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A Derivation of the 'Bohr Hypothesis' of the Bohr Hydrogen Atom Analysis

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ABSTRACT

RÉSUMÉ (Une Dérivation de « l'Hypothèse de Bohr » de l'Analyse de l'Atome Hydrogène de Bohr)

Le succès extraordinaire de l'analyse de l'Atome Hydrogène de Bohr en 1913 a vraiment lancé le commencement de l'ère de la physique moderne de la structure de l'atome. Parmi les suppositions a priori de Bohr dans son analyse est son hypothèse célèbre, bien connue et unique, qui dit que les seules orbites permises de l'électron sont telles que le moment cinétique ou angulaire de l'atome est un multiple intégral de la constante de Planck. Ce dernier a ainsi permis la suite de la solution déterminée bien connue du problème de l'Atome Hydrogène. Le succès phénoménal de l'analyse de Bohr et de la solution de Bohr du Problème de l'Atome Hydrogène pose aussi la question : « l'hypothèse » doit-elle exister—pas comme une supposition a priori et qui laisse à désirer, ou bien comme une hypothèse requise pour la solution—mais comme une conséquence naturelle des lois de la physique même? Ce dernier se manifeste en fait comme le résultat logique dans cet article, lorsque le problème et le modèle simplifié de Bohr sont vus du point de vue de l'utilisation d'un opérateur pour le moment cinétique ou angulaire, ce qui est en accord avec les lois de la physique et de la relativité (i.e. la Relativité Spéciale).

The extraordinary success of the Bohr Hydrogen Atom analysis in 1913 truly launched the beginning of the era of modern physics of the structure of the atom. Among the a' priori assumptions by Bohr in his analysis was his famous, well-known, and unique hypothesis that the only allowed orbits of the electron are such that the angular momentum of the atom is an integral multiple of Planck's Constant. The latter then allowed the well-known determinate solution of the Hydrogen Atom problem to follow. The phenomenal success of the Bohr analysis and Bohr solution of the Hydrogen Atom Problem also begs the question that the 'hypothesis' ought to exist, not as an unsatisfying a' priori assumption or hypothesis required for solution, but as a natural consequence of the laws of physics itself. The latter is indeed shown as the logical result in this paper when the problem and his simplified model are looked at from the point of view of the use of an operator for angular momentum, which is consistent with both the laws of physics and relativity (i.e., Special Relativity).

KEYWORDS: BOHR HYDROGEN ATOM, BOHR HYPOTHESIS, HYDROGEN ATOM & ELECTRON ANGULAR MOMENTUM, OPERATOR FOR ANGULAR MOMENTUM, ATOMIC STRUCTURE, SPECIAL RELATIVITY.

Section I: INTRODUCTION

Modern physics of atomic structure can be said to have begun with the truly amazing successful early quantum theories of Planck (1900) and Einstein (1905) followed by the 1913 hydrogen atom theory of Bohr (1, 2, 3). Bohr's analysis electrified the world of physics by introducing a simple picture of the hydrogen atom, a proton and its orbiting electron, which then had, as its breath-taking consequences, the highly exact prediction of the spectroscopic Rydberg Constant, as well as that of the value of the hydrogen atom's ionization energy (1, 2, 3, 4). This simple idea / model of the hydrogen atom was soon to be extended and then serve as a basic template for atoms in general of ALL the elements of the Periodic Table (tempered by the ideas of quantum mechanics).

Critical to the formulation of Bohr's hydrogen atom theory is the fundamental Bohr hypothesis that the angular momentum L of the orbiting electron can only occur in whole number "n" multiples of the Planck Constant h, or , precisely,

$$L = m v r = n h / 2 \pi \qquad (1)$$

Where m = mass of the electron
v = electron orbital velocity
r = radius of the orbit of the electron

This apparently arbitrary a' priori or ad hoc assumption of Bohr is required for the completion of the theory and for the total solution of the hydrogen atom problem. Without it, the problem is indeterminate. Yet in the intervening years since the theory was published in 1913, the Bohr Hypothesis remains as a "status quo" hypothesis ------ ie, accepted and taught as early history of modern physics and as a required 'well-established' hypothesis in order to have a solution to the hydrogen atom problem, but with no proof or strong rationale behind it.

In hindsight, there is really no mystery here to explain this 'status quo' state of the Bohr Hypothesis: with the advent of quantum mechanics in 1923 and the Schroedinger Wave Equation in 1926, Bohr's hydrogen atom theory was quickly - and quietly - relegated into the 'dustbin' of history of early quantum theory and physics …. and essentially put aside from mainstream physics. But --- kept 'alive and well' of course as an universally-taught subject (ie, as a highly simplified and approximate historical theory which nevertheless has the merits of also being easily understood) in ALL undergraduate college physics and chemistry courses to this day!!!

As an important footnote to both early quantum theory and post-Bohr work, the success of Sommerfeld's work in 1916 in this area must be noted with his introduction of elliptical orbits for the electron (and an additional quantum number) which then remarkably explained the fine structure spectrum of hydrogen (and hydrogen-like atoms, such as, singly ionized helium and doubly-ionized lithium). Unfortunately Sommerfeld's theory, like the original Bohr theory, could not be extended successfully to multi-electron atoms. Thus work in this area (ie, model approaches of Bohr and Sommerfeld) essentially ceased after the advent of quantum mechanics (3, 4).

Later post-quantum mechanics work in this area was directed towards the pedagogical theme, i.e., similar but new and interesting slants or formulations of improved teaching clarity and ease (i.e., “teach-ability”) were developed and presented in copious undergraduate textbooks on physics over the decades.

The true focus of physics research of course was understandably directed towards the new more realistic quantum theories which had “over-thrown” the Bohr hydrogen atom (and Sommerfeld) theory. And the Bohr and Sommerfeld theory, by default, were then consigned to the chapter of physics history called “early quantum theory” .

As is well known, Bohr too then moved on into quantum mechanics and worked diligently to reconcile the then two world views of physics, namely, the indeterministic world view of quantum mechanics versus the deterministic world view of classical physics, respectively. Bohr’s famous complementary and correspondence principle arguments were created just for the purpose of reconciling the two latter views (2, 3).

This paper will take a fresh look at the Bohr Hypothesis and will show, when the problem is analyzed from the point of view of an ‘operator’ for angular momentum, consistent with the laws of physics and relativity (ie, Special Relativity), that the hydrogen atom problem yields, as a natural consequence of the analysis, the ‘Bohr Hypothesis’ condition, not as a hypothesis, but as a result of the latter laws of physics itself. It is emphatically pointed out that this analysis however does not change the fact that the Bohr atom model remains a very simplistic representation of real atoms, compared of course with the more realistic model resulting from the modern theory of quantum mechanics. For example, the catastrophic atomic problem of an energy-decaying and radiating electron still remains associated with the basic Bohr model. However, it remains as a fact too, in modern physics to this present day, that the elegance of the Bohr model is still very pedagogically popular and with useful and easy imagery in understanding the atom and its application to understanding ALL of the elements of the Periodic Table, with the caveat that the model is understood to be but a simplistic representation of real atoms.

Section II: ANALYSIS OF THE ANGULAR MOMENTUM OF THE HYDROGEN ATOM

Given the Bohr Hypothesis [which specifies the angular momentum of the electron (or more precisely, of the atom system)], along with other assumptions of Bohr, the solution of the Bohr hydrogen atom proceeds per standard textbook treatment (4, 5, 6) (ie, treatments done either on the basis of the mass of the electron or the reduced mass of the atom system, respectively) and is not reproduced here. The other additional assumptions of Bohr are as follows:

1) The electron moves in a circular orbit about the center of the atom (or on the basis of its center of mass, as already cited above).

2) Radiation occurs only when an electron "jumps" from one of the allowed orbits to . one of lower energy. The difference in energy ΔE is then radiated as a . photon of frequency $f = \Delta E / h$, according to the Einstein condition. This . latter assumption leads to the well-known standard spectroscopic textbook . solution of the hydrogen atom problem (4, 5, 6).

In physics it is common to develop and use 'operators' to 'transform' from one state to another. In quantum mechanics, for example, the Hamiltonian is the common operator (4). Another common operator, consistent from a classical and /or relativistic point of view, is the one to transform work energy into force F, ie, the operator on work or kinetic energy W (or potential energy U) is [d / ds], where ds is the differential distance of travel of the net force on the system:

$$d(W) / ds \quad = \; - \; d(U) / ds \quad = \quad F \qquad (2)$$

Also to transform W or E into velocity v, the appropriate operator, consistent from a classical and/or relativistic point of view, is [d / d(p)], where p = momentum = m v . Thus, per the latter operator, applied relativistically, the famous Einstein relationship on energy - mass also immediately follows (4):

$$E \; = \; \Delta m \, c^2 \quad = \quad \int_{o}^{v} v \, d \, p \qquad (3)$$

where

$$m \; = \; mo \; / \; \sqrt{(1 \; - \; (v/c)^2)} \qquad \text{relativistic mass per Special Relativity}$$

. $\Delta m \; = \; m \; - \; mo$

mo = rest mass

For Total Energy of the system, (in this paper, treated specifically only for the simple system of the Bohr hydrogen atom model above), it will be shown that the desired operator (also consistent from the classical and/or relativistic point of view) is [d / d ω], where ω is the angular velocity. For example, for the specific case of the classical pure rotating system of rotational energy $E_{rot} \; = \; \frac{1}{2} \; I \; \omega^2$,

where I = the moment of inertia of the system, the use of the latter operator [d / dω] transforms Erot into angular momentum, accordingly, :

$$L = [\, d(E_{rot}) / d\omega \,] = d(\tfrac{1}{2} I \omega^2) / d\omega = I \omega \qquad (4)$$

The above operator for the simplified model above is also valid in the relativistic framework to transform Total Energy E into angular momentum L as well. Per Special Relativity, E tot = E is related to p as follows (2):

$$E^2 = c^2 p^2 + mo^2 c^4 \qquad (5)$$

where mo = rest mass

c = speed of light

Differentiating E with respect to p yields:

$$dE = c^2 p \, dp / E \qquad (6)$$

Introducing the independent parameter ω and its differential dω into equation (5) along with $E = mc^2$ (per standard Special Relativity result of equation (3) above) yields:

$$\frac{dE}{d\omega} = \frac{c^2 p}{E} \frac{dp}{d\omega} = v \frac{dp}{d\omega} \qquad (7)$$

$$\text{But } r p = m r^2 v / r = I_B \omega \; ; \text{ and } r \, dp = I_B \, d\omega \qquad (8)$$

where $I_B = mr^2$, moment of inertia per the above simplified Bohr model.

or

$$\frac{dp}{d\omega} = I_B / r \qquad (9)$$

Thus, equation (9) inserted into equation (7) yields the operator result on total energy E for angular momentum L for the above simplified Bohr model system and is shown to be :

$$L = \frac{dE}{d\omega} = \omega r \, I_B / r = I_B \omega \qquad (10)$$

Per the Bohr model and its assumptions, let now the orbiting electron with its (orbital) angular frequency ω connect also internal energy-wise with the Planck condition for the atom's energy and an associated frequency f ; and also, with use of Bohr's assumption (2) above, note that the respective energies of discrete different orbital states will be then related to each other as n integral multiples, i.e.,

$$E = n h f = n h \omega / 2\pi \qquad (11)$$

where $2 \pi f = \omega$ electron orbital angular frequency

It may be noted here that what quantum theory and relativity have in common is that either equation (11) or equation (5), respectively, can be used to describe the Total Energy E of the atom system.

Use the operator per equation (10) now to transform this Total Energy E of equation (11) into its associated angular momentum L , as follows, noting that the result is seen also as an implicit consequence (or a hidden result) of the Planck equation (2,5):

$$\frac{dE}{d\omega} = L = n h / 2\pi \qquad (12)$$

Equation (13) now results and is also the Bohr Hypothesis, since $I_B = m r^2$ and

$$L = I_B\omega = m r^2 (v/r) = m v r \qquad \text{(per the Bohr model)} \qquad (13)$$

Or, thusly, equation (1), i.e., the Bohr Hypothesis, results : (Q.E.D.)

$$L = m v r = n h / 2\pi \qquad (1)$$

That the above Bohr Hypothesis / condition for angular momentum is related to the Planck Equation has also as its precedent the 1912 work of Nicholson (7). Nicholson's work predates the 1913 Bohr paper (1) but essentially lays the important precedent for the groundwork of Bohr by his attempt to derive an angular momentum relationship on the quite overly general basis of the application of 'dimensional analysis' to the Planck Equation. Pais concludes (2, 3) that one of the consequences of this work is that its final result also 'quantizes' angular momentum of the electron. On this singular issue alone it is noted that Bohr himself quotes Nicholson in his own seminal paper (1) on hydrogen and also in his letter to Rutherford of 21 January 1912 (8), respectively. Pais (2) therefore concludes that “it is quite probable that Nicholson’s work influenced (Bohr) at that time”, in agreement with McCormmach’s analysis (9) on this matter of the specific influences on Bohr’s work on the hydrogen atom. Finally, Moore (10) observes that (sic) “this condition (ie, the Bohr Hypothesis) is simply another form of Planck’s hypothesis that h is the quantum of action”.

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