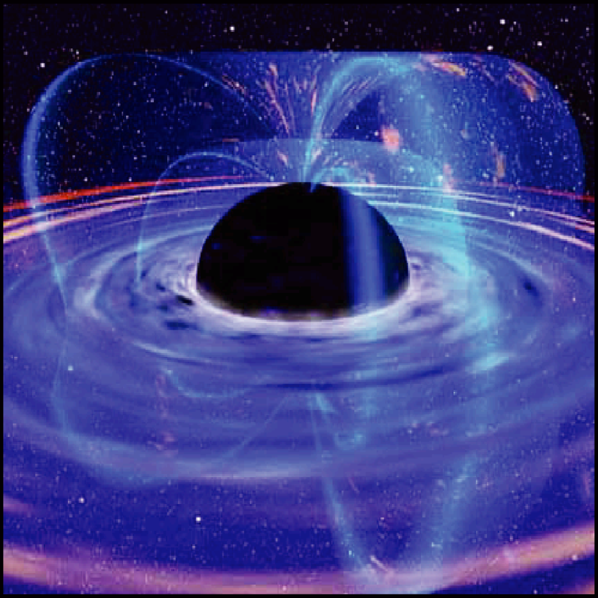
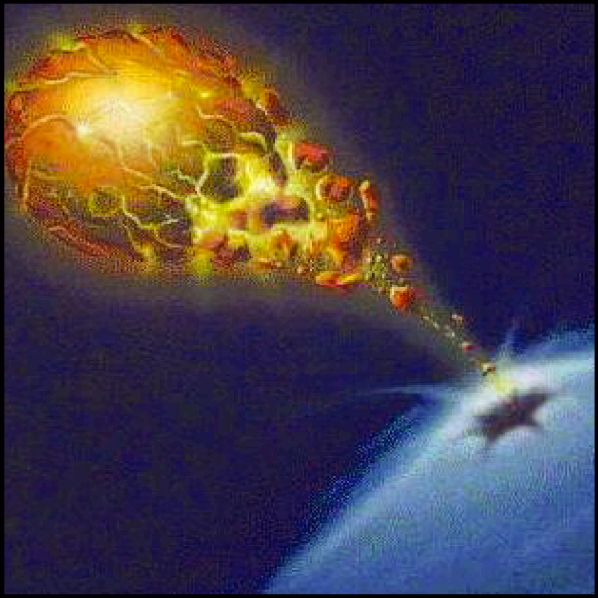
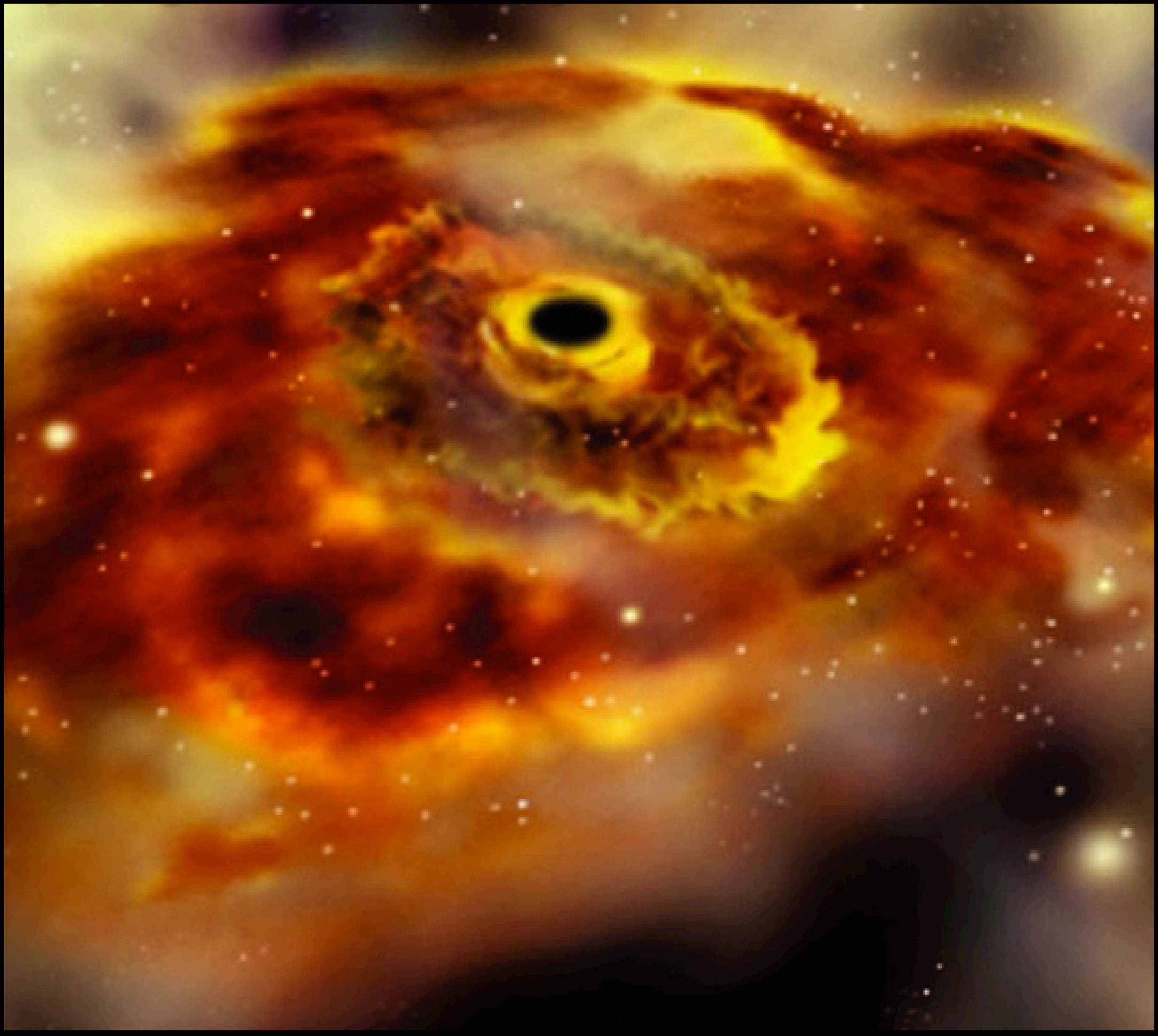
National Aeronautics and Space Administration







Black Hole Math

This collection of activities is based on a weekly series of space science problems distributed to thousands of teachers during 2004-2008 school years. They were intended as supplementary problems for students looking for additional challenges in the math and physical science curriculum in grades 6 through 8. The problems are designed to be 'one-pagers' consisting of a Student Page, and Teacher's Answer Key. This compact form was deemed very popular by participating teachers.

The topic for this collection is **Black Holes** which is a very popular, and mysterious subject among students hearing about astronomy. Students have endless questions about these exciting and exotic objects, as many of you may realize! Amazingly enough, many aspects of black holes can be understood by using simple algebra and pre-algebra mathematical skills. This booklet fills the gap by presenting black hole concepts in their simplest mathematical form.

General Approach:

The activities are organized according to progressive difficulty in mathematics. Students need to be familiar with Scientific Notation, and it is assumed that they can perform simple algebraic computations involving exponentiation, square-roots, and have some facility with calculators. The assumed level is that of Grade 10-12 Algebra II, although some problems can be worked by Algebra I students. Some of the issues of energy, force, space and time may be appropriate for students taking high school Physics.

This booklet was created by the NASA Space Math program

Dr. Sten Odenwald ***(NASA - Hinode)***

For more weekly classroom activities about astronomy and space science, visit
http://spacemath.gsfc.nasa.gov
Add your email address to our mailing list by contacting Dr. Sten Odenwald at
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Inside Credits: ***3) Black hole magnetic field XMM/Newton); 4) tidal disruption (XMM/Newton); 5) Milky Way center (NASA/Chandra) Infrared (ESA/NAOS); 6) tidal disruption (www unattributed); 70 accretion disk (www unattributed); 8) Black hole disk model (Ken Rice/UCR) artist rendition (April Hobart NASA/Chandra); 10) accretion disk (NASA/Chandra); 11) x-ray emission (Ann Field STScI);***

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Alignment with Standards

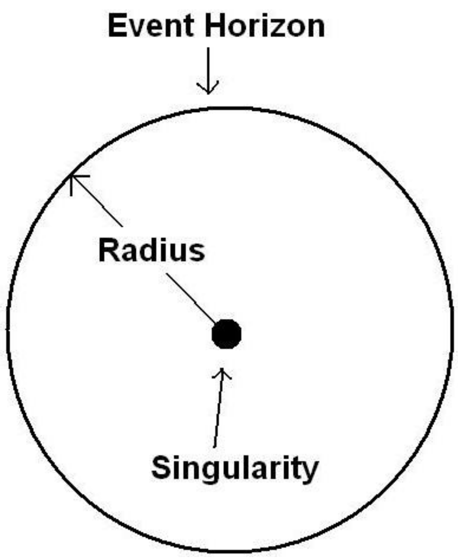
The problems have been developed to meet specific math and science benchmarks as stated in the NSF Project 2061. Project 2061's benchmarks are statements of what all students should know or be able to do in science, mathematics, and technology by the end of grades 2, 5, 8, and 12.
(http://www.project2061.org/publications/bsl/online/bolintro.htm)

The Physical Setting -
The Universe
Grade 12: Mathematical models and computer simulations are used in studying evidence from many sources in order to form a scientific account of the universe. 4A/H4

Forces of Nature:
Grade 12: Gravitational force is an attraction between masses. The strength of the force is proportional to the masses and weakens rapidly with increasing distance between them. 4G/H1

The Mathematical World -
Symbolic Relationships:
Grade 12: Any mathematical model, graphic or algebraic, is limited in how well it can represent how the world works. The usefulness of a mathematical model for predicting may be limited by uncertainties in measurements, by neglect of some important influences, or by requiring too much computation. 9B/H3

The Event Horizon



$$\text{Radius} = \frac{2\,G\,M}{c^2}$$

where $c = 3 \times 10^{10}$ cm/sec

$G = 6.67 \times 10^{-8}$ dynes cm^2 / gm^2

you get

$$\text{Radius} = 1.48 \times 10^{-28}\ \mathbf{M}\ \text{centimeters}$$

Black holes are objects that have such intense gravitational fields, they do not allow light to escape from them. They also make it impossible for anything that falls into them to escape, because to do so, they would have to travel at speeds faster than light. No forms of matter or energy can travel faster than the speed of light, so that is why black holes are so unusual!

There are three parts to a simple black hole:

Event Horizon - That's the part that we see from the outside. It looks like a black, spherical surface with a very sharp edge in space.

Interior Space - This is a complicated region where space and time can get horribly mangled, compressed, stretched, and otherwise a very bad place to travel through.

Singularity - That's the place that matter goes when it falls through the event horizon. It's located at the center of the black hole, and it has an enormous density. You will be crushed into quarks long before you get there!

Black holes can, in theory, come in any imaginable size. The size of a black hole depends on the amount of mass it contains. It's a very simple formula, especially if the black hole is not rotating. These 'non-rotating' black holes are called Schwarzschild Black Holes.

Problem 1 - The formula in red gives the Schwarzschild radius of a black hole, in centimeters, in terms of its mass, in grams. From the equation for the radius in terms of the speed of light, c, and the constant of gravity, G, verify the formula shown in red.

Problem 2 - Calculate the Schwarzschild radius, in centimeters, for Earth where

$M = 5.7 \times 10^{27}$ grams.

Problem 3 - Calculate the Schwarzschild radius, in kilometers, for the sun, where

$M = 1.98 \times 10^{33}$ grams.

Problem 4 - Calculate the Schwarzschild radius, in kilometers, for the entire Milky Way, with a mass of 250 billion suns.

Problem 5 - Calculate the Schwarzschild radius, in centimeters, for a black hole with a mass of an average human being with M = 60 kilograms.

Answer Key:

Problem 1 - The formula in red gives the Schwarzschild radius of a black hole, in centimeters, in terms of its mass, in grams. From the equation for the radius in terms of the speed of light, c, and the constant of gravity, G, verify the formula shown in red.

Answer: Radius = $2 \times (6.67 \times 10^{-8}) / (3 \times 10^{10})^2$ M

$= 1.48 \times 10^{-28}$ M centimeters

where M is the mass of the black hole in grams.

Problem 2 - Calculate the Schwarzschild radius, in centimeters, for Earth where M = 5.7×10^{27} grams.

Answer: R = 1.48×10^{-28} (5.7×10^{27}) centimeters

R = 0.84 centimeters !

Problem 3 - Calculate the Schwarzschild radius, in kilometers, for the sun, where M = 1.98×10^{33} grams.

Answer: R = 1.48×10^{-28} (1.98×10^{33}) centimeters

R = 293,000 centimeters
= 2.93 kilometers

Problem 4 - Calculate the Schwarzschild radius, in kilometers, for the entire Milky Way, with a mass of 250 billion suns.

Answer: If a black hole with the mass of the sun has a radius of 2.93 kilometers, a black hole with 250 billion times the sun's mass will be 250 billion times larger, or

R = (2.93 km / sun) x 250 billion suns = 732 billion kilometers.

Notre, the entire solar system has a radius of about 4.5 billion kilometers!

Problem 5 - Calculate the Schwarzschild radius, in centimeters, for a black hole with a mass of an average human being with M = 60 kilograms.

Answer: R = 1.48×10^{-28} (60,000) centimeters

R = 8.9×10^{-24} centimeters.

Note: A proton is only about 10^{-14} centimeters in diameter.

Time Dilation Near the Earth

$$T = t\sqrt{1 - \frac{2GM}{Rc^2}}$$

T = the time measured by someone located on a planet (seconds)

t = the time measured by someone located in space (seconds)

M = the mass of the planet (grams)

R = the distance to the far-away observer from the planet (cm)

And the natural constants are:
$G = 6.67 \times 10^{-8}$
$C = 3 \times 10^{10}$

The modern theory of gravity developed by Albert Einstein in 1915 leads to some very unusual predictions, which have all been verified by experiments.

One of the strangest ones is that two people will experience the passage of time very differently if one is standing on the surface of a planet, and the other one is in space. This is because the rate of time passing depends on the strength of the gravitational field that the observer is in.

For example, at the surface of a very dense neutron star, R = 20 km and $M = 1.9 \times 10^{33}$ grams, so

$$T = t\ (1- 0.15)^{1/2} = 0.92\ t$$

This means that for every hour that goes by on the surface of the neutron star (T = 3600 seconds), someone in space will see t = 3600 / 0.92 = 3913 seconds pass from a vantage point in space.

Problem 1 - The GPS satellites orbit Earth at a distance of R = 26,560 kilometers. If the mass of Earth is 5.9×10^{27} grams, use the formula to determine the time dilation factor. Be very careful with the small numbers in the 9th, 10th and 11th decimal places!

Problem 2 - What is the time dilation factor at Earth's surface?

Problem 3 - What is the ratio of the dilation in space to the dilation at earth's surface?

Problem 4 - At the speed of light (3×10^{10} cm/sec) how long does it take a radio signal from the GPS satellite to travel 26,560 kilometers to a hand-held GPS receiver?

Problem 5 - The excess time delay between a receiver at Earth's surface, and the GPS satellite is defined by the ratio computed in Problem 3, multiplied by the total travel time in Problem 4. What is the time delay for the GPS-Earth system?

Problem 6 - From your answer to Problem 5, how much extra time does the radio signal take compared to your answer to Problem 4?

Problem 7 - At the speed of light, how far will the radio signal travel during the extra amount of time?

Problem 8 - Is gravitational time delay an important phenomenon to include when using the GPS satellite system?

Answer Key:

Problem 1 - The GPS satellites orbit Earth at a distance of R = 26,560 kilometers. If the mass of Earth is 5.9 x 10^{27} grams, use the formula to determine the time dilation factor. Be very careful with the small numbers in the 9th, 10th and 11th decimal places!

Answer: (1 - 0.84/2.65x10^{9} cm)1/2 =
(1 - 3.1 x 10^{-10}) 1/2 =
(0.99999999969)1/2 = 0.99999999984

Problem 2 - What is the time dilation factor at Earth's surface?
(1 - 0.84/6.38x10^{8} cm)1/2 =
(1 - 1.3 x 10^{-9}) 1/2 =
(0.9999999987)1/2 = 0.99999999934

Problem 3 - What is the ratio of the dilation in space to the dilation at Earth's surface?
Answer - 0.99999999984 / 0.99999999934 = 1.0000000064

Problem 4 - How long does it take a radio signal from the GPS satellite to travel 26,560 kilometers to a hand-held GPS receiver?
Answer - Distance = 26,560 kilometers x (100,000 cm / kilometer) = 2.65 x 10^{9} centimeters.
Time = Distance / speed of light
= 2.65 x 10^{9} **cm** / 3 x 10^{10} cm/sec = 0.088 seconds.

Problem 5 - The excess time delay between a receiver at Earth's surface, and the GPS satellite is defined by the ratio computed in Problem 3, multiplied by the total travel time in Problem 4. What is the time delay for the GPS-Earth system?

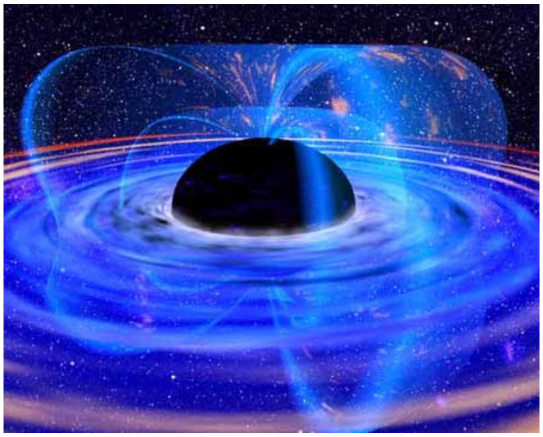
Answer - 0.088 seconds * 1.0000000064 = 0.08800000056 seconds.

Problem 6 - From your answer to Problem 5, how much extra time does the radio signal take compared to your answer to Problem 4?
Answer - 0.08800000056 - 0.088 seconds = 0.00000000056 seconds.

Problem 7 - At the speed of light, how far will the radio signal travel during the extra amount of time?
Answer = 3 x 10^{10} cm/sec x 5.6 x 10^{-10} sec = 16.8 centimeters.

Problem 8 - Is gravitational time delay an important phenomenon to include when using the GPS satellite system?
Answer: Yes!

Time Dilation Near a Black Hole



Artists illustration of a black hole with an orbiting disk of gas and dust. Friction in the disk causes matter to steadily flow inwards until it reaches the black hole event horizon. Magnetic forces in the disk cause matter to flow in complex jets and plumes. Time dilation causes delays in events taking place near the black hole compared to what distant observers will record.

Time dilation near a black hole is a lot more extreme than what the GPS satellite network experiences in orbit around Earth (See Problem 29).

$$T = t\sqrt{1 - \frac{2GM}{Rc^2}}$$

T = the time measured by someone located on a planet (seconds)

t = the time measured by someone located in space (seconds)

M = the mass of the planet (grams)

R = the distance to the far-away observer from the planet (cm)

And the natural constants are:
$G = 6.67 \times 10^{-8}$
$C = 3 \times 10^{10}$

Problem 1 - In the time dilation formula above, evaluate the quantity $2\,G\,M\,/c^2$ for a black hole with a mass of one solar mass (1.9×10^{33} grams), and convert the answer to kilometers.

Problem 2 - Re-write the formula in a more tidy form using your answer to Problem 1.

Problem 3 - In the far future, a scientific outpost has been placed in orbit around this solar-mass black hole at a distance of 10 kilometers. What will the time dilation factor be at this location?

Problem 4 - A series of clock ticks were sent out by the satellite once each hour. What will be the time interval between the clock ticks by the time they reach a distant observer?

Problem 5 - If one tick arrived at 1:00 PM at the distant observer, when will the next clock tick arrive?

Problem 6 - A radio signal was sent by the black hole outpost to a distant observer. At the frequency of the signal, when transmitted from the outpost, the individual wavelengths take 0.000001 seconds to complete one cycle. From your answer to Problem 3, how much longer will they take by the time they arrive at the distant observer?

Answer Key:

Problem 1 - In the time dilation formula above, evaluate the quantity 2 G M /c^2 for a black hole with a mass of one solar mass (1.9 x 10^{33} grams), and convert the answer to kilometers.
Answer - 2 x 6.67 x 10^{-8} x 1.9 x 10^{33} / (3 x $10^{10})^2$ = 281,600 centimeters or 2.82 kilometers.

Problem 2 - Re-write the formula in a more tidy form using your answer to Problem 1.

Answer -

$$T = t\ (1 - 2.82/R)^{1/2}$$

where R is in units of kilometers.

Problem 3 - In the far future, a scientific outpost has been placed in orbit around this solar-mass black hole at a distance of 10 kilometers. What will the time dilation factor be at this location?

Answer - $(1 - 2.82/10)^{1/2}$ = $(0.718)^{1/2}$ = 0.847

Problem 4 - A series of clock ticks were sent out by the satellite once each hour .What will be the time interval between the clock ticks by the time they reach a distant observer?

Answer - Time interval = 3600 / 0.847 = 4,250 seconds.

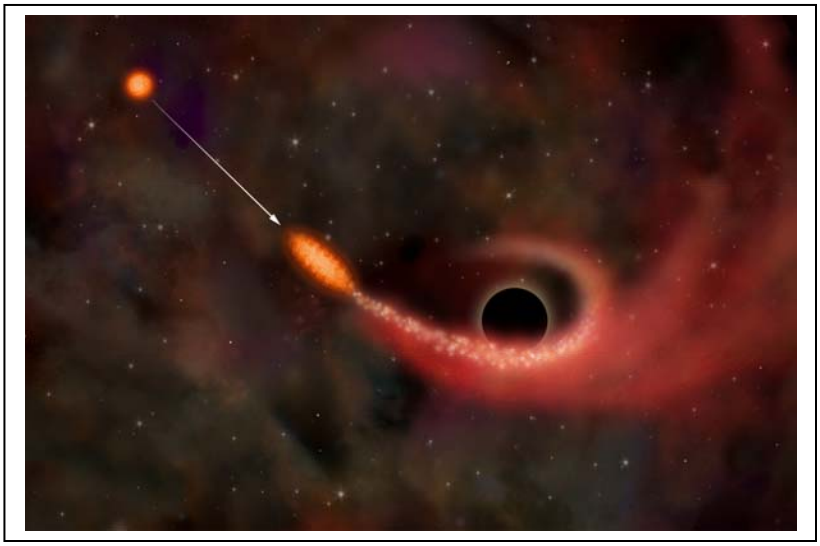
Problem 5 - If one tick arrived at 1:00 PM at the distant observer, when will the next clock tick arrive?

Answer - 1:00 PM + 4250 seconds = 1:00 PM + 1 Hour + (4250-3600) = 2:00 PM + 650 seconds = 2:10:50 PM

Problem 6 - A radio signal was sent by the black hole outpost to a distant observer. At the frequency of the signal, when transmitted from the outpost, the individual wavelengths take 0.000001 seconds to complete one cycle. From your answer to Problem 3, how much longer will they take by the time they arrive at the distant observer?

Answer - 0.000001 seconds / 0.847 = 0.00000118 seconds.

Extracting Energy from a Black Hole



Thanks to two orbiting X-ray observatories, astronomers have the first strong evidence of a supermassive black hole ripping apart a star and consuming a portion of it. The event, captured by NASA's Chandra and ESA's XMM-Newton X-ray Observatories, had long been predicted by theory, but never confirmed ... until now. Giant black holes in just the right mass range would pull on the front of a closely passing star much more strongly than on the back. Such a strong tidal force would stretch out a star and likely cause some of the star's gasses to fall into the black hole. The infalling gas has been predicted to emit just the same blast of X-rays that have recently been seen in the center of galaxy RX J1242-11 located 700 million light years from the Milky Way, in the constellation Virgo. (News Report at: http://science.msfc.nasa.gov/headlines/y2004/18feb_mayhem.htm, see also NASA report at http://chandra.harvard.edu/photo/2004/rxj1242/

Problem 1 - The Schwarschild Radius of a black hole is given by the formula R = 2.83 M, where R is the radius in kilometers, and M is that mass of the black hole in units of the sun's mass. A supermassive black hole can have a mass of 100 million times the sun. What is its Schwarschild radius in: A) kilometers B) Multiples of the Earth orbit radius called an Astronomical Unit (1 AU = 147 million kilometers). C) Compared to the orbit of Mars (1.5 U)

Problem 2 - Black holes are one of the most efficient phenomena in converting matter into energy. As matter falls inward in an orbiting disk of gas, friction heats the gas up, and the energy released can be as much as 7% of the rest mass energy of the infalling matter. The quasar 3C273 has a power output of 3.8×10^{45} ergs/second. If E = mc2 is the formula that converts mass (in grams) into energy (in ergs) and c = the speed of light, 3×10^{10} cm/sec, how many grams/second does this quasar luminosity imply?

Problem 3 - If the mass of the sun is 1.9×10^{33} grams, how many suns per year have to be consumed by the 3C273 supermassive black hole: A) at 100% conversion efficiency? B) At the black hole conversion efficiency of 7%? Note: 7% efficiency means that for every 100 grams involved, 7 grams are converted into pure energy (by $E=mc^2$)

Answer Key:

Problem 1 - The Schwarschild Radius of a black hole is given by the formula R = 2.83 M, where R is the radius in kilometers, and M is that mass of the black hole in units of the sun's mass. A supermassive black hole can have a mass of 100 million times the sun. What is its Schwarschild radius in: A) kilometers B) Multiples of the Earth orbit radius called an Astronomical Unit (1 AU = 147 million kilometers). C) Compared to the orbit of Mars (1.5 U)

Answer: A) R = 283 million kilometers B) 283 million / 147 million = 1.9 AU. C) 1.9/1.5 = 1.3 Mars Orbit. The Event Horizon would be just beyond the orbit of Mars!

Problem 2 - Black holes are one of the most efficient phenomena in converting matter into energy. As matter falls inward in an orbiting disk of gas, friction heats the gas up, and the energy released can be as much as 7% of the rest mass energy of the infalling matter. The quasar 3C273 has a power output of 3.8×10^{45} ergs/second. If $E = mc^2$ is the formula that converts mass (in grams) into energy (in ergs) and c = the speed of light, 3×10^{10} cm/sec, how many grams per year does this quasar luminosity imply if 1 year = 3.1×10^{7} seconds?

Answer: 3.8×10^{45} ergs/second x (3.1×10^{7} seconds/year) / $(3 \times 10^{10})^2$

= 1.3×10^{32} grams/year

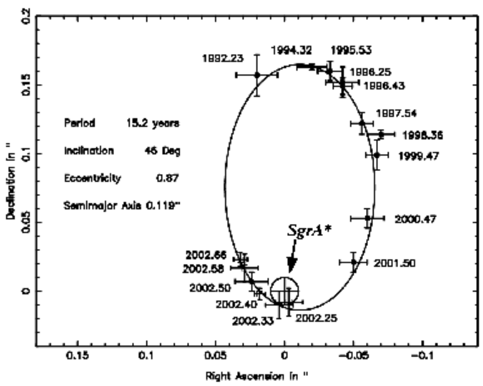
Problem 3 - If the mass of the sun is 1.9×10^{33} grams, how many suns per year have to be consumed by the 3C273 supermassive black hole: A) at 100% conversion efficiency? B) At the black hole conversion efficiency of 7%?

Answer: A) (1.3×10^{32} grams/year) / (1.9×10^{33} grams/sun) = 0.07 suns per year

B) 7% efficiency means that for every 100 grams involved, 7 grams are converted into pure energy (by $E=mc^2$). So,

0.07 suns per year / (7/100) = 1.0 suns per year for 7% efficiency.

The Milky Way Black Hole



At the center of our Milky Way Galaxy lies a black hole, called Sagittarius A*, with over 2.6 million times the mass of the Sun. Once a controversial claim, this astounding conclusion is now virtually inescapable and based on observations of stars orbiting very near the galactic center.

Astronomers patiently followed the orbit of a particular star, designated S2. Their results convincingly show that S2 is moving under the influence of the enormous gravity of an unseen object which must be extremely compact, and contain huge amounts of matter -- a supermassive black hole. The drawing shows the orbit shape.

The Chandra image above shows the x-ray light from a region of space a few light years across. The black hole is invisible, but is near the center of this image. The gas near the center produces x-ray light as it is heated. Many of the 'stars' in the field probably have much smaller black holes near them that are producing the x-ray light from the gas they are consuming.

Chandra image: http://chandra.harvard.edu/photo/2003/0203long/

Problem 1 - Kepler's Third Law can be used to determine the mass of a body by measuring the orbital period, T, and orbit radius, R, of a satellite. If R is given in units of the Astronomical Unit (AU) and T is in years, the relationship becomes $R^3 / T^2 = M$, where M is the mass of the body in multiples of the sun's mass. In these units, for Earth, R = 1.0 AU, and T = 1 year, so M = 1.0 solar masses. In 2006, the Hubble Space Telescope, found that the star Polaris has a companion, Polaris Ab, whose distance from Polaris is 18.5 AU and has a period of 30 years. What is the mass of Polaris?

Problem 2 - The star S2 orbits the supermassive black hole Sagittarius A*. Its period is 15.2 years, and its orbit distance is about 840 AU. What is the estimated mass of the black hole at the center of the Milky Way?

Answer Key:

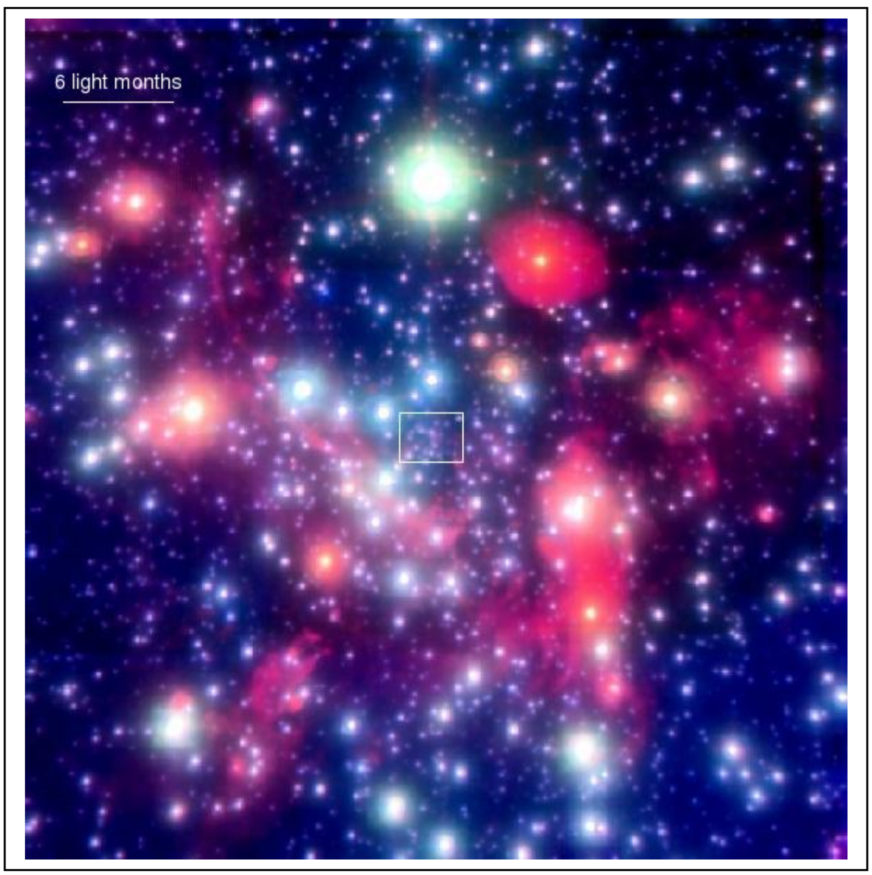
Problem 1 - Kepler's Third Law can be used to determine the mass of a body by measuring the orbital period, T, and orbit radius, R, of a satellite. If R is given in units of the Astronomical Unit (AU) and T is in years, the relationship becomes $R^3 / T^2 = M$, where M is the mass of the body in multiples of the sun's mass. In these units, for Earth, R = 1.0 AU, and T = 1 year, so M = 1.0 solar masses. In 2006, the Hubble Space Telescope, found that the star Polaris has a companion, Polaris Ab, whose distance from Polaris is 18.5 AU and has a period of 30 years. What is the mass of Polaris?

Answer: $M = (18.5)^3 / (30)^2 = 7.0$ solar masses.

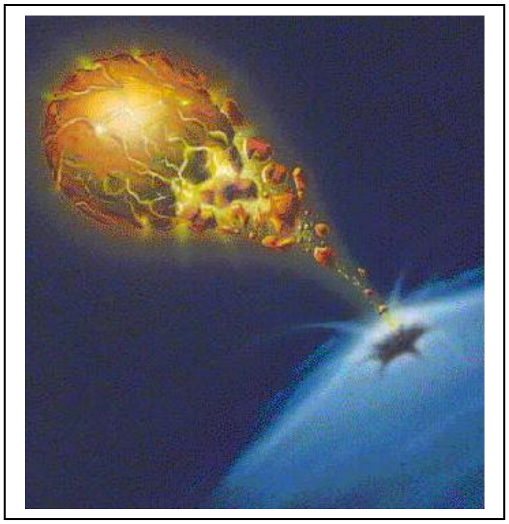
Problem 2 - The star S2 orbits the supermassive black hole Sagittarius A*. Its period is 15.2 years, and its orbit distance is about 840 AU. What is the estimated mass of the black hole at the center of the Milky Way?

Answer: $M = (840)^3 / (15.2)^2 = 2.6 \times 10^6$ solar masses.

The infrared image below shows the central few light years of the Milky Way. The box contains the location of the supermassive black hole and Sagittarius A*. (Courtesy ESA - NAOS)



Black Holes and Tidal Forces



$$a = \frac{2GM}{R^2}\frac{d}{R}$$

A tidal force is a difference in the strength of gravity between two points. The gravitational field of the moon produces a tidal force across the diameter of Earth, which causes the Earth to deform. It also raises tides of several meters in the solid Earth, and larger tides in the liquid oceans.

If the tidal force is stronger than a body's cohesiveness, the body will be disrupted. The minimum distance that a satellite comes to a planet before it is shattered this way is called its Roche Distance. The artistic image to the left shows what tidal disruption could be like for an unlucky moon.

A human falling into a black hole will also experience tidal forces. In most cases these will be lethal! The difference in acceleration between the head and feet could be many thousands of Earth Gravities. A person would literally be pulled apart! Some physicists have termed this process spaghettification!

Problem 1 - The equation lets us calculate the tidal acceleration across a body with a length of d. The acceleration of gravity on Earth's surface is 979 cm/sec^2. The tidal acceleration between your head and feet is given by the above formula. For M = the mass of Earth (5.9×10^{27} grams), R = the radius of Earth (6.4×10^{8} cm) and the constant of gravity whose value is G = 6.67×10^{-8}, calculate a if d = 2 meters.

Problem 2 - What is the tidal acceleration across the full diameter of Earth?

Problem 3 - A stellar-mass black hole has the mass of the sun (1.9×10^{33} grams), and a radius of 2.8 kilometers. A) What would be the tidal acceleration across a human at a distance of 100 kilometers? B) Would a human be spaghettified?

Problem 4 - A supermassive black hole has 100 million times the mass of the sun (1.9×10^{33} grams), and a radius of 280 million kilometers. What would be the tidal acceleration across a human at a distance of 100 kilometers from the event horizon of the supermassive black hole?

Problem 5 - Which black hole could a human enter without being spagettified?

Answer Key:

Problem 1 - The equation lets us calculate the tidal acceleration across a body with a length of d. The acceleration of gravity on Earth's surface is 979 cm/sec^2. The tidal acceleration between your head and feet is given by the above formula. For M = the mass of Earth (5.9×10^{27} grams), R = the radius of Earth (6.4×10^{8} cm) and the constant of gravity whose value is G = 6.67×10^{-8}, calculate a if d = 2 meters.

Answer: $a = 2 \times (6.67 \times 10^{-8}) \times (5.9 \times 10^{27}) \times 200 / (6.4 \times 10^{8})^3$

$= 0.000003 \times (200)$

$= 0.0006$ cm/sec^2

Problem 2 - What is the tidal acceleration across the full diameter of Earth?

Answer: $d = 1.28 \times 10^{9}$ cm, so $a = 0.000003 \times 1.28 \times 10^{9}$ = 3,840 cm/sec^2

Problem 3 - A stellar-mass black hole has the mass of the sun (1.9×10^{33} grams), and a radius of 2.8 kilometers. A) What would be the tidal acceleration across a human at a distance of 100 kilometers? B) Would a human be spaghettified?

Answer: $a = 2 \times (6.67 \times 10^{-8}) \times (1.9 \times 10^{33}) \times 200 / (1.0 \times 10^{7})^3$

= 50,700,000 cm/sec^2

Yes, this is equal to 50,700,000/979 = 51,700 times the acceleration of gravity, and a human would be pulled apart and 'spaghettified'

Problem 4 - A supermassive black hole has 100 million times the mass of the sun (1.9×10^{33} grams), and an event horizon radius of 280 million kilometers. What would be the tidal acceleration across a d=2 meter human at a distance of 100 kilometers from the event horizon of the supermassive black hole?

Answer: $a = 2 \times (6.67 \times 10^{-8}) \times (1.9 \times 10^{41}) \times 200 / (2.8 \times 10^{13})^3$

= 0.00023 cm/sec^2

Note that R + 2 meters is essentially R if R = 280 million kilometers.

Problem 5 - Which black hole could a human enter without being spaghettified?

Answer: The supermassive black hole, because the tidal force is far less than what a human normally experiences on the surface of Earth. That raises the question that as a space traveler, you could find yourself trapped by a supermassive black hole unless you knew exactly what its size was before hand .You would have no physical sensation of having crossed over the 'one way' surface before it was too late.

Falling Into a Black Hole

An object that falls into a black hole will cross the Event Horizon, and speed up as it gets closer. This is like a ball traveling faster and faster as it is dropped from a tall building. Suppose the particle fell 'from infinity', how fast would it be traveling? We can answer this question by considering the concepts of kinetic energy (K.E.) and gravitational potential energy (P.E.):

$$K.E. = 1/2\ m\ V^2$$

$$P.E. = G\ M\ m\ /\ R$$

The kinetic energy that the particle with mass, m, will gain as it falls, will depend on the total potential energy it has lost in traveling from infinity to a distance R. By setting the two equations equal to each other, we can relate the kinetic energy a particle gains as it falls to its current distance of R from the center of mass. The quantity, M, is the mass of the gravitating body the particle is falling towards. G is the constant of gravitation which equals 6.67×10^{-8}.

$$1/2\ m\ V^2 = G\ M\ m\ /\ R$$

We can then solve for the speed, V, in terms of R

$$V = (2GM/R)^{1/2}$$

Problem 1 - Suppose a body falls to Earth and strikes the ground. How fast will it be traveling when it hits if $M = 5.9 \times 10^{24}$ kilograms and R = 6,378 kilometers? Explain why this is the same as Earth's escape velocity?

Problem 2 - NASA's ROSSI satellite was used in 2004 to determine the mass and radius of a neutron star in the binary star system named EXO 0748-676, located about 30,000 light-years away in the southern sky constellation Volans, or Flying Fish. The neutron star was deduced to have a mass of 1.8 times the sun, and a radius of 11.5 kilometers. A) How fast, in km/sec, will a particle strike the surface of the neutron star if the mass of the sun is 1.9×10^{33} grams? B) In percent, what will the speed be compared to the speed of light, 300,000 km/sec?

Problem 3 - The star HD226868 is a binary star with an unseen companion. It is also the most powerful source of X-rays in the sky second to the sun - it's called Cygnus X-1. Astronomers have determined the mass of this companion to be 8.7 times the sun. As a black hole, its Event Horizon radius would be R = 2.93 x 8.7 = 25.5 kilometers. A) How fast, in km/sec, would a body be traveling as it passed through the Event Horizon? B) In percentage compared to the speed of light?

Answer Key:

Problem 1 - Suppose a body falls to Earth and strikes the ground .How fast will it be traveling when it hits? Explain why this is the same as Earth's escape velocity?

Answer - R = 6,378 kilometers. M = 5.9 x 10^{24} kilograms.
V = (2 x 6.67 x 10^{-8} x 5.9 x 10^{29}/6.4 x $10^{8})^{1/2}$ = 1.1 x 10^{6} cm/sec or 11 kilometers/second. The particle 'fell' from infinity, so this means that if you gave a body a speed of 11 km/sec at Earth's surface, it would be able to travel to infinity and 'escape' from Earth.

Problem 2: NASA's ROSSi satellite was used in 2004 to determine the mass and radius of a neutron star in the binary star system named EXO 0748-676, located about 30,000 light-years away in the southern sky constellation Volans, or Flying Fish. The neutron star was deduced to have a mass of 1.8 times the sun, and a radius of 11.5 kilometers. A) How fast, in km/sec, will a particle strike the surface of the neutron star if the mass of the sun is 1.9 x 10^{33} grams? B) In percent, what will the speed be compared to the speed of light, 300,000 km/sec?

Answer: A) Mass = 1.8 x 1.9 x 10^{33} grams = 3.4 x 10^{33} grams. V = (2 x 6.67x10^{-8} x 3.4 x 10^{33}/1.15 x $10^{6})^{1/2}$ = 1.98 x 10^{10} cm/sec = 198,000 km/sec.

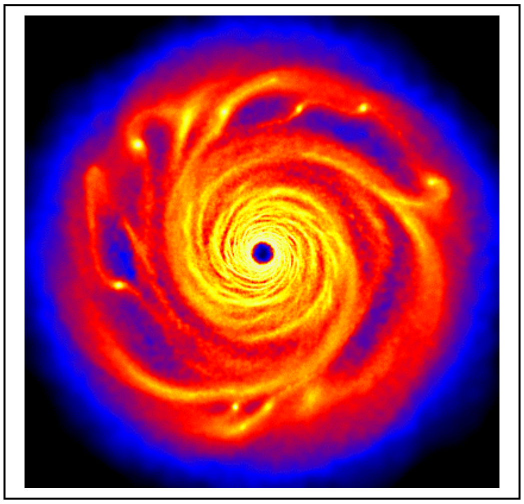
B) 198,000/300,000 = 66% of the speed of light!

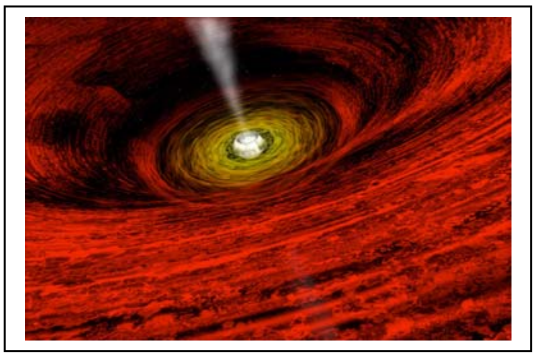
Problem 3 - The star HD226868 is a binary star with an unseen companion. It is also the most powerful source of X-rays in the sky second to the sun - it's called Cygnus X-1. Astronomers have determined the mass of this companion to be 8.7 times the sun. As a black hole, its Event Horizon radius would be R = 2.93 km x 8.7 = 25.5 kilometers. A) How fast, in km/sec, would a body be traveling as it passed through the Event Horizon? B) In percentage compared to the speed of light?

Answer: A) Mass = 8.7 x 1.9 x 10^{33} grams = 1.7 x 10^{34} grams. V = (2 x 6.67x10^{-8} x 1.7 x 10^{34}/2.55 x $10^{6})^{1/2}$ = 2.98 x 10^{10} cm/sec = 298,000 km/sec.

B) 298,000/300,000 = 99% of the speed of light!

Black Holes...Hot Stuff!





Top -Computer model of an accretion disk model.
(Courtesy: Ken Rice - UCR)

Bottom - Artist's painting of an accretion disk
(Courtesy: April Hobart, NASA/Chandra/CXC)

The farther a particle falls towards a black hole, the faster it travels, and the more kinetic energy it has. Kinetic energy is mathematically defined as K.E. = 1/2 m V^2 where m is the mass of a the particle and v is its speed.

Suppose all this energy was converted into heat energy by friction as the particle falls, and that this added energy causes nearby gases to heat up. How hot will they get? The equivalent amount of thermal energy carried by a single particle is

$$T.E = 3/2\ kT$$

where k = Boltzman's Constant 1.38 x 10^{-16} erg/deg. If we set K.E = T.E we get

$$T = 3\ mV^2 / k$$

If all the particles in a gas carried this same kinetic energy, then we would say the gas has a temperature of T degrees Kelvin. We also know that the potential energy of the particle is given by

$$P.E = 2\ G\ M\ m / R$$

So if we set P.E = T.E we also get

$$T = 4/3\ G\ M\ m / k\ R$$

Problem 1 - The formula T = 4/3 G M m/kR gives the approximate temperature of hydrogen gas (m = 1.6 x 10^{-24} grams) in an accretion disk around a black hole. What is the temperature for a solar-mass black hole disk near the orbit of Earth? (R = 1.47 x 10^{13} cm, M = 1.9 x 10^{33} grams, for G = 6.67 x 10^{-8})?

Problem 2 - How hot would the disk be at the distance of Neptune (R = 4.4 x 10^{14} cm)?

Problem 3 - X-rays are the most common forms of energy produced at temperatures above 100,000 K. Visible light is produced at temperatures above 2,000 K. What would you expect to see if you studied the accretion disk around a black hole?

Answer Key:

Problem 1 - The formula T = 4/3 G M m/kR gives the approximate temperature of hydrogen gas (m = 1.6 x 10^{-24} grams) in an accretion disk around a black hole. What is the temperature for a solar-mass black hole disk near the orbit of Earth? (R = 1.47 x 10^{13} cm, M = 1.9 x 10^{33} grams, for G = 6.67 x 10^{-8})?

Answer: 4/3 x 6.67 x 10^{-8} x 1.9 x 10^{33} x 1.6 x 10^{-24} / (1.38 x 10^{-16} x 1.47 x 10^{13})
= 133,000 K.

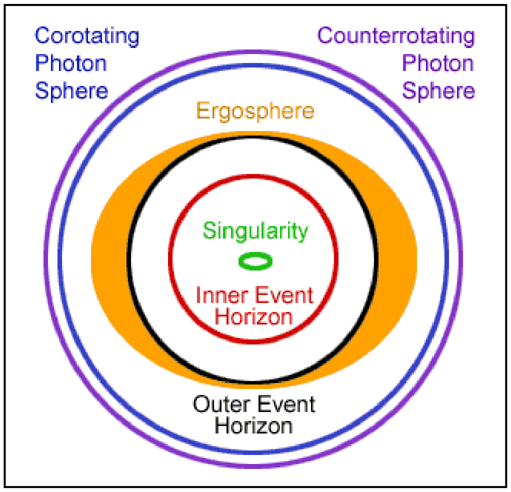
Problem 2 - How hot would the disk be at the distance of Neptune (R = 4.4 x 1014 cm)?

Answer: 4/3 x 6.67 x 10^{-8} x 1.9 x 10^{33} x 1.6 x 10^{-24} / (1.38 x 10^{-16} x 4.4 x 10^{14}) = 4,400 K.

Problem 3 - X-rays are the most common forms of energy produced at temperatures above 100,000 K. Visible light is produced at temperatures above 2,000 K. What would you expect to see if you studied the accretion disk around a black hole?

Answer: The inner disk region would be an intense source of x-rays and visible light, because the gas is mostly at temperatures above 100,000 K. the outer disk, the gas is much cooler and emits mostly visible light.

Black Holes.....What's inside?



Outside a black hole, we have the normal universe of space, time, matter and energy we have all come to know. But inside, things are very different. We know this because the same mathematics that predicts black holes should exist, also predicts what to find inside them. One of the biggest surprises is the way that time and space, themselves, behave.

Inside a black hole, the roles played by time and space reverse themselves. Before we are crushed to death by the Singularity, we have limited freedom to intentionally move through space, but some freedom in how we travel through time...in our last moments of life!

Problem 1 - In relativity, space and time are part of a single 4-dimensional thing called spacetime. There are 3 dimensions to space and 1-dimension to time. Every point, called an Event, has three coordinates to describe its location in space, and one extra coordinate to describe its location in time. We write these as an ordered set like A(x,y,z,t) or B(x,y,z,t). Write the ordered set for the following event called A: I travel north 5 miles, east 3 miles, and up 1 miles, at 9:00AM on February 16, 2008.

Problem 2 - Suppose I travel from one event with coordinates A(3 km, 6 km, 2 km, 5:00PM) to another event B(5 km, 7 km, 3 km, 8:00PM). How far did I travel in space in each direction during the time interval from 5:00 PM to 8:00 PM?

Problem 3 - Use the Pythagorean Theorem to calculate the actual distance in space in Problem 2.

Problem 4 - The 4-dimensional, 'hyper' distance between the events is found by using the 'hyperbolic' Pythagorean Theorem formula $D^2 = -c^2T^2 + X^2 + y^2 + z^2$ where c is the speed of light (c = 300,000 km/sec). Calculate the hyper-distance, D^2, between Events A and B in Problem 2.

Problem 5 - Based on your answer, which part of D^2 makes the largest contribution to the hyper-distance, the time-like part, T, or the space-like part (x,y,z)?

Problem 6 - Inside a black hole, the formula for D^2 changes to $D^2 = c^2T^2 - x^2 - y^2 - z^2$ Suppose Events A and B are now happening inside the black hole. What is the hyper-distance between them?

Extra for Experts:

Problem 7 - If an observer defines 'time' as the part of D^2 that has a negative sign, and 'space' as the part that has the positive sign, can you explain what happens as the traveler passes inside the black hole?

Answer Key:

Problem 1 - In relativity, space and time are part of a single 4-dimensional thing called spacetime. There are 3 dimensions to space and 1-dimension to time. Every point, called an Event, has three coordinates to describe its location in space, and one extra coordinate to describe its location in time. We write these as an ordered set like A(x,y,z,t) or B(x,y,z,t). Write the ordered set for the following event called A: I travel north 5 miles, east 3 miles, and up 1 miles, at 9:00AM on February 16, 2008.
Answer - A(5,3,1, 9:00AM 2/16/2008)

Problem 2 - Suppose I travel from one event with coordinates A(3 km, 6 km, 2 km, 5:00PM) to another event B(5 km, 7 km, 3 km, 8:00PM). How far did I travel in space during the time interval from 5:00 PM to 8:00 PM?
Answer: Take the difference in the x, y and z coordinates to get x = 5-3 = 2 km; y=6-3 = 3 km and z=3-2 = 1 km.

Problem 3 - Use the Pythagorean Theorem to calculate the actual distance in space in Problem 2.
Answer: This only involves the x, y and z coordinate differences we found in Problem 2: Distance = $(2^2 + 3^2 + 1^2)^{1/2} = (14)^{1/2}$ or 3.7 kilometers.

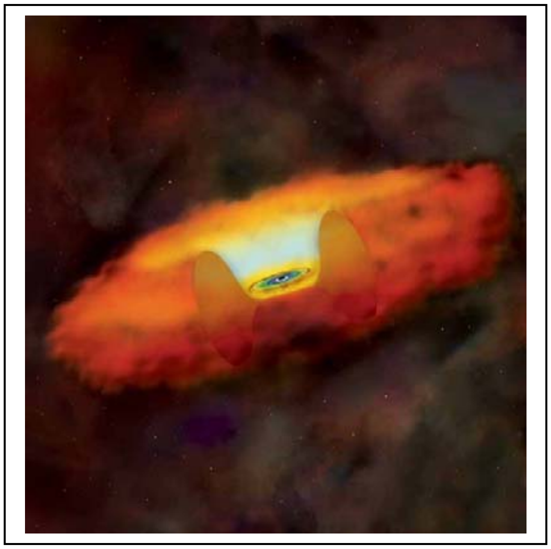
Problem 4 - The 4-dimensional, 'hyper' distance between the events is found by using the 'hyperbolic' Pythagorean Theorem formula $D^2 = -c^2T^2 + X^2 + y^2 + z^2$ where c is the speed of light (c = 300,000 km/sec). Calculate the hyper-distance, D^2, between Events A and B in Problem 2.
Answer: First, the time difference is 8:00PM - 5:00 PM = 3 hours. This equals 10,800 seconds. Then from the formula, where all units are in kilometers, we get $D^2 = -(300{,}000\ \text{km/sec})^2 (10{,}800\ \text{sec})^2 +2^2 + 3^2 + 1^2 =$ $- 1.05 \times 10^{19}$ kilometers.

Problem 5 - Based on your answer, which part of D^2 makes the largest contribution to the hyper-distance, the time-like part, T, or the space-like part (x,y,z)?
Answer - D^2 is negative, so it is the time-like part that makes the biggest difference.

Problem 6 - Inside a black hole, the formula for D2 changes to $D^2 = c^2T^2 - X^2 - y^2 - z^2$
Suppose Events A and B are now happening inside the black hole. What is the hyper-distance between them?
Answer - $D^2 = (300{,}000\ \text{km/sec})^2 (10{,}800\ \text{sec})^2 - 2^2 - 3^2 - 1^2 =$ $+1.05 \times 10^{19}$ kilometers.

Problem 7 - If an observer defines 'time' as the part of D^2 that has a negative sign, and 'space' as the part that has the positive sign, can you explain what happens as the traveler passes inside the black hole? Answer - Outside the black hole, the T variable we defined to be time is part of the negative-signed component to D^2, and x,y,z are the space variables and are the positive part of D^2. When we enter the black hole, the T variable becomes part of the positive space-like part of D^2, and x, y and z are part of the negative part of D^2. This means that the roles of space and time have been reversed inside a black hole!

Black Hole Power



Black holes are sometimes surrounded by a disk of orbiting matter. This disk is very hot. As matter finally falls into the black hole from the inner edge of that disk, it releases about 7% of its rest-mass energy in the form of light. Some of this energy was already lost as the matter passed through, and heated up, the gases in the surrounding disk. But the over-all energy from the infalling matter is about 7% of its rest-mass in all forms (heat+ light).

The power produced by a black hole is phenomenal, with far more energy per gram being created than by ordinary nuclear fusion, which powers the sun.

Illustration of black hole accretion disk cut-away, showing the central black hole. Courtesy NASA/Chandra/ M.Weiss (CXC)

Problem 1 - The Event Horizon of a black hole has a radius of 2.93 M kilometers, where M is the mass of the black hole in multiples of the sun's mass. Assume the Event Horizon is a spherical surface, so its surface area is $S = 4 \pi R^2$. What is the surface area of A) a stellar black hole with a mass of 10 solar masses? B) a supermassive black hole with a mass of 100 million suns?

Problem 2 - What is the volume of a spherical shell with the surface area of the black holes in Problem 1, with a thickness of one centimeter?

Problem 3 - If the density of gas near the horizon is 10^{10} atoms/cc of hydrogen, how much matter is in each black hole shell, if the mass of a hydrogen atom is 1.6×10^{-24} grams?

Problem 4 - If $E = m c^2$ is the rest mass energy, E, in ergs, for a particle with a mass of m in grams, what is the rest mass energy equal to the masses in Problem 3 if $c = 3 \times 10^{10}$ cm/sec is the speed of light and only 7% of the mass produced energy?

Problem 5 - Suppose the material was traveling at 1/2 the speed of light as it crossed the horizon, how much time does it take to travel one centimeter if $c = 3 \times 10^{10}$ cm/sec is the speed of light?

Problem 6 - The power produced is equal to the energy in Problem 4, divided by the time in Problem 5. What is the percentage of power produced by each black hole compared to the sun's power of 3.8×10^{33} ergs/sec?

Answer Key:

Problem 1 - The Event Horizon of a black hole has a radius of 2.93 M kilometers, where M is the mass of the black hole in multiples of the sun's mass. Assume the Event Horizon is a spherical surface, so its surface area is $S = 4 \pi R^2$. What is the surface area of A0 a stellar black hole with a mass of 10 solar masses? B) a supermassive black hole with a mass of 100 million suns?

Answer; A) The radius is 2.93 x 10 = 29.3 kilometers. The surface area S = 4 x 3.14 x $(2.93 \times 10^6)^2 = 1.1 \times 10^{14}$ cm^2, B) 2.93×10^{13} cm S = 4 x 3.14 x $(2.93 \times 10^{13})^2 = 1.1 \times 10^{28}$ cm^2,

Problem 2 - What is the volume of a spherical shell with the surface area of the black holes in Problem 1, with a thickness of one centimeter?
Answer: Stellar black hole, V = S x 1 cm = 1.1×10^{14} cm^3; Supermassive black hole, V = 1.1×10^{28} cm^3.

Problem 3 - If the density of gas near the horizon is 10^{10} atoms/cc of hydrogen, how much matter is in each black hole shell, if the mass of a hydrogen atom is 1.6 x 10-24 grams?

Answer - Stellar: M = $(1.1 \times 10^{14}$ cm^3) x $(1.0 \times 10^{10}$ atoms/cm^3) x $(1.6 \times 10^{-24}$ grams/atom = 1.76 grams. Supermassive: M = $(1.1 \times 10^{28}$ cm^3) x $(1.0 \times 10^{10}$ atoms/cm^3) x $(1.6 \times 10^{-24}$ grams/atom = 1.76×10^{14} grams.

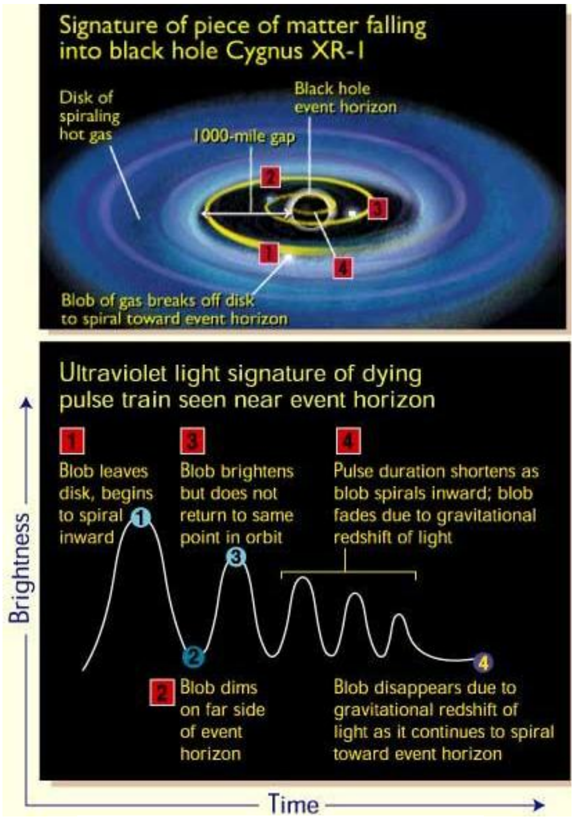
Problem 4 - If $E = m c^2$ is the rest mass energy, E, in ergs, for a particle with a mass of m in grams, what is the rest mass energy equal to the masses in Problem 3 if c = 3×10^{10} cm/sec is the speed of light and only 7% of the mass produced energy?
Answer: Stellar: E = 0.07 x (1.76 grams x $(3 \times 10^{10})^2$) = 1.1×10^{20} ergs.;
Supermassive E = 0.07 x $(1.76 \times 10^{14}$ grams x $(3 \times 10^{10})^2$) = 1.1×10^{34} ergs

Problem 5 - Suppose the material was traveling at 1/2 the speed of light as it crossed the horizon, how much time does it take to travel one centimeter if c = 3×10^{10} cm/sec is the speed of light?
Answer; 1 cm / 3×10^{10} cm/sec = 3.3×10^{-11} seconds.

Problem 6 - The power produced is equal to the energy in Problem 4, divided by the time in Problem 5. What is the percentage of power produced by each black hole compared to the sun's power of 3.8×10^{33} ergs/sec?
Answer Stellar: 1.1×10^{20} ergs / 3.3×10^{-11} seconds = 3.3×10^{30} ergs/sec
Percent = 100% x $(3.3 \times 10^{30} / 3.8 \times 10^{33})$ = 0.08 %

Supermassive: 1.1×10^{34} ergs / 3.3×10^{-11} seconds = 3.3×10^{44} ergs/sec
= $(3.3 \times 10^{44} / 3.8 \times 10^{33})$ = 86 billion times the sun's power!

Black Hole...Fade-out!



$$L = L_0 e^{\frac{-2T}{3\sqrt{3}\ 2M}}$$

As seen from a distance, not only does the passage of time slow down for someone falling into a black hole, but the image fades to black!

This happens because, during the time that the object reaches the event horizon and passes beyond, a finite number of light particles (photons) will be emitted. Once these have been detected to make an image, there are no more left because the object is on the other side of the event horizon and the photons cannot escape. A star, collapsing to a black hole, will be going very fast as it collapses, then appear to slow down as time 'dilates'. Meanwhile, the image will become redder and redder, until it literally fades to black!

Photographs taken by the Hubble Space Telescope of the black-hole candidate called Cygnus XR-1 detected two instances where a hot gas blob appeared to be slipping past the event horizon for the black hole. Because of the gravitational stretching of light, the fragment disappeared from Hubble's view before it ever actually reached the event horizon. The pulsation of the blob, an effect caused by the black hole's intense gravity, also shortened as it fell closer to the event horizon. Without an event horizon, the blob of gas would have brightened as it crashed onto the surface of the accreting body. See The Astrophysical Journal, 502:L149–L152, 1998 August 1

Diagram courtesy Ann Field (STScI)

Problem 1 - The simple formula predicts the exponential decay of the light from matter falling in to a black hole. T is the time in seconds measured by distant observer, and M is the mass of the black hole in units of solar masses. How long does it take for the light to fall to half its initial luminosity (i.e. power in units of watts) given by L_0 for a M =1.0 solar mass, stellar black hole?

Problem 2 - How long will your answer be, in years, for a supermassive black hole with M = 100 million times the mass of the sun?

Problem 3 - A supermassive black hole 'swallows' a star. If the initial luminosity, L_0, of the star is 2.5 times the sun's, how long will it take before the brightness of the star fades to 0.0025 Suns, and can no longer be detected from Earth?

Answer Key:

Problem 1 - The simple formula predicts the exponential decay of the light from matter falling in to a black hole. T is the time in seconds measured by distant observer, and M is the mass of the black hole in units of solar masses. How long does it take for the light to fall to half its initial luminosity (i.e. power in units of watts) given by L_0 for a M =1.0 solar mass, stellar black hole??

Answer - Set L = 1/2 L_0, and M = 1.0, then solve for T. The formula is 0.5 = e (-0.19 T)
Tale the natural logarithm of both sides to get -0.69 = -0.19 T so T = 3.6 seconds.

Problem 2 - How long will your answer be, in years, for a supermassive black hole with M = 100 million times the mass of the sun?

Answer, The formula will be 0.5 = e (- 1.9 x 10^{-9} T) so -0.69 = -1.9 x 10^{-9} T, and T = 3.6 x 10^{8} seconds. If there are 3.1 x 10^{7} seconds in 1 year, T = 11.5 years.

Problem 3 - A supermassive black hole 'swallows' a star. If the initial luminosity, L_0, of the star is 2.5 times the sun's, how many years will it take before the brightness of the star fades to 0.0025 Suns, and can no longer be detected from Earth?

Answer: 0.0025 L_{sun} = 1.0 L_{sun} e (-1.9x10-9 T)
Ln(0.0025) = -1.9x10^{-9} T
-6.0 = -1.9x10^{-9} T
T = 6.0/1.9x10^{-9}
T = 3.2 x 10^{9} seconds
T = 3.2 x 10^{9} seconds x (1.0 year /3.1 x 10^{7} seconds)
T = 103 years.

A note from the Author,

Since they first came into public view in the early 1970s, black holes have been a constant source of curiosity and mystery for millions of adults and children. No astronomer has had the experience of visiting a classroom, and NOT being asked questions about these weird objects with which we share our universe.

Beyond answering that they are objects with such intense gravity that even light cannot escape them, we tend to be at a loss for what to say next. The mathematics of Einstein's General Theory of Relativity are extremely complex even for advanced undergraduates in mathematics, so we tend to resort to colorful phrases and actual hand-waving to describe them to eager students.

Actually, there are many important aspects of black holes that can be readily understood by using pre-algebra (scientific notation), Geometry (concepts of space and coordinates, Pythagorean Theorem), Algebra I (working with simple formulae), and Algebra II (working with asymptotic behavior).

This book is a compilation of some of my favorite problems in black hole physics. They will introduce the student to the important concept of the event horizon, time dilation, and how energy is extracted from a black hole to create many kinds of astronomical phenomena. Some of these problems may even inspire a student to tackle a Science Fair or Math Fair problem!

Black holes are indeed something of a mystery, but many of there most well-kept secrets can be understood with just a little mathematics. I hope the problems in this book will inspire students to learn more about them!

Sten Odenwald

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