25. QUANTUM CHROMODYNAMICS

25.1. The QCD Lagrangian

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Quantum Chromodynamics (QCD), the gauge field theory which describes the strong interactions of colored quarks and gluons, is one of the components of the SU(3)×SU(2)×U(1) Standard Model. A quark of specific flavor (such as a charm quark) comes in 3 colors; gluons come in eight colors; hadrons are color-singlet combinations of quarks, anti-quarks, and gluons. The Lagrangian describing the interactions of quarks and gluons is (up to gauge-fixing terms)

$$L_{\rm QCD} = -\frac{1}{4}F^{(a)}_{\mu\nu}\,F^{(a)\mu\nu} + i\sum_q \overline{\psi}^i_q\,\gamma^\mu\,(D_\mu)_{ij}\,\psi^j_q - \sum_q m_q\,\overline{\psi}^i_q\,\psi_{qi}\ , \tag{25.1}$$

$$F^{(a)}_{\mu\nu} = \partial_\mu\,A^a_\nu - \partial_\nu\,A^a_\mu + g_s\,f_{abc}\,A^b_\mu\,A^c_\nu\ , \tag{25.2}$$

$$(D_\mu)_{ij} = \delta_{ij}\,\partial_\mu - ig_s\,\sum_a \frac{\lambda^a_{i,j}}{2}A^a_\mu\ , \tag{25.3}$$

where g_s is the QCD coupling constant, and the f_{abc} are the structure constants of the SU(3) algebra (the λ matrices and values for f_{abc} can be found in Sec. 21 of this *Review*). The $\psi^i_q(x)$ are the 4-component Dirac spinors associated with each quark field of (3) color i and flavor q and the $A^a_\mu(x)$ are the (8) Yang-Mills (gluon) fields. A complete list of the Feynman rules which derive from this Lagrangian, together with some useful color-algebra identities, can be found in Ref. 1.

The principle of "asymptotic freedom" (see below) determines that the renormalized QCD coupling is small only at high energies, and it is only in this domain that high-precision tests—similar to those in QED—can be performed using perturbation theory. Nonetheless, there has been in recent years much progress in understanding and quantifying the predictions of QCD in the nonperturbative domain, for example in soft hadronic processes and on the lattice [2]. This short review will concentrate on QCD at short distances (large momentum transfers), where perturbation theory is the standard tool. It will discuss the processes that are used to determine the coupling constant of QCD. Other recent reviews of the coupling constant measurements may be consulted for a different perspective [3].

25.2. The QCD coupling and renormalization scheme

The renormalization scale dependence of the effective QCD coupling $\alpha_s = g_s^2/4\pi$ is controlled by the β-function:

$$\mu\frac{\partial\alpha_s}{\partial\mu} = -\frac{\beta_0}{2\pi}\,\alpha_s^2 - \frac{\beta_1}{4\pi^2}\,\alpha_s^3 - \frac{\beta_2}{64\pi^3}\,\alpha_s^4 - \cdots\ , \tag{25.4a}$$

$$\beta_0 = 11 - \frac{2}{3}n_f\ , \tag{25.4b}$$

$$\beta_1 = 51 - \frac{19}{3}n_f\ , \tag{25.4c}$$

$$\beta_2 = 2857 - \frac{5033}{9}n_f + \frac{325}{27}n_f^2\ , \tag{25.4d}$$

here n_f is the number of quarks with mass less than the energy scale μ. In solving this differential equation for α_s, a constant of integration is introduced. This constant is the one fundamental constant of QCD that must be determined from experiment. The most sensible choice for this constant is the value of α_s at a fixed-reference scale μ_0, but it is more conventional to introduce the dimensional parameter Λ since this provides a parametrization of the μ dependence of α_s. The definition of Λ is arbitrary. One way to define it (adopted here) is to write a solution of Eq. (25.4) as an expansion in inverse powers of $\ln(\mu^2)$:

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0\,\ln(\mu^2/\Lambda^2)}\left[1 - \frac{2\beta_1}{\beta_0^2}\,\frac{\ln\left[\ln(\mu^2/\Lambda^2)\right]}{\ln(\mu^2/\Lambda^2)} + \frac{4\beta_1^2}{\beta_0^4\ln^2(\mu^2/\Lambda^2)} \times\left(\left(\ln\left[\ln(\mu^2/\Lambda^2)\right] - \frac{1}{2}\right)^2 + \frac{\beta_2\beta_0}{8\beta_1^2} - \frac{5}{4}\right)\right] \tag{25.5a}$$

The last term in this expansion is

$$\mathcal{O}\left(\frac{\ln^2\left[\ln(\mu^2/\Lambda^2)\right]}{\ln^3(\mu^2/\Lambda^2)}\right) \tag{25.5b}$$

and is usually neglected in the definition of Λ. We choose to include it even though its effect on $\alpha_s(\mu)$ is smaller than the experimental errors. For a fixed value of $\alpha_s(M_Z)$, the inclusion of this term shifts the value of Λ by ~ 15 MeV.

This solution illustrates the *asymptotic freedom* property: $\alpha_s \to 0$ as $\mu \to \infty$. Alternative definitions of Λ are possible. We adopt this as the standard. Values given by experiments using other definitions are adjusted as needed to meet our definition.

In the above discussion we have ignored quark-mass effects, *i.e.*, we have assumed an idealized situation where quarks of mass greater than μ are neglected completely. In this picture, the β-function coefficients in the expression for α_s change by discrete amounts as flavor thresholds are crossed when integrating the differential equation for α_s. It follows that, for a relationship such as Eq. (25.5) to remain valid for all values of μ, crossed. Hence Λ must also change as flavor thresholds are crossed. This leads to the concept of a different Λ for each range of μ corresponding to an effective number of massless quarks: $\Lambda \to \Lambda^{(n_f)}$. There is some arbitrariness in how this relationship is set up. As an idealised case consider QCD with $n_f - 1$ massless quarks and one quark of mass M. Now imagine an experiment at energy scale μ; for example, this could be $e^+e^- \to$ *hadrons* at center of mass energy μ. If $\mu \gg M$, the mass M is negligible and the process well described by QCD with n_f massless flavors and its parameter $\Lambda^{(n_f)}$ up to terms of order M^2/μ^2. Conversely if $\mu \ll M$, the heavy quark plays no role and the process well described by QCD with $n_f - 1$ massless flavors and its parameter $\Lambda^{(n_f-1)}$ up to terms of order μ^2/M^2. If $\mu \sim M$ the effects of the quark mass are process dependent and cannot be absorbed into the running coupling.

A mass scale μ' is chosen where the relationship between $\Lambda^{(n_f-1)}$ and $\Lambda^{(n_f)}$ will be fixed. μ' should be of order M and the relationship should not depend on it. A prescription has been given [4] which has this property. We use this procedure choosing $\mu' = M_Q$, where M_Q is the mass of the value of the running quark mass defined in the $\overline{\rm MS}$ scheme (see the note on quark masses for more details) *i.e.* where $M_{\overline{\rm MS}}(M_Q) = M_Q$. Then [4]

$$\begin{aligned}
\beta_0^{n_f-1}\ln\left(\frac{\Lambda^{(n_f)}}{\Lambda^{(n_f-1)}}\right)^2 &= (\beta_0^{n_f} - \beta_0^{n_f-1})\ \ln\left(\frac{M_Q}{\Lambda^{(n_f)}}^2\right)\\
&+ 2\left(\frac{\beta_1^{n_f}}{\beta_0^{n_f}} - \frac{\beta_1^{n_f-1}}{\beta_0^{n_f-1}}\right)\ \ln\left[\ln\left(\frac{M_Q}{\Lambda^{(n_f)}}\right)^2\right]\\
&- \frac{2\beta_1^{n_f-1}}{\beta_0^{n_f-1}}\ \ln\left(\frac{\beta_0^{n_f}}{\beta_0^{n_f-1}}\right)\\
&+ \frac{4\frac{\beta_1^{n_f}}{(\beta_0^{n_f})^2}\left(\frac{\beta_1^{n_f}}{\beta_0^{n_f}} - \frac{\beta_1^{n_f-1}}{\beta_0^{n_f-1}}\right)\ \ln\left[\ln\left(\frac{M_Q}{\Lambda^{n_f}}\right)^2\right]}{\ln\left(\frac{M_Q}{\Lambda^{(n_f)}}\right)^2}\\
&+ \frac{\frac{1}{\beta_0^{n_f}}\left[\left(\frac{2\beta_1^{n_f}}{\beta_0^{n_f}}\right)^2 - \left(\frac{2\beta_1^{n_f-1}}{\beta_0^{n_f-1}}\right)^2 - \frac{2\beta_2^{n_f}}{\beta_0^{n_f}} + \frac{2\beta_2^{n_f-1}}{\beta_0^{n_f-1}} - \frac{14}{9}\right]}{\ln\left(\frac{M_Q}{\Lambda^{(n_f)}}\right)^2}
\end{aligned} \tag{25.6}$$

This result is valid to order α_s^3 (or alternatively to terms of order $1/\ln^2[(M_Q/\Lambda^{(n_f)})^2]$).

In the previous edition of this note an alternative matching was used [5]. This procedure required the equality $\alpha_s(\mu)^{(n_f)} =$