Exponentiation in power series fields

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Let K be a totally ordered field, and v be the natural valuation on K (i.e. the valuation induced by the archimedean equivalence relation). Note that the residue field $\overline{K}$ is always archimedean, i.e. a subfield of the reals. We say K is **exponential** if there is an order preserving isomorphism from the (ordered) additive group of the field onto its (ordered) multiplicative group of positive elements. We study criteria for a nonarchimedean ordered field K to be exponential. More precisely, if $\overline{K}$ is an exponentially closed subfield of the reals, we study necessary and sufficient conditions on the value group $v(K)$ so that one can extend the exponential on $\overline{K}$ to an exponential on K.

We then apply our results to the particularly interesting class of generalized power series fields. These fields are important in Geometry, Calculus, and Algebra. They were also studied by Levi-Civita in an attempt to develop function theory over nonarchimedean fields. This study was pursued by H. Hahn, L. Neder, D. Laugwitz, A. Robinson and many others. One of the first concerns there is to be able to define elementary functions (e.g. the exponential function exp) on those fields. It was already known to Levi-Civita that $\exp(x)$ (as well as other analytic functions) is definable for x infinitesimal, through the Taylor expansion of the function. But the problem of defining a total and surjective exponential remained open.

Using our above analysis of the value groups of exponential fields, we show that it is impossible to have a total exponential on a power series field, thus answering the above open problem to the negative. On the other hand, we shall observe that every power series field is "almost exponential", that is, it admits a partial but surjective exponential, or equivalently a logarithm defined for all positive elements, but not surjective. Since the surjectivity is missing, we take countable unions of power series fields to make the logarithm surjective. We thus obtain a nonarchimedean model K_ω of real exponentiation (i.e. exp on K_ω satisfies the same elementary theorems as the real exponential function), which is not itself a power series field, but is a countable union of power series fields K_n, $n \in \mathbb{N}$. Similar constructions were recently used by L. van den Dries, D. Marker and A. Macintyre to solve, for instance, an open problem due to Hardy, concerning the asymptotic behavior of certain functions.