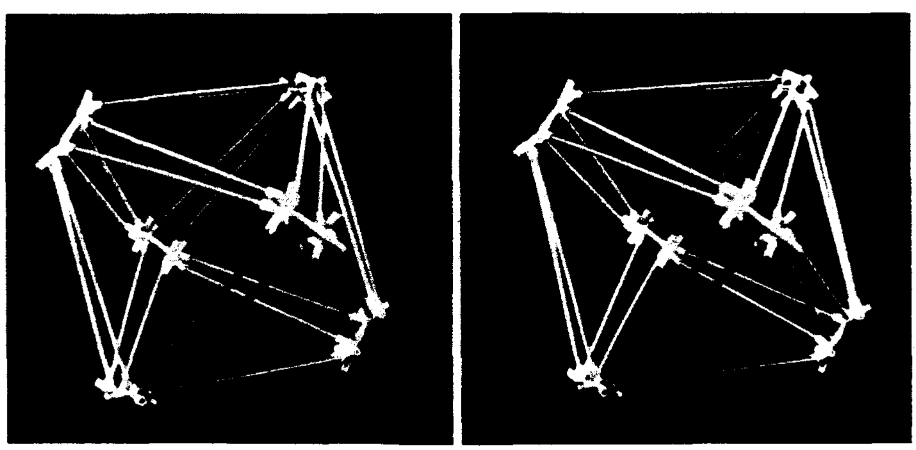
*See Figure 9.

Figure 6 (continued)
"Jitterbug"



The Icosahedron Phase

As the top triangle is lowered toward the opposite triangle, equatorial rotation moves the vertices into the Icosahedron phase. NOTE: That each of the pairs of opposite triangles, although in motion, maintain an axial relationship.



The Octahedron Phase

The completion of the cycling results in the phase of an Octahedron.

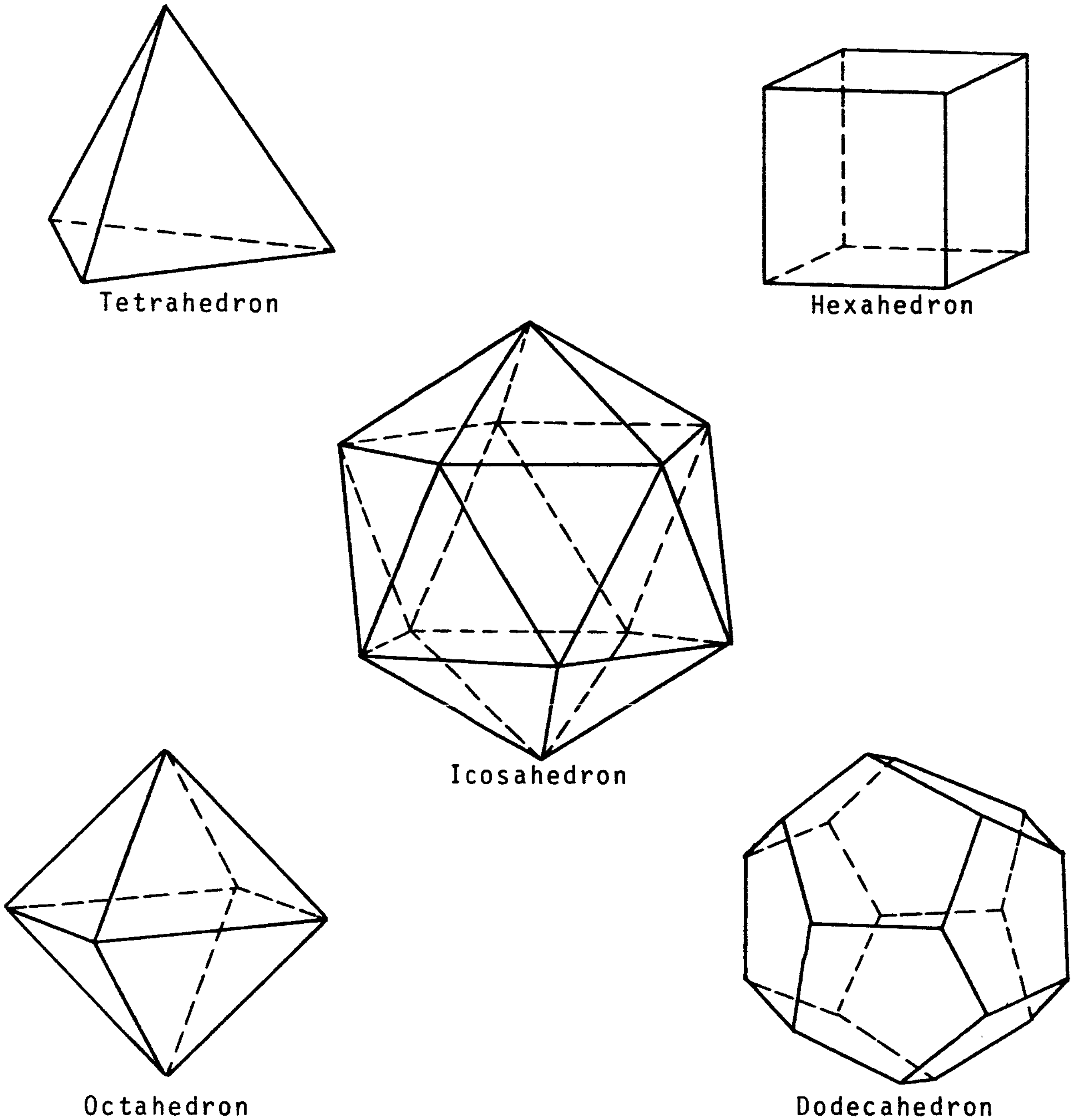


Figure 7

Platonic Solids

A convex Polyhedron is said to be "regular" if its faces are regular and equal, while its vertices are all surrounded alike. Eulers's theorem states that $V + F = E + 2$, where V, F, E are respectively the number of vertices, faces, and edges. It can be shown that there are only five regular convex Polyhedra. These Platonic solids are the Tetrahedron, Cube (Hexahedron), Octahedron, Icosahedron, and (Pentagonal) Dodecahedron.

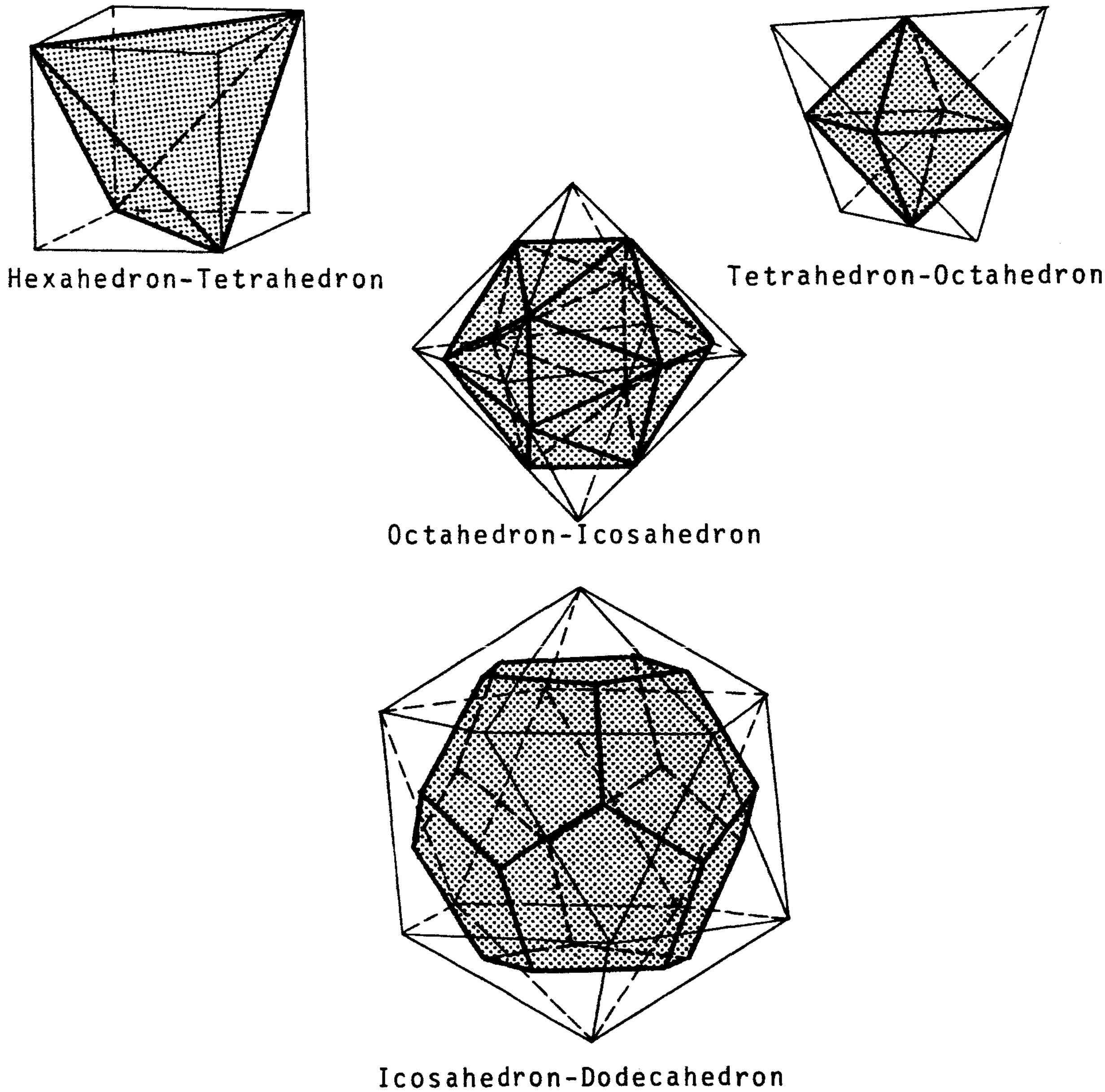


Figure 8

Classical Transformation of Platonic Solids

The Cube is converted into a Tetrahedron by an alternate removal of vertices. The Tetrahedron is converted into an Octahedron by truncation of its vertices. The Octahedron is transformed into the Icosahedron by a somewhat more complex truncation, and similarly the Icosahedron is converted into the Dodechedron. A prior knowledge of the forms is necessary before the complete series of transformation may be accomplished.

TABLE I

DEFINING RULES FOR REGULAR AND SEMI-REGULAR POLYHEDRA*

1. A Regular Polyhedron must enclose a volume of space with a surface composed of only one kind of plane polygon.
2. These plane polygons must be equilateral, equiangular, and rectilinear.
3. The polygons must mutually join at their edges and vertices so as to completely fill a single imaginery spherical surface passing through the joined vertices.
4. In a similar fashion, a Semi-regular Polyhedron must be totally composed of plane polygons as defined in 2--but now, more than one kind of polygon may be used in a single polyhedron.
5. There must be the same numbers and kinds of polygons, joined in the same order (or its enantiomorph), at each of the vertices of the polyhedral surface.
6. For Regular and Semi-regular Polyhedra, the corner angles which join at a single vertex must total in aggregate, less than 360°.
7. The plane of any polygon, if extended, must not pass through the interior volume of the polyhedron. And, a plane passed through the polyhedron at random will always have a single closed polygon at its line of intersection with the polyhedral surface.

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*Stuart 1., p. 5.

(TABLE II)

CHARACTERISTICS OF THE
REGULAR AND SEMI-REGULAR POLYHEDRA

Name	Equilateral Triangle	Square	Pentagons	Hexagons	Octagons	Decagon	Designation Schiäfli Symbol*	Faces	Edges	Vertices
Tetrahedron	4	-	-	-	-	-	3^3	4	6	4
Cube	-	6	-	-	-	-	4^3	6	12	8
Octahedron	8	-	-	-	-	-	3^4	8	12	6
Dodecahedron	-	-	12	-	-	-	5^3	12	30	20
Icosahedron	20	-	-	-	-	-	3^5	20	30	12
Truncated Tetrahedron	4	-	-	4	-	-	$3 . 6^2$	8	18	12
Cuboctahedron	8	6	-	-	-	-	$(3 . 4)^2$	14	24	12
Truncated Cube	8	-	-	-	6	-	$3 . 8^2$	14	36	24
Truncated Octahedron	-	6	-	8	-	-	$4 . 6^2$	14	36	24
Small rhomicubocta-hedron	8	18	-	-	-	-	$3 . 4^3$	26	48	24
Great rhomicubocta-hedron	-	12	-	8	8	-	4 . 6 . 8	26	72	48
Snub Cube	32	6	-	-	-	-	$3^4 . 4$	38	60	24
Icosadodecahedron	20	-	12	-	-	-	$(3 . 5)^2$	32	60	30
Truncated dodecahedron	20	-	-	-	-	12	$3 . 10^2$	32	90	60
Truncated icosahedron	-	-	12	20	20	-	$5 . 6^2$	32	90	60
Small rhomicosidodeca-hedron	20	30	12	-	-	-	3 . 4 . 5 . 4	62	120	60
Great rhomicosidodeca-hedron	-	30	-	20	20	12	4 . 6 . 10	62	180	120
Snub dodecahedron	80	-	12	-	-	-	$3^4 . 5$	92	150	60

*Kravitz 1., p. 119.

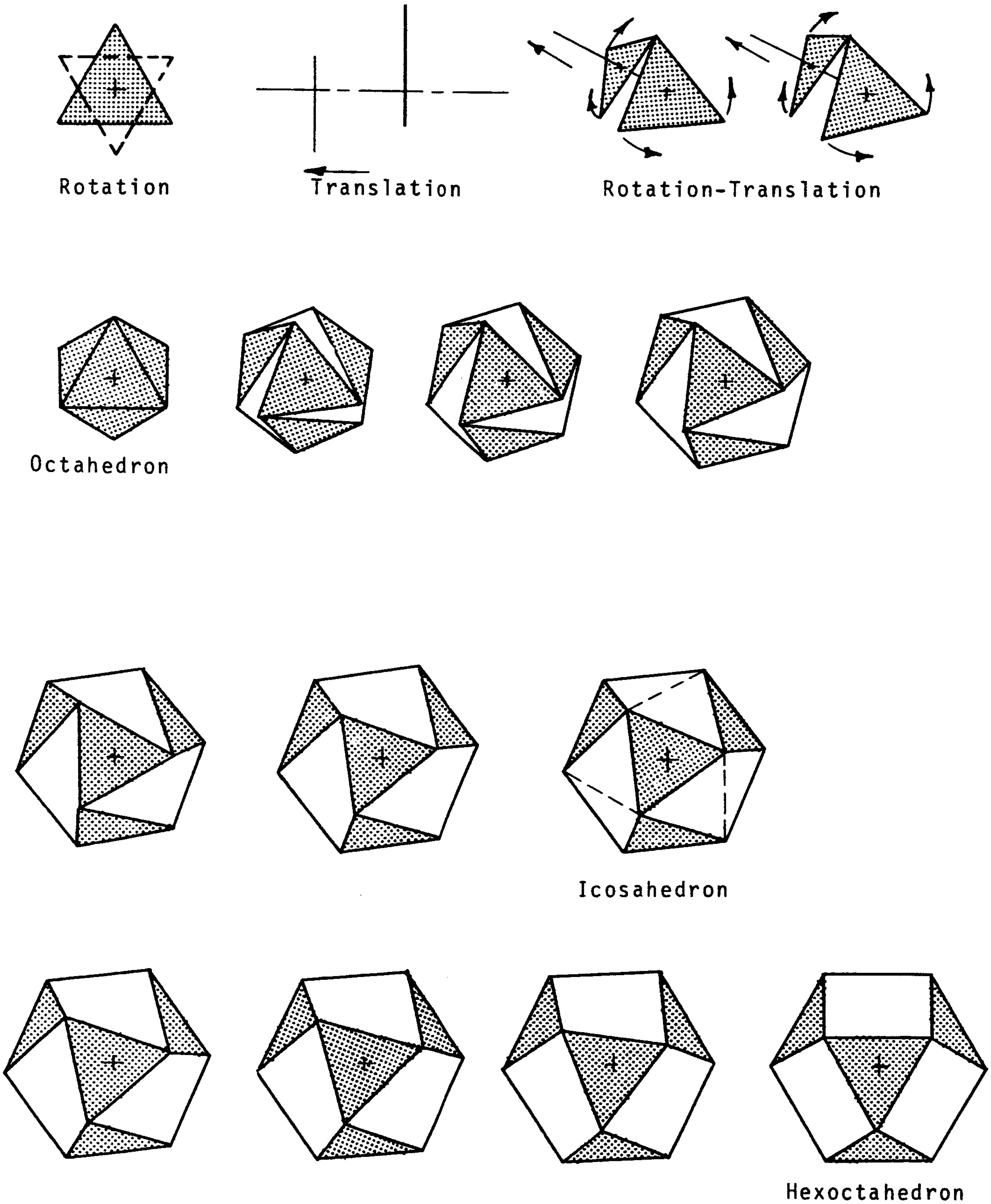

Figure 9

Transformations

III THE RESEARCH CONDUCTED

The Problem

A known configuration and the number of faces exist prior to the transformation of a polyhedron. At the completion of the transformation the size and number of original faces still exist; however, they have been translated into a new position creating a new polyhedron form and introducing new faces that had not existed previously. The new faces created during the transformation take the form of void areas and, being unique in themselves, have definable limitations.

The problem was to conduct a conceptual investigation of Rotation-Translation Transformation of Platonic Polyhedra to determine means of filling the void areas created during the transformation. As a separate entity to the problem, patterns and similarities among the transformations are illustrated showing commonness and orderliness among them.

Significance of the Problem

The most ideal situation for expandable structures would be to compress the system into an extremely small, lightweight package for transportation purposes and deploy it into the desired configuration after delivery to the desired location. This may be accomplished by using the concept of Rotation-Translation Transformation of Polyhedra. The rigid expandable structure is made up of rigid components compacted into a small package which can rearrange themselves to provide for a greater surface area and/or enclosed volume (Tables III and IV).

Expandable structures have been receiving considerable attention in the last few years by NASA. As NASA points out, the possible applications of expandable structures which have been proposed in the last few years include almost every conceivable aerospace structure. Such areas are:*

1. Launch vehicles
2. Unmanned space vehicles
3. Manned space vehicles
4. Re-entry structures
5. Vehicle extensions and secondary structures
6. Exploration base structures

The specific applications in which expandable structures may be used are limitless and need not be discussed here. However, very definite advantages are pertinent. The principal advantage of expandable structures in all application areas is their ability to be packaged into a relatively small, efficiently shaped volume. Ease of erection is an advantage associated with packaging and can also be attained. Very often expandable structures show a decided weight saving advantage over either rigid or erectable configurations. One advantage which is of specific interest here and is unique to the transformation concept is that of dimensional

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*Koleum, H. E. 1., pp. 89-92.

stability. Dimensional stability includes more than the elastic and inelastic deflections of a structure under load.* Membrane structures, used so extensively in many expandable systems, resist normal loads by radial changes in shape so that the load is supported everywhere in tension. This would also be true with the use of the transformation concept.

In order to make the transformation concept a practical instrument, much research needs to be done with it. It is from this need that this study stems in an attempt to answer one of the many questions arising from the initial concept.

Delimitations

The polyhedral forms investigated were limited to the Platonic and Archimedian Polyhedra as described in Tables I and II.

The investigation was further limited to the Rotation-Translation Transformation of Platonic Polyhedra.

Only rigid components were used in the filling of the void areas.

Procedure

The solution to the problem was approached by:

I. *Investigation of the platonic solids used in the initial transformation.* During this phase of the study, the

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*Wright, F. N. 1., pp. 11-16.

platonic solids were defined and pertinent characteristics and similarities noted. The angles and surface configuration studies were determined for the necessary drawings and models that were constructed.

II. *Investigation of Rotation-Translation Transformation of Platonic Polyhedra.* The underlying theory of transformation was defined and limited. Transformation calculations were determined to further aid in the collection of data necessary for determining similarities and characteristics existing among the forms. This data was noted and used further for the construction of the drawings and models.

III. *Investigation of the polyhedron forms created by the transformation of the platonic and Archimedian Polyhedra.* The forms created by the transformation were investigated to facilitate defining and determining the characteristics and similarities existing among these forms. The investigator attempted to find coordinates for the vertices of these solids and examined the cases where one could be inscribed in another. Further calculations were conducted to aid in the accumulation of the needed data.

IV. *Construction of drawings and models.* Using the data previously found, drawings and constructions of models showing the transformation series was completed.

V. *Investigation of the void areas created during the* *transformation.* This phase was conducted throughout the previous phases of this investigation. With the aid of the calculations made and the drawings and models constructed, means of filling the void areas created during the transformations were determined.

VI. *Interpretation of data.* The data found during the investigations conducted during previous phases was analyzed for possible similarities and patterns arising. This analysis was conducted through the use of various tools which were deemed necessary throughout the study.

(Table III)

Surface Area Increase of the Transformations
Element Unit = 1

Polyhedron Transformation	Area of Polyhedron	Increase in Area of Polyhedron
Tetrahedron	1.7	
↓		1.7
Octahedron	3.5	
Octahedron	3.5	
↓		6.0
Hexoctahedron	9.5	
Hexahedron	6.0	
↓		3.5
Hexoctahedron	9.5	
Icosahedron	8.7	
↓		20.6
Icosadodecahedron	29.3	
Dodecahedron	20.6	
↓		8.7
Icosadodecahedron	29.3	
	30 25 20 15 10 5 0	5 10 15 20 25 30

(Table IV)

Volume Increase of the Transformations
Element Unit = 1

Polyhedron Transformation	Volume of Polyhedron	Increase in Volume of Polyhedron
Tetrahedron	0.1	
↓		0.3
Octahedron	0.5	
Octahedron	0.5	
↓		1.9
Hexoctahedron	2.4	
Hexahedron	1.0	
↓		1.4
Hexoctahedron	2.4	
Icosahedron	2.2	
↓		12.8
Icosadodecahedron	15.0	
Dodecahedron	7.7	
↓		7.3
Icosadodecahedron	15.0	
	18 15 12 9 6 3 0	0 3 6 9 12 15 18

IV THE CONCEPTUAL INVESTIGATION

The Polyhedron

Hoppe, in 1882, coined the word *polytope:* a geometrical figure bounded by portions of lines, planes, or hyperplanes; in two dimensions it is a *polygon,* in three a *polyhedron*.* The foundations for this subject were laid, however, by the Greeks over two thousand years ago, with the findings of Euclid. Others such as Klein and Schiafli and Coxeter introduced much to the concepts of the polytope. The area of interest in this report is that of the polyhedral forms of the polytypes, specifically, the *regular* (Platonic) and *Quasi-Regular* polyhedra.

The five, so-called Platonic polyhedra, from the simplest group of the polyhedra. In Euclid's writings, The Elements, explanation and definition is given to the five regular solids as known to the ancient world. These convex polyhedra are said to be regular if they each have regular and equal faces, if they are congruent, and if their vertices are regular polyhedral angles. (Table I, Chapter II) The five regular polyhedra are: (1) *Tetrahedron,* (2) *Hexahedron (cube),* (3) *Octahedron,* (4) *Icosahedron,* and (5) *Dodecahedron.* Also illustrated are the two quasi-regular polyhedra: (6) *Cuboctahedron,* and (7) *Icosadodecahedron.*†

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*Coxeter, H. S. M. 1., p. ix.
†See Figure 10.

Figure 10

Regular and Quasi-Regular Polyhedra

The quasi-regular polyhedra are solids having regular faces, while their vertex figures, though not regular, are cyclic and equiangular: encriptible in circles and alternate sides.* These two polyhedral forms along with five regular polyhedra round out the family of polyhedra encompassing this study.

As was illustrated in Figure 9 (Chapter II), the transformation was derived from a combination of two of the three "primitive" transformations: *translation* (in a certain direction, through a given distance), and *rotation* (about a certain line or *axis*, through a given angle).† Note the *duality* of the five regular polyhedra.‡ The tetrahedron is self-dual; the hexahedron is a dual of the octahedron; octahedron, dual of a hexahedron; the icosahedron, dual of

*Coxeter, H. S. M. 1., pp. 17-20.

†Coxeter, H. S. M. 1., pp. 33-38.

‡Candy, H. M. 1., pp. 76-82.

the dodecahedron; and the dodecahedron is a dual of the icosahedron.*

Table V shows the transformations of the Platonic polyhedra and illustrates the use of the duals. The combination of duals will transform into another polyhedral form containing the totals of their faces, thus increasing both surface area and volume. Two tetrahedra transform into the octahedron; the octahedron and hexahedron combine to make the quasi-regular polyhedron: hexoctahedron and the icosahedron and dodecahedron form the quasi-regular polyhedron: icosadodecahedron.†

From investigation of these polyhedral forms, concepts of transformations, and the concept of dualities; three basic conceptual solutions were derived for filling the void areas created during the transformation of the Platonic polyhedra: *Double-rotation-translation transformation, Folding rotation-translation transformation, and Fanning rotation-translation transformation.*

Double Rotation-Translation Transformation

The *double rotation-translation transformation* concept makes use of the characteristics of duality. As illustrated in Figures 19-21, the dual of the polyhedral form will fill the void areas created during the transformation of the ini-

*See Figures 11-13.

†See Figures 14-18.

tial polyhedron. (i.e. the tetrahedron with its dual tetrahedron transform together to fill each other's void areas)* The characteristics that are common to both forms and make this filling possible are: the element lengths are the same for the initial polyhedron and the filling polyhedron and vice versa; and the rotation and translation of faces in each transformation are equal. It must be noted here, however, that with this concept there is surface interference with the dual pairs: tetrahedron, tetrahedron; and octahedron, hexahedron.† But, with the dual pair dodecahedron, icosahedron, the icosahedron can be contained within the dodecahedron and no surface interference results.‡

Folding Rotation-Translation Transformation and Fanning Rotation-Translation Transformation

The two filling concepts, *Folding and Fanning rotation-translation transformation,* make use of the *Folding concept* and *Fan concept* of expandable structures. As illustrated in Figures 22-26 the appropriate polygon face is hinged and either fanned or folded into the void area thus filling the voids created during the transformation of the polyhedron. With the aid of the polyhedral nets, the surface hinging positions can be readily seen. It must be noted that the hinging of square faces, in the case of the octahedron trans-

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*See Figure 19.

†See Figures 19-20.

‡See Figure 21.

formation, and the hinging of the pentagonal faces in the case of the icosahedron transformation, became very awkward and would perhaps be difficult to pack.* The polygons, however, that are used to fill the void areas in these cases may be sectioned into triangles and hinged from each side of the existing faces and locked into their square or pentagonal position after the transformation.†

*See Figures 24 and 25.

†See Figure 27.

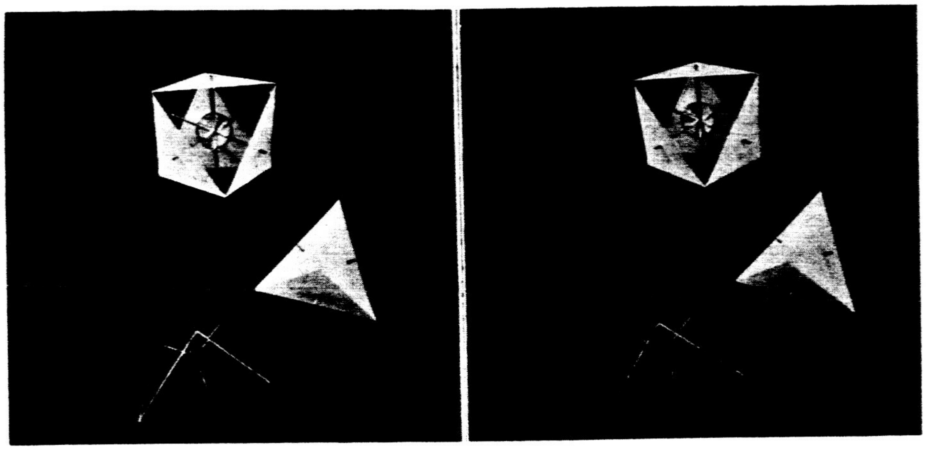


Figure 11

Dual Family Tetrahedron-Tetrahedron

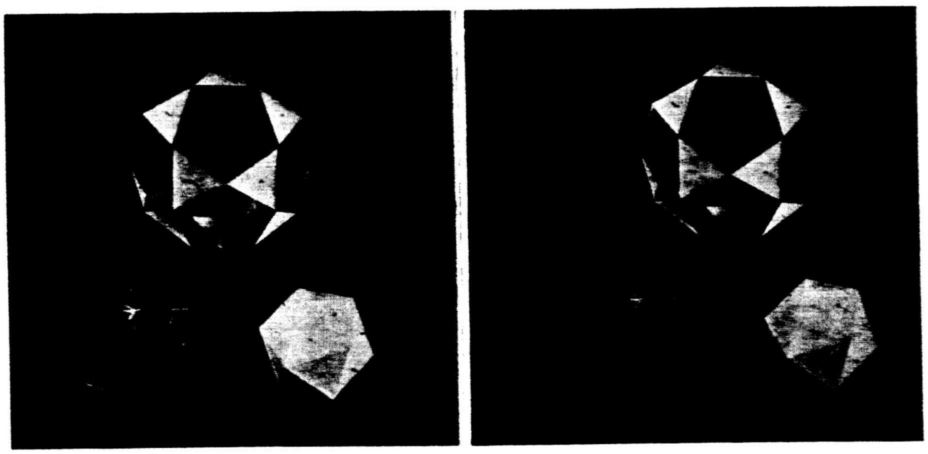


Figure 12

Dual Family Dodecahedron-Icosahedron

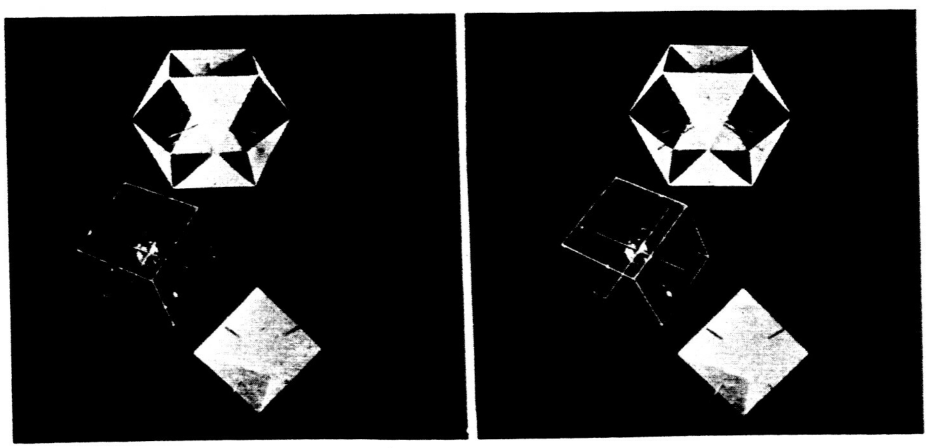


Figure 13

Dual Family Hexahedron-Octahedron

TABLE V

TRANSFORMATIONS OF PLATONIC POLYHEDRA

TETRAHEDRON →		OCTAHEDRON*
OCTAHEDRON →	ICOSAHEDRON →	HEXOCTAHEDRON†
HEXAHEDRON →		HEXOCTAHEDRON‡
ICOSAHEDRON →		ICOSADODECAHEDRON"
DODECAHEDRON →		ICOSADODECAHEDRON**

	VERTICES	FACES	EDGES	△	□	⬠
TETRAHEDRON	4	4	6	4	-	-
OCTAHEDRON	6	8	12	8	-	-
HEXAHEDRON	8	6	12	-	6	-
ICOSAHEDRON	12	20	30	20	-	-
DODECAHEDRON	20	12	30	-	-	12
HEXAHEDRON	12	14	24	8	6	-
ICOSADODECAHEDRON	30	32	60	20	-	12

	FACES EXISTING		FACES Existing	FACES Void
TETRAHEDRON	△ -4 →	OCTAHEDRON	△ -4 →	△ -4
OCTAHEDRON	△ -8 →	HEXOCTAHEDRON	△ -8	□ -6
HEXAHEDRON	□ -6 →	HEXOCTAHEDRON	□ -6	△ -8
ICOSAHEDRON	△-20 →	ICOSADODECAHEDRON	△-20	⬠-12
DODECAHEDRON	⬠-12 →	ICOSADODECAHEDRON	⬠-12	△-20

---

*See Figure 14

†See Figure 15

‡See Figure 16

"See Figure 17

**See Figure 18.

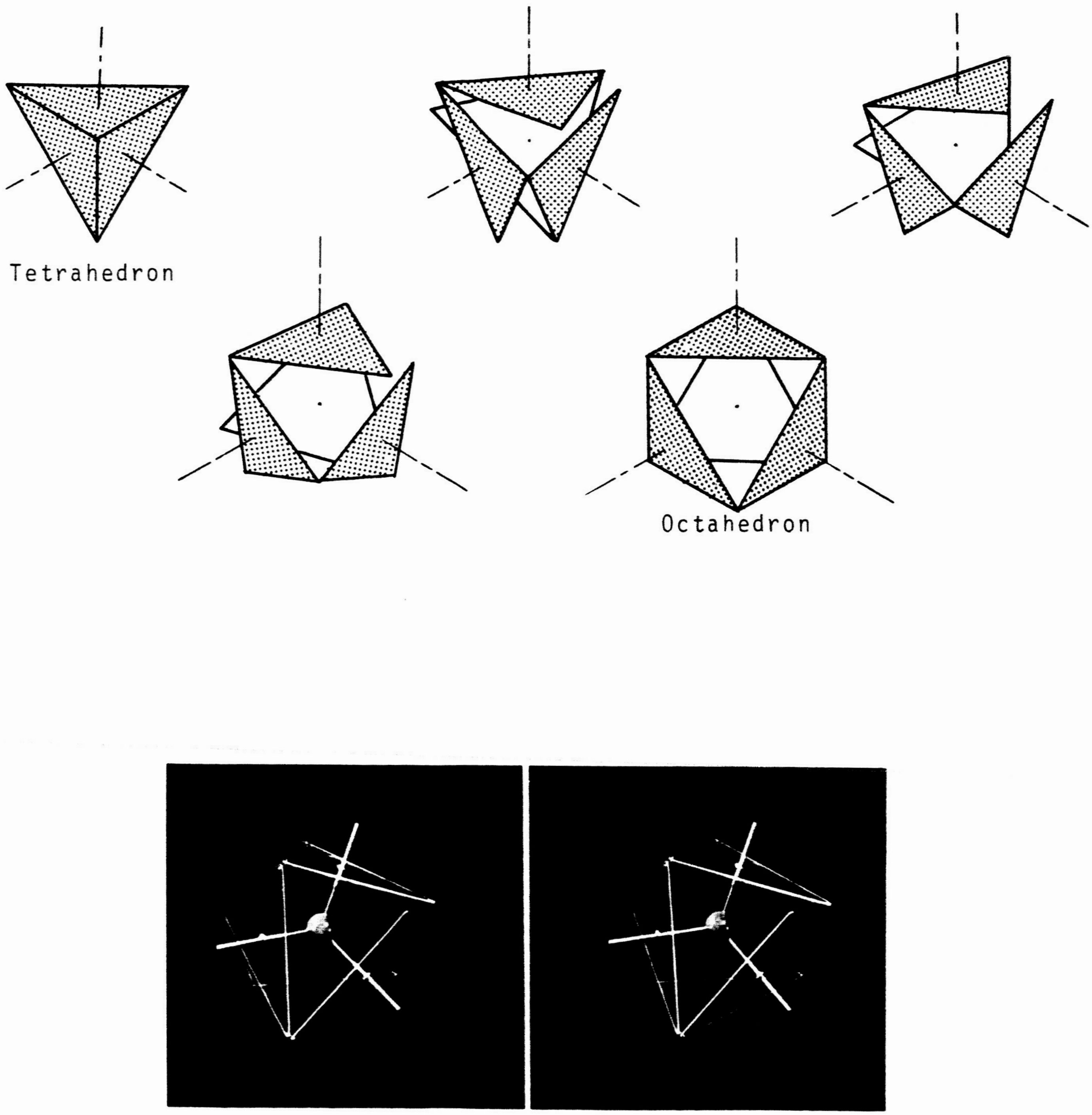
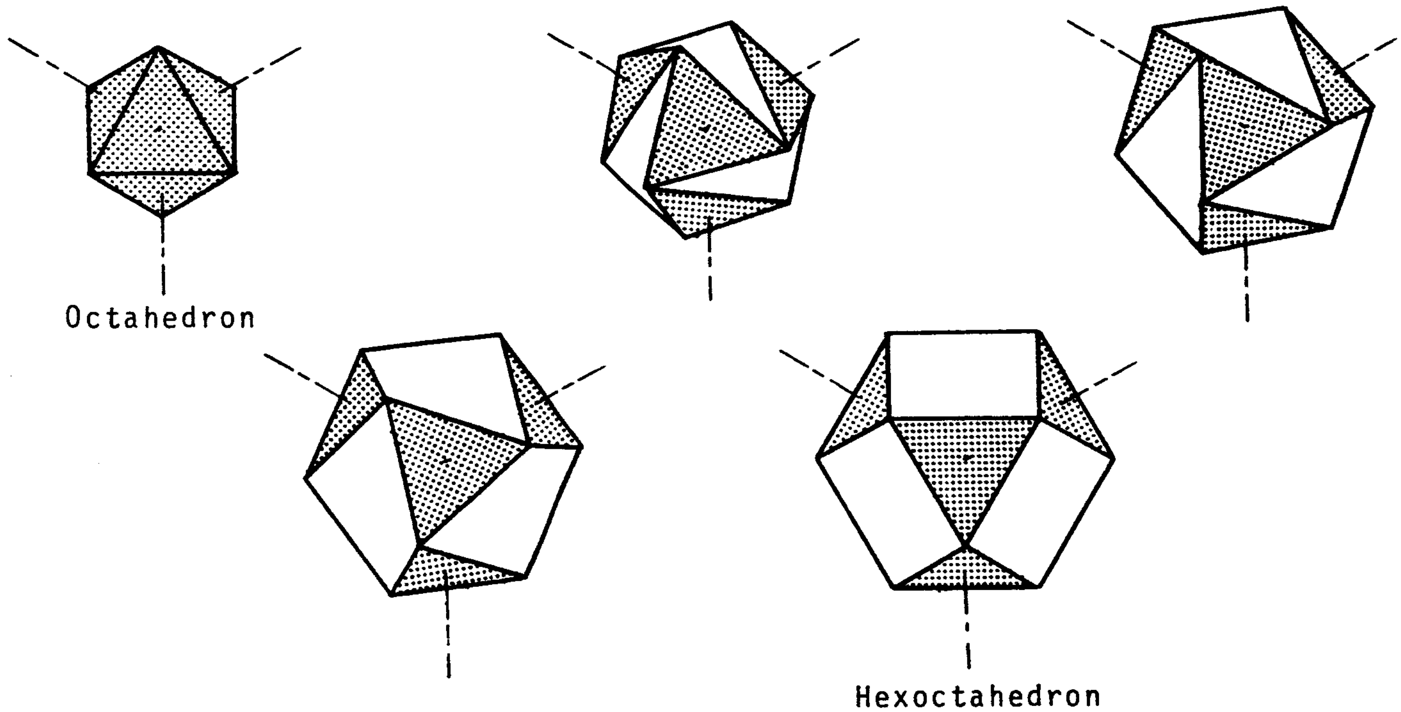


Figure 14

Transformation of Tetrahedron



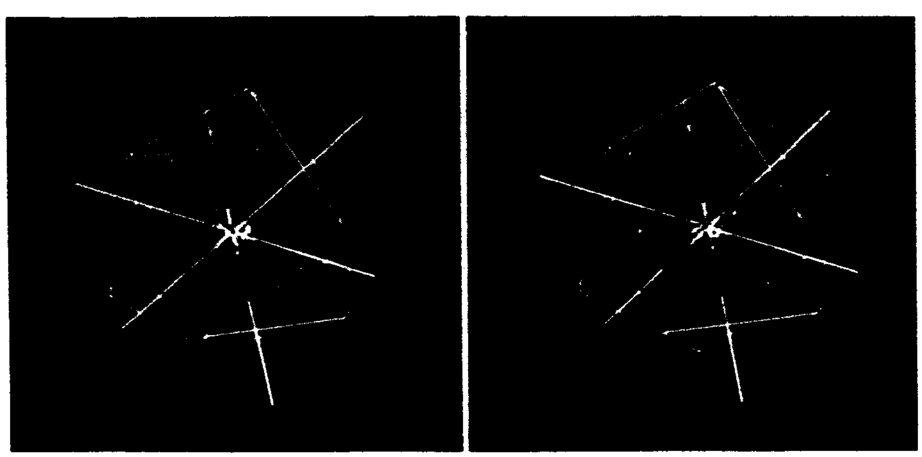
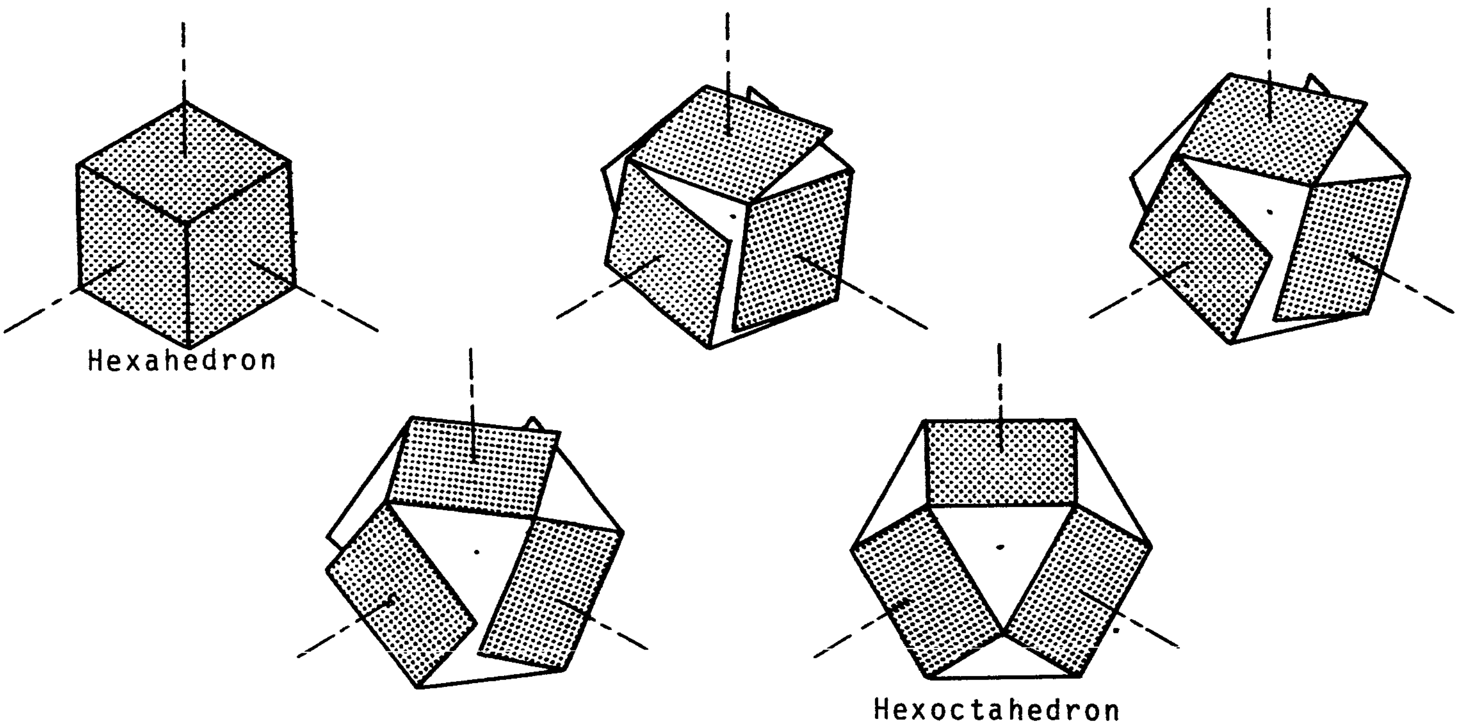


Figure 15

Transformation of Octahedron



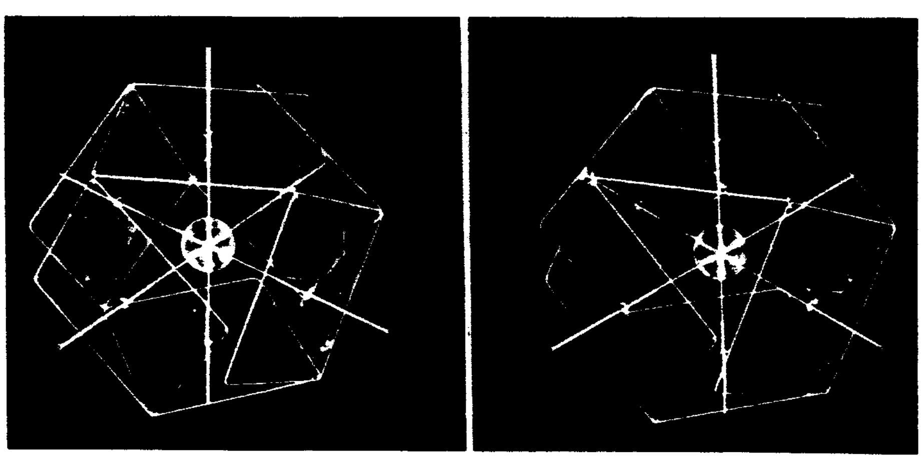


Figure 16

Transformation of Hexahedron

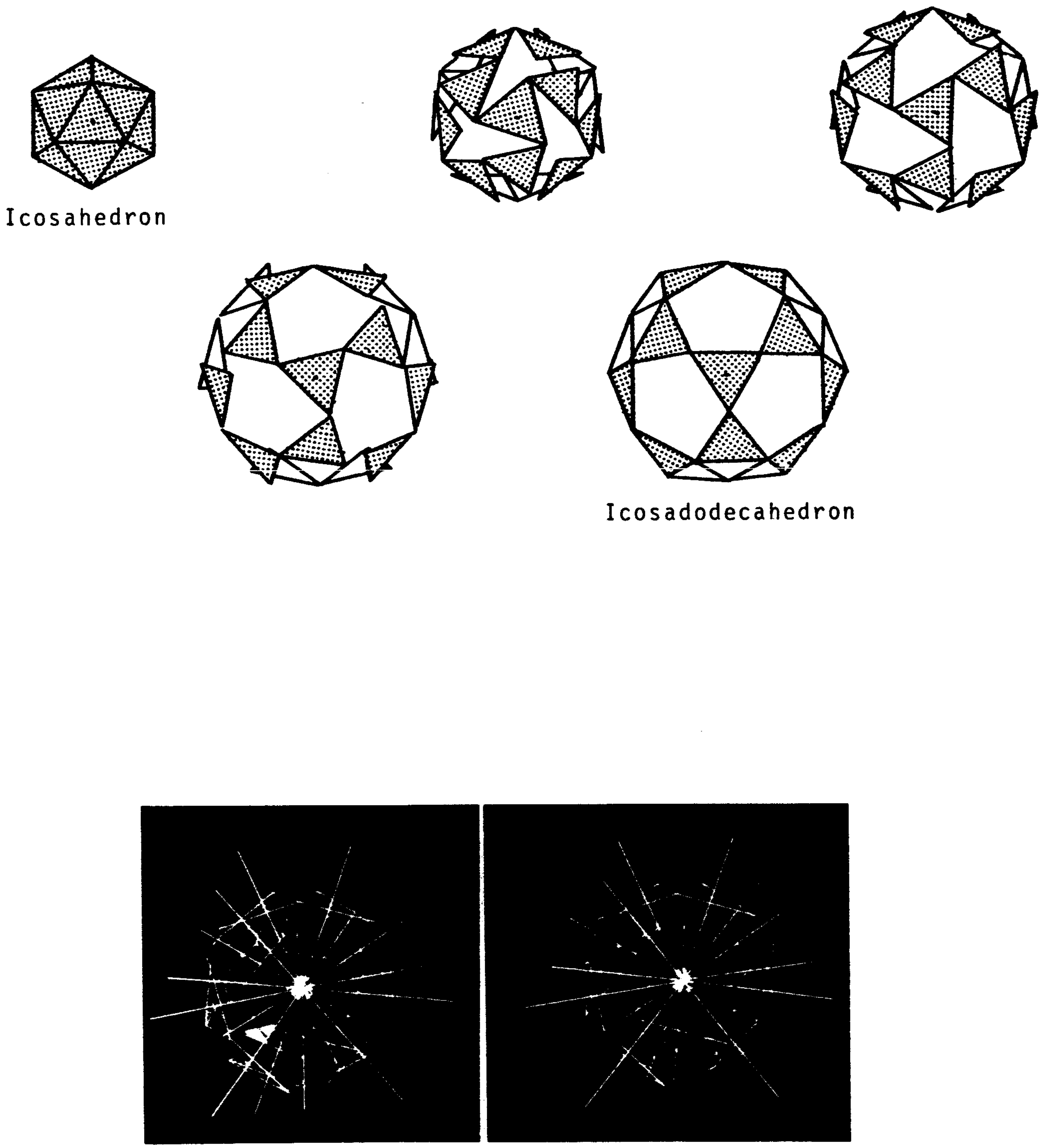


Figure 17

Transformation of Icosahedron

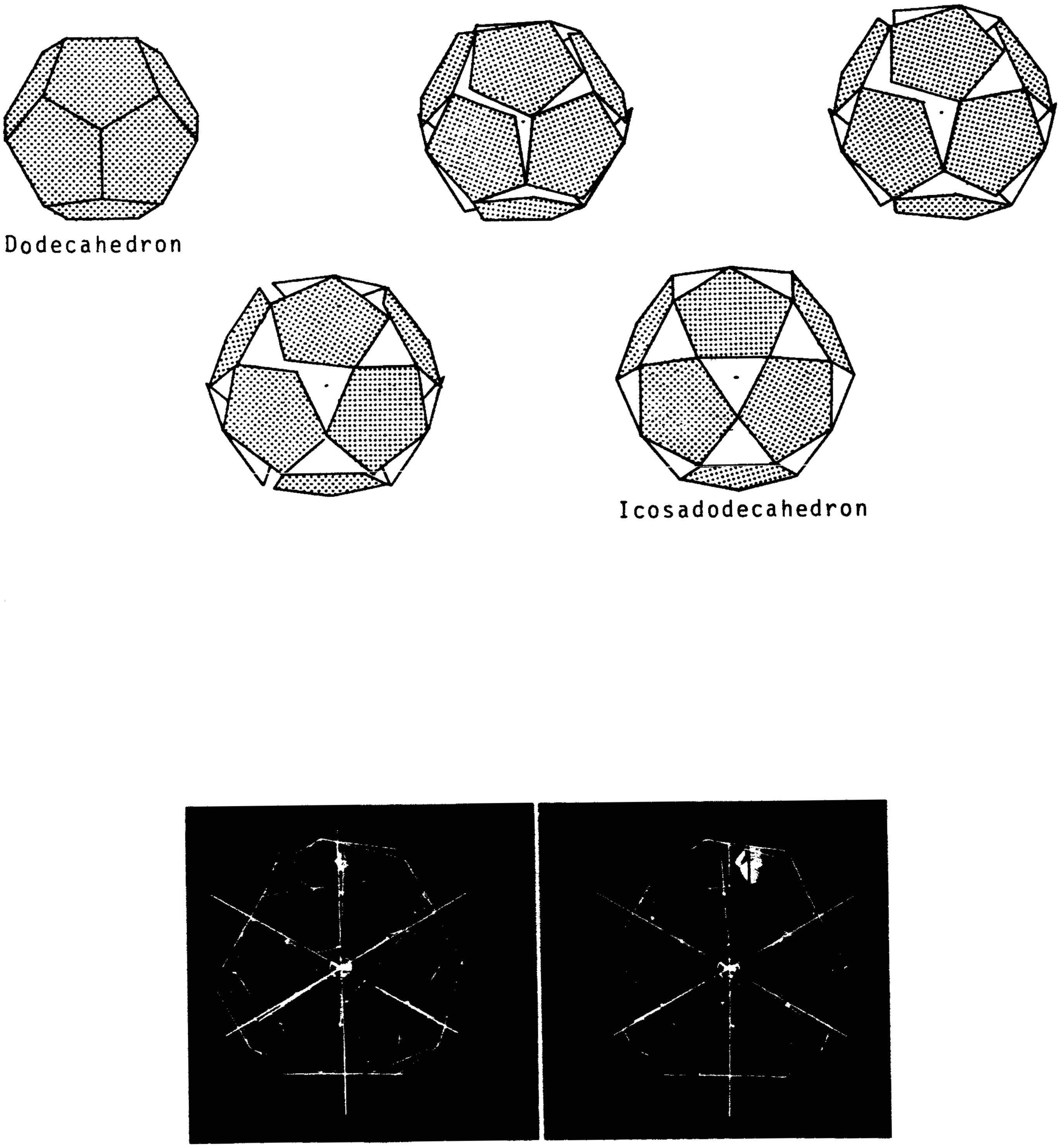


Figure 18

Transformation of Dodecahedron

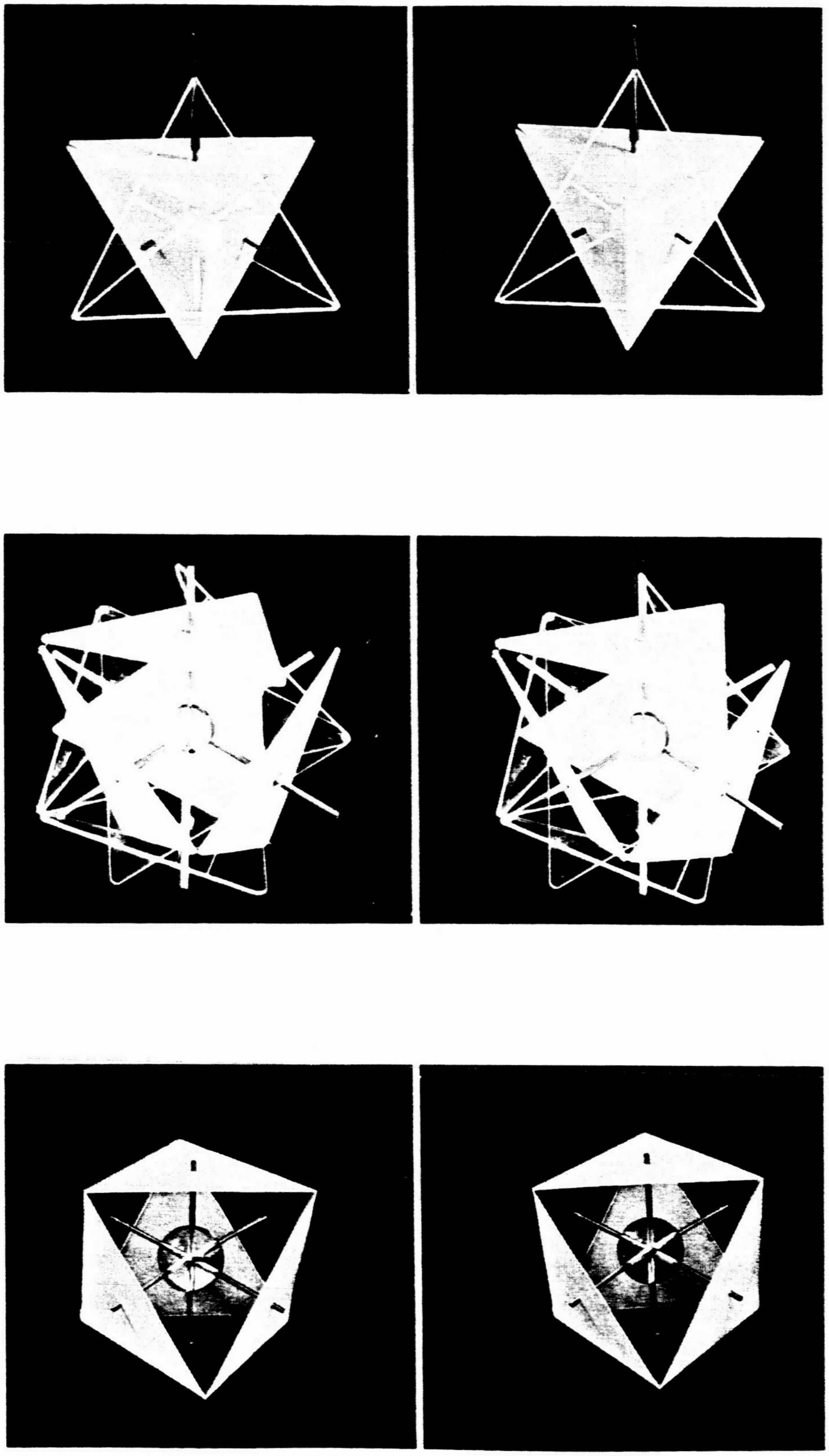


Figure 19

Double Transformation of Tetrahedron-Tetrahedron

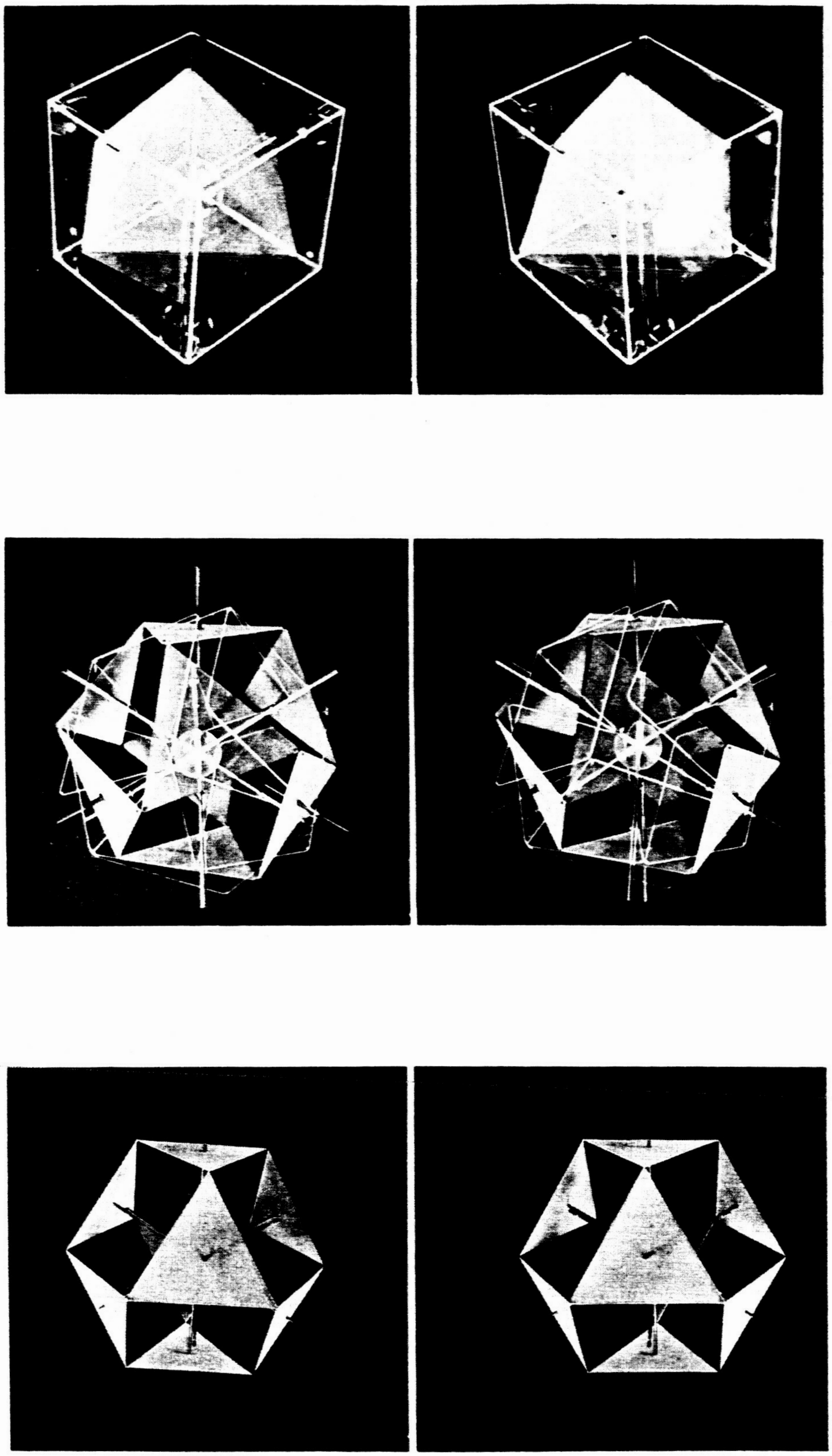


Figure 20

Double Transformation of Hexahedron-Octahedron

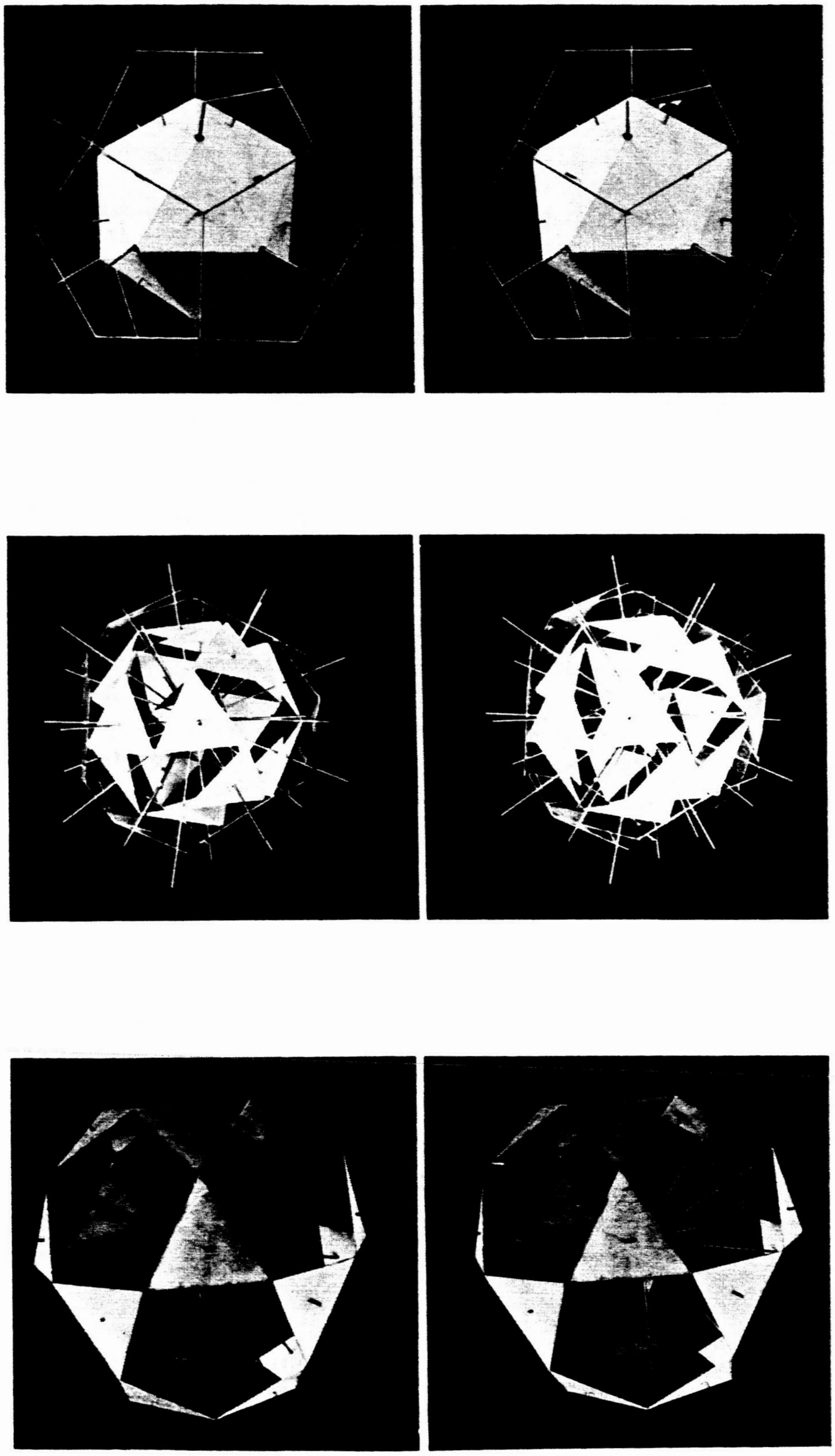
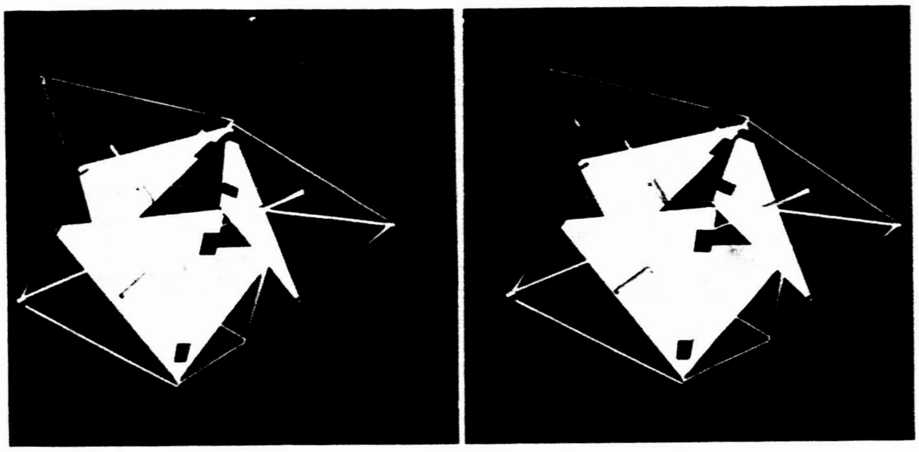
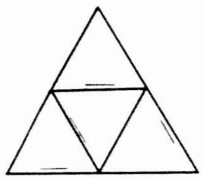


Figure 21

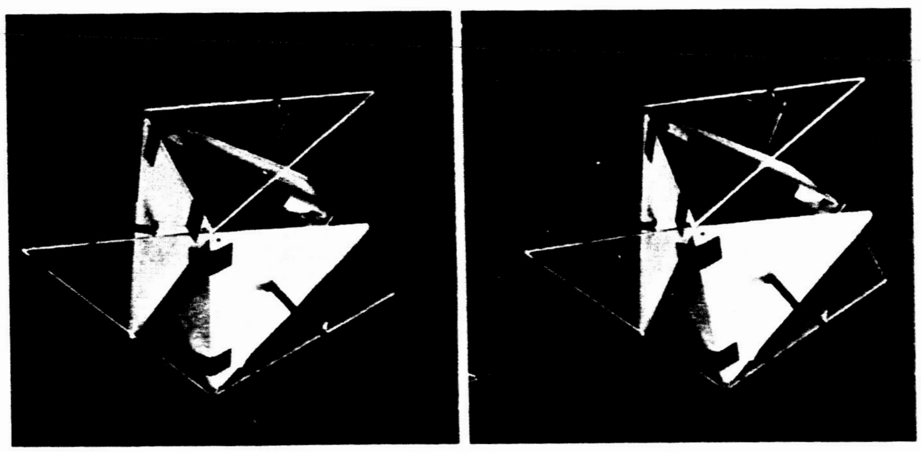
Double Transformantion of Icosahedron-Dodecahedron



Folding



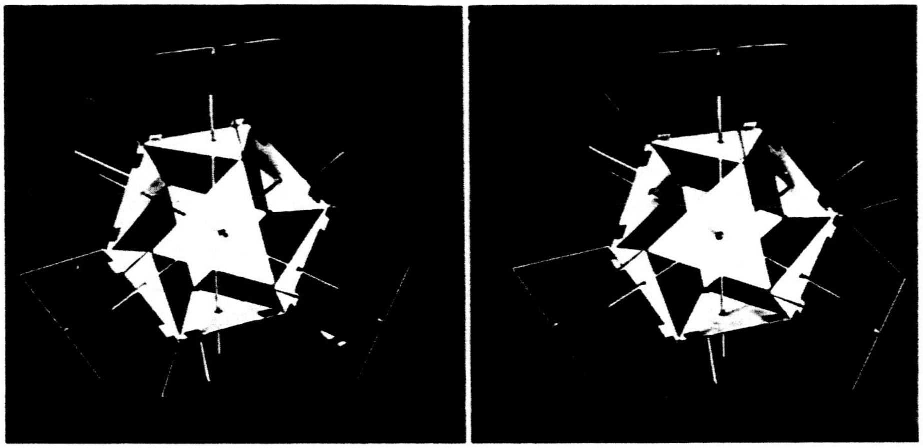
Net



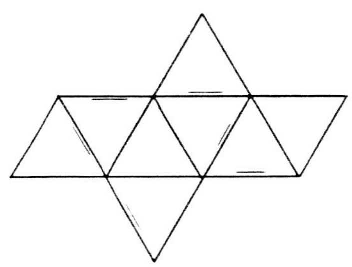
Fanning

Figure 22

Folding-Fanning Transformation of Tetrahedron



Folding



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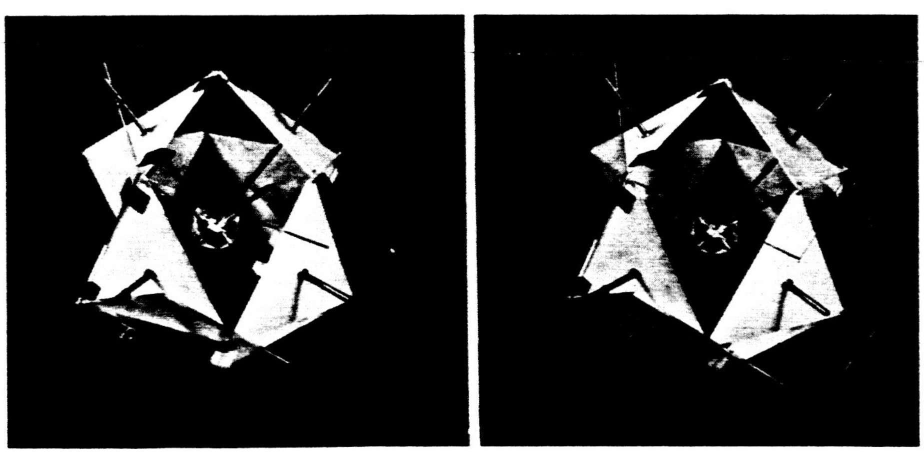
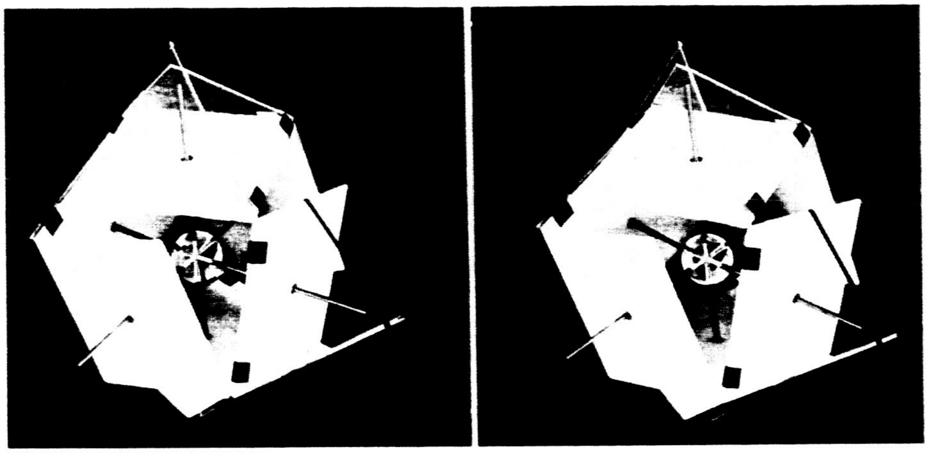
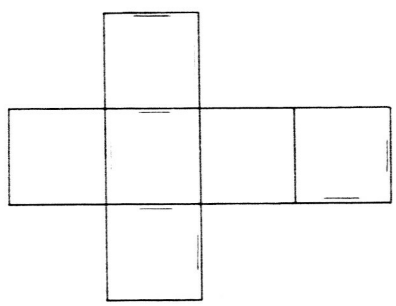


Fanning
Figure 23

Folding-Fanning Transformation of Octahedron



Folding



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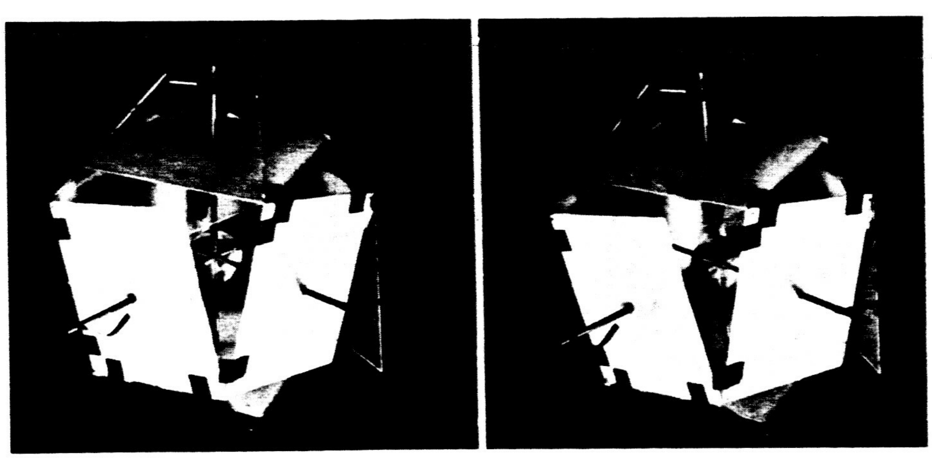
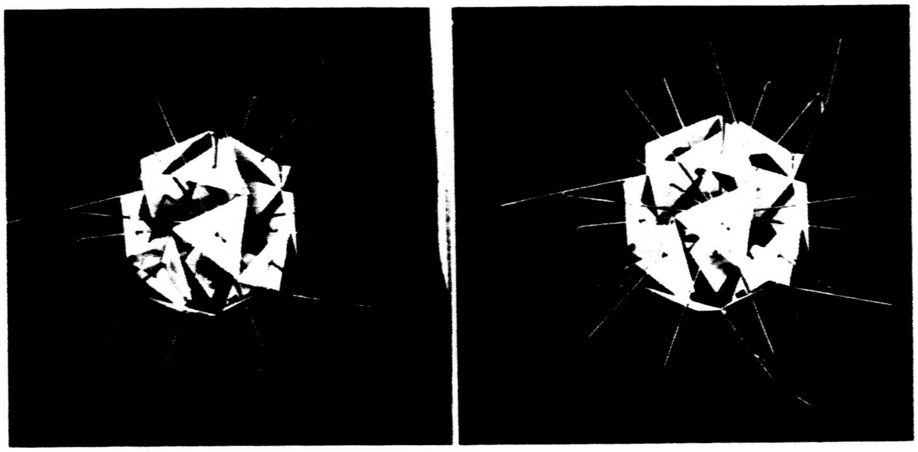


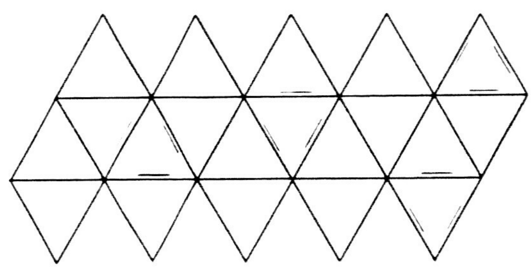
Fanning

Figure 24

Folding-Fanning Transformation of Hexahedron



Folding



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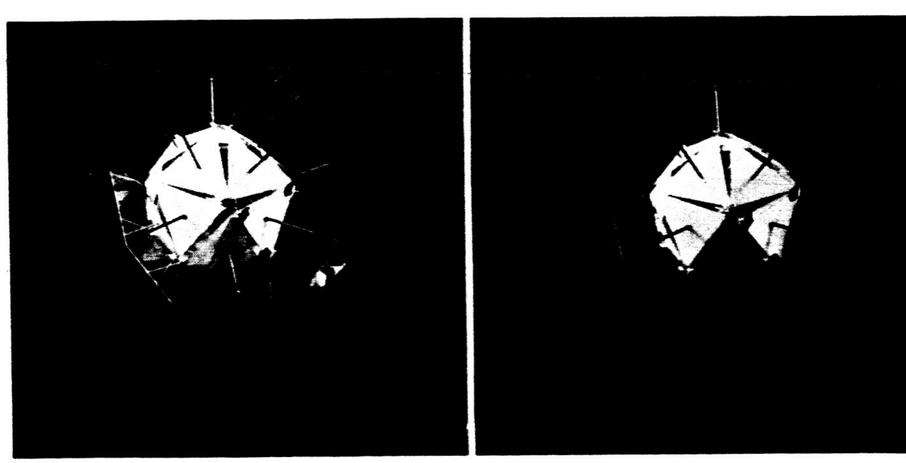
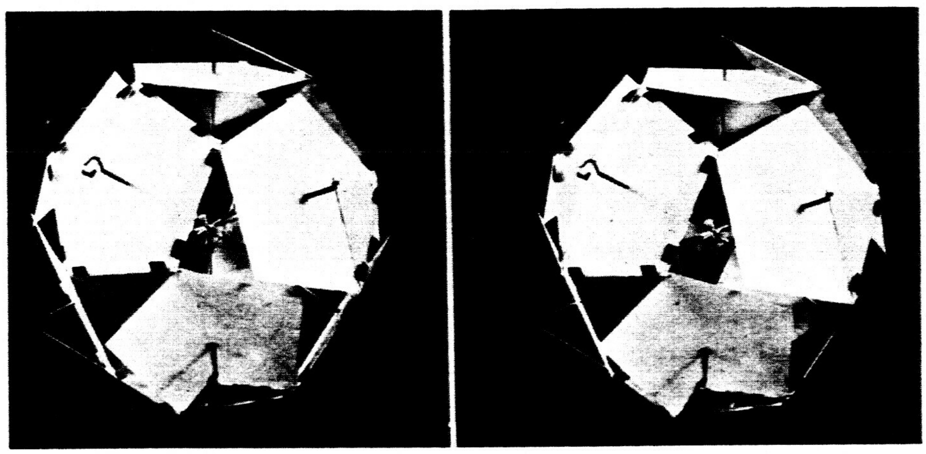
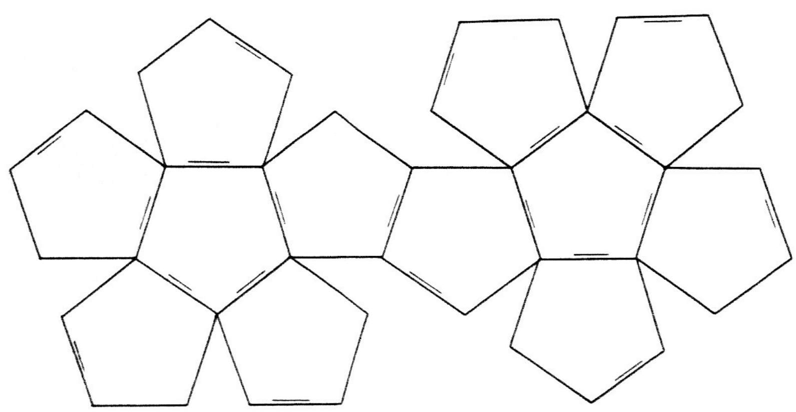


Fanning
Figure 25

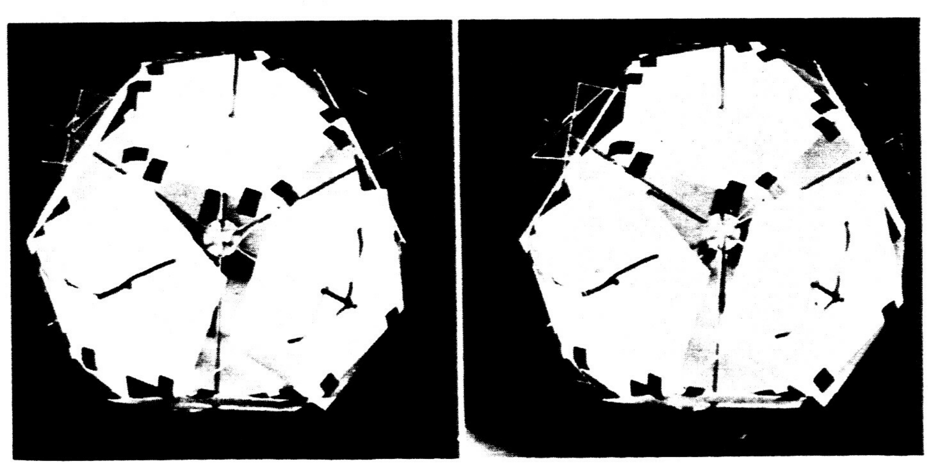
Folding-Fanning Transformation of Icosahedron



Folding



Net



Fanning

Figure 26

Folding-Fanning Transformation of Dodecahedron

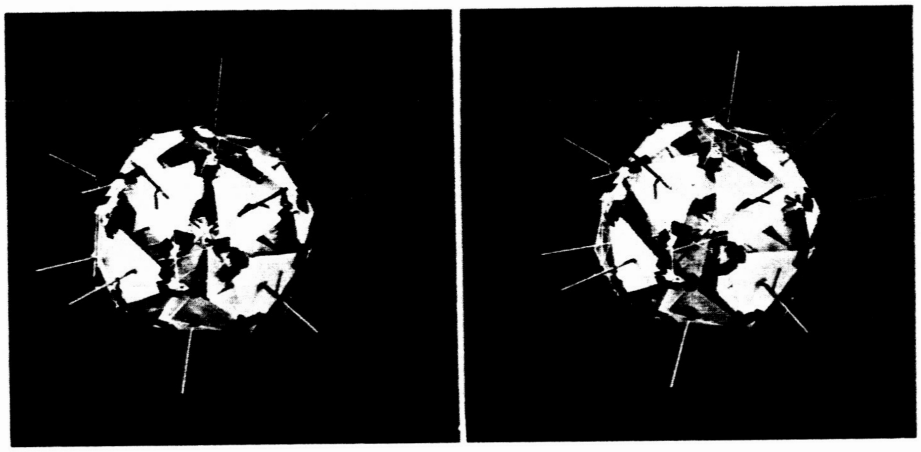


Figure 27

Polygon Sectioning For Filling Voids

V SUMMARY AND CONCLUSIONS

Man has continually preoccupied himself with seeking various kinds of order within his environment. One group of scholars who studied orderliness was the Greeks who envisioned what we call *polyhedra.* Another display of order was found by Haechel. The *radiolaria* (microscopic skeletons of sea creatures) that he discovered had many of the properties of polyhedra occuring in nature. And in more recent times the study of *closest packing of spheres* revealed close relationships to polyhedra.

In 1947, R. Buckminster Fuller presented, through his studies of closest packing, the *"Jitterbug"* which illustrated a new concept of the orderliness of polyhedra. Through the studies stemming from the *"Jitterbug"* evolved the transformation concept of polyhedra. In contrast to the classical transformation the rotation-translation transformation concept has an orderly characteristic about it. The rotation-translation transformation concept introduces a new means of expanding a structure. Expansion is accomplished by allowing each face of a polyhedron to rotate about its axis, translate upon its axis, and maintain connection with one of its paired vertices; by doing so one polyhedron form is transformed into another form with more surface area and a larger volume.

Prior to the translation a known configuration and number of faces exist, and after the transformation another configuration exists with an increased number of faces. The configuration after the transformation maintains only the number of faces that it began with and the increased number exists as voids. This study revolved around conceptual means for filling these voids created during the transformation. Three concepts for filling the voids were presented: double rotation-translation transformation, fanning rotation-translation transformation, and folding rotation-translation transformation.

The first concept takes advantage of polyhedra duals and the transformation concept itself, and uses the duals to fill each other. The disadvantages to this concept is that there is surface interference existing except in the case of the icosahedron,dodecahedron dual.

The latter two concepts make use of the "classical" *folding* and *fan* concepts. Faces are hinged to the existing surfaces and are either folded or fanned into position. A disadvantage existing around these two concepts is the awkwardness of hinging the square polygons in the case of the octahedron transformation and the hinging of the pentagonal polygons in the case of the icosahedron transformation. By dividing these awkward surfaces, however, into triangular polygons and locking them into position after the transformation the awkwardness is deleted.

PROJECTED RESEARCH

With the introduction of the transformation concept, an entire new area was opened for research. This report dealt only with one minute entity of the concept. The research done, however, introduced many problem areas yet to be investigated.

Several conceptual studies proposed for further research stem from the investigation discussed in this report. One such area indicated is an investigation of the forms created during the transformation of polyhedra as illustrated in Figure 28. Instead of the face rotation-translation transformation discussed in this report, a concept of element face rotation-translation transformation is used.

Another area needing study is that of the Archimedian or semi-regular polyhedral forms which may ultimately lead into the familiar Geodistic Icosahedron or Fuller domes. It was noted during the investigation that similarities did appear among the regular polyhedron and the geodistic configurations used in construction of spherical domes.

Another interesting concept that may warrant investigation is that of polyhedral cycling. It was noted in filling the void areas created during the transformation that if an appropriate number of faces were used instead of a single face to fill the void, a cycling effect took place. The faces to be used need to have the same length of element as the existing surfaces. They also need to be equal in number to the existing exposed sides. (i.e. the octahedron transforms into the hexoctahedron leaving square void areas. Instead of filling the voids with square surfaces, they are filled with four triangles that match the existing surfaces. The result is the original octahedron form only enlarged in volume and surface area. The original transformation axis is retained and the cycling can be repeated.)* It is conceivable that through locking and retransforming the cycling could take place as many times as physically or mechanically possible.

There is also need for hardware investigations to answer the problems of jointing, hinging, rotating, translating and transforming the polyhedra. There are unique characteristics which warrant investigation in these directions to aid in applications of the transformation concept.

It is quite evident that there is much needed research to make the transformation concept an applicable one for practical expandable structures.

*See Figure 29.

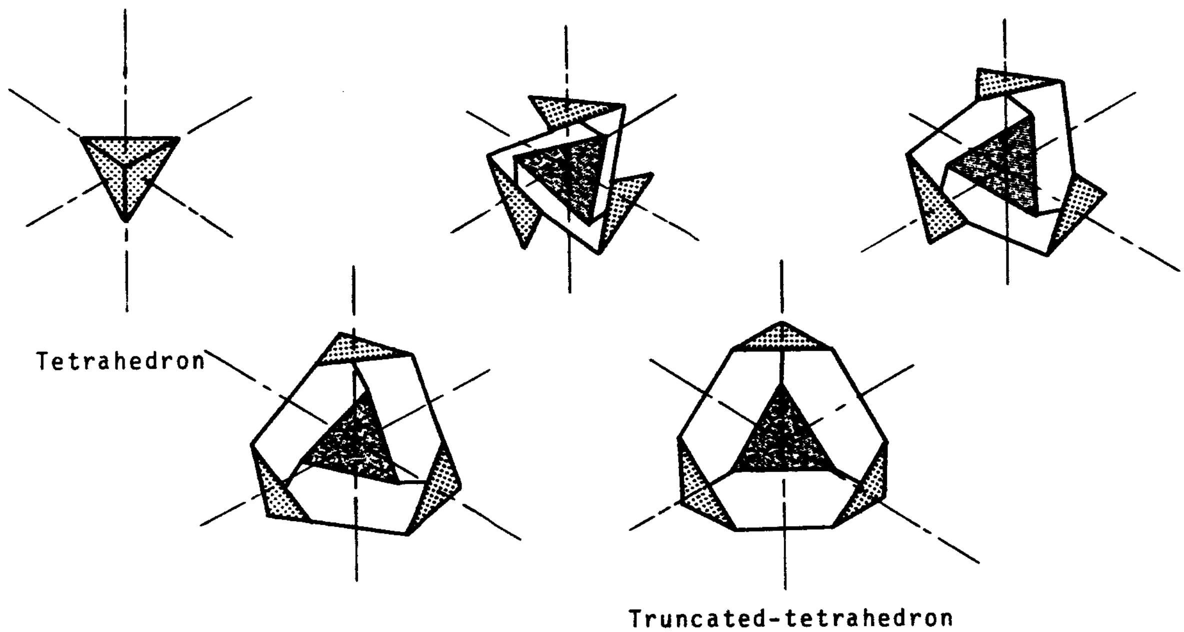


Figure 28

Element Face Rotation-Translation Transformation

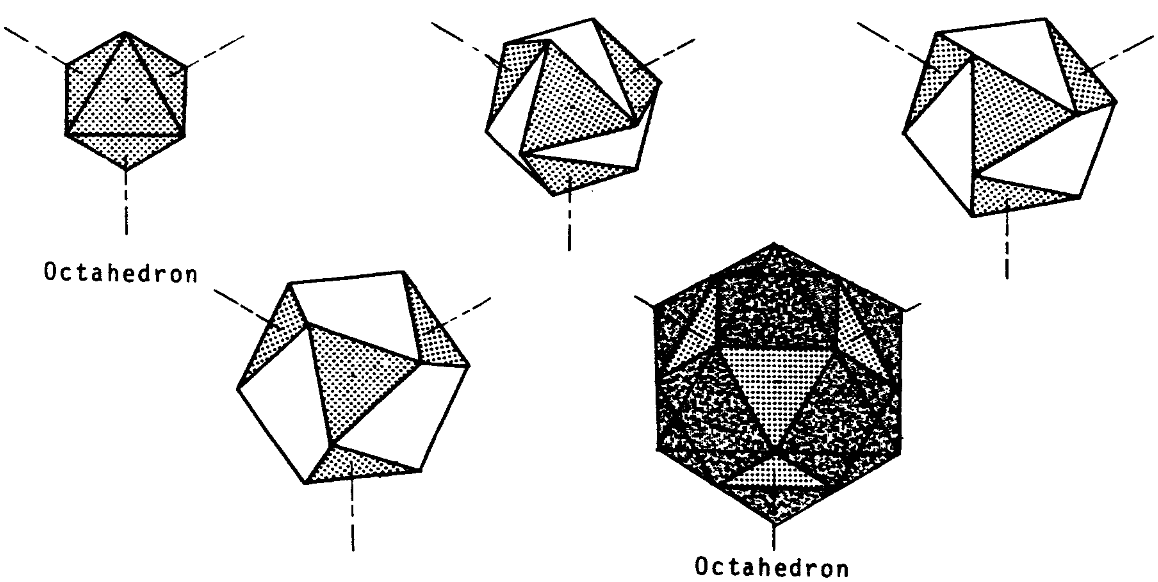


Figure 29

Polyhedron Transformation Cycle

REFERENCES

Aerospace Expandable Structures. 1. : Conference Transactions, (ASTIA No. Ad 432 0 6), AF Aero Propulsion Laboratory; AF Flight Dynamics Laboratory: NCR Sugar Camp, Dayton, Ohio, October 1963.

Abraham, Lewis H. 1. : Important Structural Research Problems for the Support of Future Space Missions. NASA TND-2059, 1963.

_____ 2. : Structural Designs of Missiles and Spacecraft. McGraw-Hill Book Co., Inc., 1962.

Albert, Adrian 1. : Solid Analytic Geometry. McGraw-Hill Book Co., 1949.

Allen, K. W. 1. : "Polyhedral Plathrate Hydrates. VIII. The Geometry of the Polyhedra." The Journal of Chemical Physics. Vol. 41, No. 3, 1 August 1961, pp. 840-844.

Brumfiel, Charles F., Eicholz, Robert E., and Shanks, Merrill E. 1. : Geometry. Addison-Wesley Publishing Company, Inc., 1962.

Candy, H. M.; Rollell, A. P. 1. : Mathematical Models. Oxford University Press, 1952.

Coxeter, H. S. M. 1. : Contributions of Geometry to the Main Stream of Mathematics. The Development of Mathematics, The Oklahoma Agriculture and Mechanical College, 1955.

_____ 2. : Regular Polytopes. The MacMillan Company, 1963.

Coxeter: Da Vol: Petrie 1. : The Fifty-nine Icosahedra. University of Toronto Studies (Mathematical Series), Vol. III, No. 6, 1938.

Coxeter, H. S. M.; and Moser, W. O. J. 1. : "Generators and Relations for Discrete Groups." Ergebnisse Der Mathematik und Ihrer Grenzgebiete. Berlin, Springer-Verlag, 1957, Heft 14.

Faben, Colin 1. : Candela: The Shell Builder. Reinhold Publishing Corporation, 1963.

Forbes, F. W. 1. : Expandable Structures for Aerospace Applications. (NASA No. N63-14222), AF Aero Propulsion Laboratory, Research and Technical Division, 1963.

______ 2.: Expandable Structures for Space Applications. (NASA No. N65-14423; ASTIA No. AD60-7541), AF Aero Propulsion Laboratory, Research and Technical Division, July 1964.

Fuller, R. Buckminster 1. : "Omnidirectional Halo." No More Secondhand God and other Writings. Southern Illinois University Press, 1963, pp. 130-163.

______ 2.: World Design Science Decade 1965-1975, The Design Initiative. Phase I (1964), Document 2, World Resources Inventory, Southern Illinois University, 1963.

Heath, T. L. 1. : The Works of Archimedes. Dover Publications, Inc., 1912.

Hilker, W. R.; Fujoka, R. S. 1. : A Variable Geometry Space Structures Concept - Basic Characteristics and Properties. (NASA No. N63-12355), Northrop Space Laboratories, October 1962.

Kasper, John S.; Lonsdale, Kathleen (ed.) 1. : "International Tables for X-ray Crystallography." Vol. II. Mathematical Tables. Birmingham, England: The Kynoch Press, 1959, pp. 43-49.

Klee, Victor 1. : Diameters of Polyhedral Graphs. Mathematical Note No. 307, Mathematics Research Laboratory Boeing Scientific Research Laboratories, June 1963, (Also available as ASTIA No. AD411312).

______ 2. : Paths of Polyhedra I. Mathematical Note No. 372, Mathematics Research Laboratory Boeing Scientific Research Laboratories, October 1964, (Also available as ASTIA No. AD609571).

______ 3. : Paths of Polyhedra II. Mathematical Note No. 374, Mathematics Research Laboratory Boeing Scientific Research Laboratories, October 1964, (Also available as ASTIA No. AD609498).

Klein, Felix 1. : Famous Problems of Elementary Geometry. Archibald, R. C., Transl., G. E. Stechert and Co., 1930.

______ 2. : Lectures on the Icosahedron. Norrice G. G. Transl. Dover Publishing, Inc., 1956.

______ 3. : _Elementary Mathematics from an Advanced Standpoint_. Hedrick, E. R.; Noble, C. A., Transl., The MacMillan Company, 1939.

Knox, P. M. Jr.; Corell, A.; Lane, E. 1. : _Expandable Space Structure_. Interim Technical Documentary Report No. ASD - TDR - 7 - 943a (1) (ASTIA No. AD296899), Martin Company, Aerospace Division of Martin, Marietta Corporation, Denver, Colorado, January 1963.

Koleum, H. E. 1. : "NASA Explores Space Station Problems," _Aviation Week_, (26 June 1961).

Kravitz, Sidney 1. : "Your Turn Tessellations and Polyhedrons," _Product Engineering_. 20 July 1964, pp. 119-112.

Lebovits, M.; Supata, A. L.; King, C. F.; Young, J. W. 1. : _Conceptual Investigations of Variable Geometry Space Structures_. (NASA No. N6410309; CR - 52464). Final Report, Northrop Space Laboratories, May 1963.

Lyusternik, L. A. 1. : _Convex Figures and Polyhedra_. Dover Publications, Inc., 1963.

Marks, Robert W. 1. : _The Dymaxion World of Buckminster Fuller_. Reinhold Publishing Corporation, 1960.

McHale, John 1. : _R. Buckminster Fuller_. George Braziller, Inc., 1962.

Meserve, Bruce 1. : _Fundamental Concepts of Geometry_. Cambridge: Addison-Wesley Publishing Co., Inc., 1955.

Motzkin, Theodore S. 1. : _The Evenness of the Number of Edges of a Convex Polyhedron_. proceedings of the National Academy of Sciences, Vol. 52 No. 1, July, 1964, pp. 44-45, (Also available as ASTIA No. AD452853).

Ore, Oystein 1. : _Graphs and Their Uses_. Random House, 1963.

Riyves, Z. 1. : _About Projection Properties of the Orthocentric Tetrahedron_. An est SSR. Izvestiya. Seriya Fiziko-Matematicheskikh i Tekknicheskish Nauk, Vol. 9, 1 November 1960, pp. 69-74, (Also available as ASTIA No. AD606965).

Salvador, Mario; Heller, Robert 1. : _Structure in Architecture_. Prentice-Hall Inc., 1963.

Stuart, Duncan 1. : "Polyhedral and Mosaic Transformations." The Student Publications of the School of Design. North Carolina State of the University of North Carolina, Vol. 12, No. I, 1963.

Tinnan, L. M.; Weber, E. A. 1. : Self-Deploying Space Station. Final Report, NASA CR-52 707, (NASA No. N6411728), 31 October 1962.

Tutton, A. E. H. 1. : Crystallography and Practical Crystal Measurement. MacMillan and Co., Luncted, London, England: Vol. I, 1922.

Wright, F. N. 1. : Concepts and Applications of Expandable Structures. Report No. SM-42809, (NASA No. N64-33640), Douglas Aircraft Company Independent Research and Development Program, Missile and Space Systems Division, Douglas Aircraft Company, Inc., May 1963.

APPENDIX

PROPERTIES OF THE PLATONIC SOLIDS + 2

In the following tables showing the properties of the regular polyhedra, the vertices are given in terms of the Cartesian co-ordinates. In each polyhedron the most important dimensions are given in terms of a simple choice of dimension edge. The dimensions are expressed in terms of the units chosen for the edge.* Each one is then computed for a unit edge of 4.

(TABLE VI)

CUBE

V = 8   F = 6   E = 12

Dihedral ∢ = 90°

Ventrices	(1/2, 1/2, 1/2)		(2, 2, 2)	
Edge	1	1	4	4
Face diagonal	$\sqrt{2}$	1.41421	$4\sqrt{2}$	5.65684
Body diagonal	$\sqrt{3}$	1.73205	$4\sqrt{3}$	6.92820
Area of Face	1	1	16	16
Volume	1	1	64	64

*Kasper 1, pp. 43-49.

(TABLE VII)

TETRAHEDRON

V = 4 F = 4 E = 6

Dihedral ∢ = 70°32'

Vertices	$(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$ $(\frac{-1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}})$	$(\frac{1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}})$ $(\frac{-1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$	$(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}})$ $(\frac{-2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, \frac{-2}{\sqrt{2}})$	$(\frac{2}{\sqrt{2}}, \frac{-2}{\sqrt{2}}, \frac{-2}{\sqrt{2}})$ $(\frac{-2}{\sqrt{2}}, \frac{-2}{\sqrt{2}}, \frac{2}{\sqrt{2}})$
Centre to mid edge	$\frac{1}{2\sqrt{2}}$	0.35355	$\frac{2}{\sqrt{2}}$	1.41420
Centre to centre of face	$\frac{1}{2\sqrt{6}}$	0.20412	$\frac{2}{\sqrt{6}}$	0.81648
Centre to vertex	$\sqrt{\frac{3}{8}}$	0.61237	$4\sqrt{\frac{3}{8}}$	2.44948
Edge	1	1.00000	4	4.00000
Mid edge to centre of face	$\frac{1}{2\sqrt{3}}$	0.28868	$\frac{2}{\sqrt{3}}$	1.15472
Mid edge to vertex	$\frac{\sqrt{3}}{2}$	0.86603	$2\sqrt{3}$	3.46412
Mid edge to opposite mid edge	$\frac{1}{\sqrt{2}}$	0.70711	$\frac{4}{\sqrt{2}}$	2.82844
Height (vertex to centre of opposite face)	$\sqrt{\frac{2}{3}}$	0.81650	$4\sqrt{\frac{2}{3}}$	3.25600
Area of face	$\frac{\sqrt{3}}{4}$	0.43301	$4\sqrt{3}$	6.92816
Volume	$\frac{1}{6\sqrt{2}}$	0.11785	$\frac{32}{3\sqrt{2}}$	7.54240

(TABLE VIII)

OCTAHEDRON

V = 6 F = 8 E = 12

Dihedral ∢ = 109°28'

Vertices	$(\pm\frac{1}{\sqrt{2}}, 0, 0)$ $(0, 0, \pm\frac{1}{\sqrt{2}})$	$(0, \pm\frac{1}{\sqrt{2}}, 0)$	$(\pm\frac{4}{\sqrt{2}}, 0, 0)$ $(0, 0, \pm\frac{4}{\sqrt{2}})$	$(0, \pm\frac{4}{\sqrt{2}}, 0)$
Edge	1	1	4	4
Centre to vertex	$\frac{1}{\sqrt{2}}$	0.70711	$\frac{4}{\sqrt{2}}$	2.82844
Centre to mid edge	$\frac{1}{2}$	0.50000	2	2.00000
Centre to centre of face	$\frac{1}{\sqrt{6}}$	0.40825	$\frac{4}{\sqrt{6}}$	1.63300
Mid edge to near vertex	$\frac{\sqrt{3}}{2}$	0.86603	$2\sqrt{3}$	3.46412
Mid edge to distant vertex	$\frac{\sqrt{5}}{2}$	1.11803	$2\sqrt{5}$	4.47212
Area of face	$\frac{\sqrt{3}}{4}$	0.43301	$4\sqrt{3}$	6.92816
Volume	$\frac{\sqrt{2}}{3}$	0.47140	$\frac{64\sqrt{2}}{3}$	30.16960

(TABLE IX)

ICOSAHEDRON

V = 12   F = 20   E = 30

Dihedral ∡ = 138°12'

$^{\dagger}\rho = \frac{1+\sqrt{5}}{2} = 1.61803;\ \frac{1}{\rho} = \rho-1;\ \rho^2 = \rho+1;\ \rho\sqrt{5} = \rho+2$

Vertices†	$\left(0,\ \pm\frac{\rho}{2},\ \pm\frac{1}{2}\right)$ $\left(\pm\frac{\rho}{2},\ \pm\frac{1}{2},\ 0\right)$	$\left(\pm\frac{1}{2},\ 0,\ \pm\frac{\rho}{2}\right)$	$\left(0,\ \pm2\rho,\ \pm2\right)$ $\left(\pm2\rho,\ \pm2,\ 0\right)$	$\left(\pm2,\ 0,\ \pm2\rho\right)$
Edge	1	1	4	4
Centre to vertex	$\frac{5^{1/4}\sqrt{\rho}}{2}$	0.95106	$2(5^{1/4})\sqrt{\rho}$	3.80424
Centre to mid edge	$\frac{\rho}{2}$	0.80902	2ρ	3.23608
Centre to centre of face	$\frac{\rho^2}{2\sqrt{3}}$	0.75576	$\frac{2\rho}{\sqrt{3}}$	3.02304
Area of face	$\frac{\sqrt{3}}{4}$	0.43301	$4\sqrt{3}$	6.92816
Volume	$\frac{5\rho^2}{6}$	2.18169	$\frac{160\rho^2}{3}$	139.62816

(TABLE X)

REGULAR (PENTAGONAL) DODECAHEDRON

V = 20   F = 12   E = 30

Dihedral ∡ = 116°34'

$^{\dagger}\rho = \frac{1+\sqrt{5}}{2} = 1.61803;\ \frac{1}{\rho} = \rho-1;\ \rho^2 = \rho+1;\ \sqrt{5} = \rho+2$

Vertices†	$\left(0,\ \pm\frac{1}{2},\ \pm\frac{\rho^2}{2}\right)$ $\left(\pm\frac{\rho^2}{2},\ 0,\ \pm\frac{1}{2}\right)$	$\left(\pm\frac{1}{2},\ \pm\frac{\rho^2}{2},\ 0\right)$ $\left(\pm\frac{\rho}{2},\ \pm\frac{\rho}{2},\ \pm\frac{\rho}{2}\right)$	$\left(0,\ \pm2,\ \pm2\rho^2\right)$ $\left(\pm2\rho^2,\ 0,\ \pm2\right)$	$\left(\pm2,\ 2\rho^2,\ 0\right)$ $\left(\pm2\rho,\ \pm2\rho,\ \pm2\rho\right)$
Edge	1	1	4	4
Centre to vertex	$\frac{\rho\sqrt{3}}{2}$	1.40126	$2\rho\sqrt{3}$	5.60504
Centre to mid edge	$\frac{\rho^2}{2}$	1.30902	$2\rho^2$	5.23608
Centre to centre of face	$\frac{\rho^{5/2}}{2(5^{1/4})}$	1.11352	$\frac{2\rho^{5/2}}{5^{1/4}}$	4.45408
Area of face	$\frac{5^{3/4}\rho^{3/2}}{4}$	1.72048	$4(5^{3/4}\ \rho^{2/3})$	27.52768
Volume	$\frac{\rho^4\sqrt{5}}{2}$	7.66312	$32\rho^2\sqrt{5}$	490.43968

(TABLE XI)

CUBOCTRAHEDRON

V = 12 F = 14 E = 24

Dihedral $\angle$ = 125°16'

Edge	1	1	4	4
Centre to vertex	1	1	4	4
Centre to mid edge	$\frac{\sqrt{3}}{2}$	0.86603	$2\sqrt{3}$	3.46412
Centre to centre of square face	$\frac{1}{\sqrt{2}}$	0.70711	$\frac{4}{\sqrt{2}}$	2.82844
Centre to centre of triangular face	$\frac{2}{3}$	0.81650	$4\frac{2}{3}$	3.26600
Vertex to centre of square face	$\frac{1}{\sqrt{2}}$	0.70711	$\frac{4}{\sqrt{2}}$	2.82844
Vertex to centre of triangular face	$\frac{1}{\sqrt{3}}$	0.57735	$\frac{4}{\sqrt{3}}$	2.30940
Area of square face	1	1	16	16
Area of triangular face	$\frac{\sqrt{3}}{4}$	0.43301	$4\sqrt{3}$	6.92816
Volume	$\frac{5\sqrt{2}}{3}$	2.35702	$\frac{320\sqrt{2}}{3}$	150.84928

(TABLE XII)

ICOSADODECAHEDRON

V = 30 F = 36 E = 60

Dihedral $\angle$ = 142°37'

$\rho = \frac{1\ \sqrt{5}}{2} = 1.61803;\ \frac{1}{\rho} = \rho-1;\ \rho^2 = \rho+1;\ \rho\sqrt{5} = \rho+2$

Edge	1	1	4	4
Centre to vertex	ρ	1.6180	4ρ	6.4720
Centre to mid edge	$\frac{3.0777}{2}$	1.5388	$2 \cdot 3.0777$	6.1552
Centre to centre of triangular face	$\sqrt{\rho^2-\left(\frac{1}{\sqrt{3}}\right)^2}$	1.5083	$4\sqrt{\rho^2-\left(\frac{1}{\sqrt{3}}\right)^2}$	6.0332
Centre to centre of pentagonal face	$\sqrt{\rho^2-\left(\frac{1.7613}{2}\right)^2}$	1.3764	$4\sqrt{\rho^2-\left(\frac{1.7613}{2}\right)^2}$	5.5056
Vertex to centre of triangular face	$\frac{1}{\sqrt{3}}$	0.57735	$\frac{4}{\sqrt{3}}$	2.2940
Vertex to centre of pentagonal face	$\frac{1.7013}{2}$	0.85065	$2 \cdot 1.7013$	3.4020
Mid edge to centre of triangular face	$\frac{1}{2\sqrt{3}}$	0.28867	$\frac{2}{\sqrt{3}}$	1.14468
Mid edge to centre of pentagonal face	$\frac{1\ 3764}{2}$	0.68819	$2 \cdot 1.3764$	2.75276
Area of pentagonal face	$\frac{5^{3/4}\ \rho^{3/2}}{4}$	1.72048	$4(5^{3/4}\ \rho^{3/2})$	27.52768
Area of triangular face	$\frac{\sqrt{3}}{4}$	0.43301	$4\sqrt{3}$	6.92816
Volume	$5^{3/4}\rho\sqrt{\rho^2-\left(\frac{1.7013}{2}\right)^2} + \frac{5}{3}\sqrt{\rho^2-\left(\frac{1}{\sqrt{3}}\right)^2}$	14.9915	$4^3\left(5^{3/4}\ \rho\sqrt{\rho^2-\left(\frac{1.7013}{2}\right)^2} + \frac{5}{3}\sqrt{\rho^2-\left(\frac{1}{\sqrt{3}}\right)^2}\right)$	959.4560

(TABLE XIII)

PROPERTIES OF THE PLATONIC SOLIDS SPHERICALLY CIRCUMSCRIBED

	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Vertices V.	4	8	6	20	12
Faces F.	4	6	8	12	20
Edges E.	6	12	12	30	30
Diameter of Sphere	d	d	d	d	d
Edge (ℓ)	0.81655d	0.57735d	0.70710d	0.35682d	0.52572d
Area of face A_f	$0.2887d^2$	$0.3333d^2$	$0.2165d^2$	$0.2191d^2$	$0.1197d^2$
Area of Sphere A_s	πd^2	πd^2	πd^2	πd^2	πd^2
Area of Sphere "face" A_c	$0.7854d^2$	$0.5236d^2$	$0.3927d^2$	$0.2618d^2$	$0.1571d^2$
Volume of Sphere V_s	$0.5236d^3$	$0.5236d^3$	$0.5236d^3$	$0.5236d^3$	$0.5236d^3$
Volume of Phoyhedra Vp	$0.0642d^3$	$0.1925d^3$	$0.1666d^3$	$0.3479d^3$	$0.3170d^3$