# SECOND-ORDER <br> PLANETARY THEORY <br> PART I CASE FILE <br> S. E. HAMID <br> COPY 



Smithsonian Astrophysical Observatory SPECIAL REPORT 302

## Research in Space Science

SAO Special Report No. 302

## SECOND-ORDER PLANETARY THEORY

## Part I: Outline of the Method

S. E. Hamid

## TABLE OF CONTENTS

Section Page
ABSTRACT ..... v
1 INTRODUCTION ..... 1
2 THE EQUATIONS OF MOTION ..... 3
3 THE DEVELOPMENT OF $G_{2 x}, G_{2 y}, G_{2 z}$ ..... 7
4 THE DECOMPOSITION OF $\delta x_{2}, \delta y_{2}, \delta z_{2}$ ..... 23
5 NUMERICAL APPLICATION ..... 31
6 NUMERICAL RESULTS ..... 33
7 ACKNOWLEDGMENTS ..... 81
REFERENCES ..... 83

## LIST OF TABLES

Table Page
1 Fourier representation of $\delta x_{2 m j k}$ (periodic part) ..... 36
2 Fourier representation of $\delta y_{2 m j k}$ (periodic part) ..... 52
3 Fourier representation of $\delta z_{2 m j k}$ (periodic part). ..... 68
4 Fourier representation of $\delta x_{2}$ (mixed part) ..... 72
5 Fourier representation of $\delta y_{2}$ (mixed part) ..... 73
6 Fourier representation of $\delta z_{2}$ (mixed part) ..... 74
7 Comparison of numerical integration with the analytical ..... 75
solution


#### Abstract

The analytical procedure for computing second-order perturbations in rectangular coordinates, according to Brouwer's theory of planetary motion, is given. Single- and double-harmonic analyses and the multiplication of Fourier series with numerical coefficients are used in the computations. In the series multiplication, a variable tolerance is considered, enabling us to avoid the difficulties arising from a small divisor.

Also presented is an example computing that part of the second-order perturbation of Mars containing the masses of Jupiter and Saturn. The analytical solution of this perturbation is compared with the numerical integration of the differential equations defining this perturbation. The numerical integration covered the interval from 0 to 40,000 days. The comparison shows an agreement within $1 \times 10^{-9}$.

\section*{RESUME}

La procédure analytique de calcul des perturbations de second ordre en coordonnées rectangulaires, d'après la théorie de Brouwer des mouvements planétaires, est exposée. Des analyses d'harmoniques simples et doubles et la multiplication de séries de Fourier avec des coëfficients numériques sont utilisées dans les calculs. Dans la multiplication de séries on tient compte d'une tolérance variable qui permet d'éviter les difficultés provenant d'un petit diviseur.

Un exemple de calcul de la perturbation de second ordre de Mars due aux masses de Jupiter et Saturne est également présenté. La solution analytique de cette perturbation est comparée avec l'intégration numérique des équations différentielles qui définissent cette perturbation. L'intégration numérique couvre l'intervalle de 0 à 40.000 jours. La comparaison montre une concordance de $1 \times 10^{-9}$.


## Резнме

Излагается аналитический способ вычисления возбуждений второго порндка в прямоугольньх ноординатах, согласно Брауэрской теории планетного двихения. Одно- и двух-гармонические анализы и умножение серии Фоурьера с числовыми ноэффициентами использованы в расчетах. При умножении серии была учитана переменная толеранция, позволяя избежать трудности, вознинающие благодаря малому разделителю.

Такке приведен пример вычисления Марсовых возбуждений второго порлдка, содержащих массы Юпитера и Сатурна. Аналитичесное решение этих возбуждений сравнено с числовой интеграцией дифференциальньх уравнений, определяющих зтих возбуждений. Числовая интеграция покрыла интервал от $0^{\text {до }} 40.000$ дней. Сравнение поназывает согласие в пределах $\mathrm{I} \times \mathrm{I}^{-5}$.

# SECOND-ORDER PLANETARY THEORY 

Part I: Outline of the Method

S. E. Hamid

## 1. INTRODUCTION

The author has successfully applied Brouwer's theory of general perturbation in rectangular coordinates to obtain a first-order planetary theory for all the principal planets except Pluto (Hamid, 1968). The advantage of Brouwer's theory over other planetary theories is its convenience when higher order perturbations are considered.

In this report, the adaptation of the theory in the computation of secondorder perturbations is discussed. General computer programs have been developed for the computation of the different second-order terms of planetary perturbations. These programs have been applied for the planet Mars to compute the second-order perturbations factored by the product of the masses of Jupiter and Saturn. The numerical results obtained have been tested successfully against the numerical integration of the differential equations satisfying these perturbations.

[^0]
## 2. THE EQUATIONS OF MOTION

Consider a set of rectangular axes, the $x$ axis corresponding to the direction from the sun to the perihelion of the orbit of the perturbed planet at a given epoch, and the $z$ axis perpendicular to that orbital plane at this epoch. Then, the perturbations $\delta x, \delta y, \delta z$ in the rectangular coordinates satisfy the following set of differential equations:

$$
\begin{align*}
& \frac{d^{2} \delta x}{d t^{2}}+\mu \frac{\delta x}{r_{0}^{3}}-\frac{3 \mu x_{0}}{r_{0}^{5}}\left(x_{0} \delta x+y_{0} \delta y\right)=G_{x} \\
& \frac{d^{2} \delta y}{d t^{2}}+\mu \frac{\delta y}{r_{0}^{3}}-\frac{3 \mu y_{0}}{r_{0}^{5}}\left(x_{0} \delta x+y_{0} \delta y\right)=G_{y} \\
& \frac{d^{2} \delta z}{d t^{2}}+\mu \frac{\delta z}{r_{0}^{3}}=G_{z} \tag{1}
\end{align*}
$$

The quantities ( $x_{0}, y_{0}, z_{0}$ ) are the coordinates of the planet, with its unperturbed orbit assumed at epoch, while $r_{0}$ denotes the heliocentric distance of the planet.

The functions $G_{x}, G_{y}, G_{z}$ can be separated into different parts of descending order of magnitude, the first part giving rise to first-order perturbations, the second part to second-order perturbations, and so on. We denote these parts by $G_{1 x}, G_{1 y}, G_{1 z} ; G_{2 x}, G_{2 y}, G_{2 z} ; \ldots$ and let $\delta x_{1}, \delta y_{1}$, $\delta z_{1} ; \delta x_{2}, \delta y_{2}, \delta z_{2} ; \ldots$ be the corresponding perturbations in the rectangular coordinates.

The first-order perturbations $\delta x_{1}, \delta y_{1}, \delta z_{1}$ will satisfy equations (1) when the values of $G_{x}, G_{y}, G_{z}$ are put equal to $G_{l x}, G_{l y}, G_{l z}$, and similarly for higher order perturbations.

We consider the second-order perturbations, which are written as

$$
\begin{align*}
& \frac{d^{2} \delta x_{2}}{d t^{2}}+\mu \frac{\delta x_{2}}{r_{0}^{3}}-\frac{3 \mu x_{0}}{r_{0}^{5}}\left(x_{0} \delta x_{2}+y_{0} \delta y_{2}\right)=G_{2 x}, \\
& \frac{d^{2} \delta y_{2}}{d t^{2}}+\mu \frac{\delta y_{2}}{r_{0}^{3}}-\frac{3 \mu y_{0}}{r_{0}^{5}}\left(x_{0} \delta x_{2}+y_{0} \delta y_{2}\right)=G_{2 y}, \\
& \frac{d^{2} \delta z_{2}}{d t^{2}}+\mu \frac{\delta z_{2}}{r_{0}^{3}}=G_{2 z} . \tag{2}
\end{align*}
$$

The solution of equations (2), given by Brouwer and Clemence (1961), takes the following form:

$$
\begin{aligned}
\delta x_{2}= & \frac{\partial x_{0}}{\partial L_{0}} \int\left(\frac{\partial x_{0}}{\partial \omega_{0}} G_{2 x}+\frac{\partial y_{0}}{\partial \omega_{0}} G_{2 y}\right) d t-\frac{\partial x_{0}}{\partial \omega_{0}} \int\left(\frac{\partial x_{0}}{\partial L_{0}} G_{2 x}+\frac{\partial y_{0}}{\partial L_{0}} G_{2 y}\right) d t \\
& +\frac{\partial x_{0}}{\partial \xi_{0}} \int\left(\frac{\partial x_{0}}{\partial \eta_{0}} G_{2 x}+\frac{\partial y_{0}}{\partial \eta_{0}} G_{2 y}\right) d t-\frac{\partial x_{0}}{\partial \eta_{0}} \int\left(\frac{\partial x_{0}}{\partial \xi_{0}} G_{2 x}+\frac{\partial y_{0}}{\partial \xi_{0}} G_{2 y}\right) d t \\
& -3 \mu^{2} L_{0}^{-4} \frac{\partial x_{0}}{\partial \omega_{0}} \iint\left(\frac{\partial x_{0}}{\partial \omega_{0}} G_{2 x}+\frac{\partial y_{0}}{\partial \omega_{0}} G_{2 y}\right) d t^{2}, \\
\delta y_{2}= & \frac{\partial y_{0}}{\partial L_{0}} \int\left(\frac{\partial x_{0}}{\partial \omega_{0}} G_{2 x}+\frac{\partial y_{0}}{\partial \omega_{0}} G_{2 y}\right) d t-\frac{\partial y_{0}}{\partial \omega_{0}} \int\left(\frac{\partial x_{0}}{\partial L_{0}} G_{2 x}+\frac{\partial y_{0}}{\partial L_{0}} G_{2 y}\right) d t \\
& +\frac{\partial y_{0}}{\partial \xi_{0}} \int\left(\frac{\partial x_{0}}{\partial \eta_{0}} G_{2 x}+\frac{\partial y_{0}}{\partial \eta_{0}} G_{2 y}\right) d t-\frac{\partial y_{0}}{\partial \eta_{0}} \int\left(\frac{\partial x_{0}}{\partial \xi_{0}} G_{2 x}+\frac{\partial y_{0}}{\partial \xi_{0}} G_{2 y}\right) d t \\
& -3 \mu^{2} L_{0}^{-4} \frac{\partial y_{0}}{\partial \omega_{0}} \iint\left(\frac{\partial x_{0}}{\partial \omega_{0}} G_{2 x}+\frac{\partial y_{0}}{\partial \omega_{0}} G_{2 y}\right) d t^{2},
\end{aligned}
$$

$$
\begin{equation*}
\delta z_{2}=q_{2} \int q_{1} G_{2 z} d t-q_{1} \int q_{2} G_{2 z} d t \tag{3}
\end{equation*}
$$

For the definition of the different partial derivatives of the coordinates $x_{0}$, $y_{0}$ and of the quantities $q_{1}, q_{2}$, in equations (3), see Brouwer and Clemence (1961).
3. THE DEVELOPMENT OF $G_{2 x}, G_{2 y}, G_{2 z}$

The expressions for $G_{2 x}, G_{2 y}, G_{2 z}$ have the following form:

$$
\begin{aligned}
G_{2 x}= & \frac{\partial^{2} R_{0}}{\partial x_{k}^{2}} \delta x_{k}+\frac{\partial^{2} R_{0}}{\partial x_{k} \partial y_{k}} \delta y_{k}+\frac{\partial^{2} R_{0}}{\partial x_{k} \partial z_{k}} \delta z_{k} \\
& +\sum_{j=1}^{n}\left(\frac{\partial^{2} R_{0}}{\partial x_{k} \partial x_{j}} \delta x_{j}+\frac{\partial^{2} R_{0}}{\partial x_{k} \partial y_{j}} \delta y_{j}+\frac{\partial^{2} R_{0}}{\partial x_{k} \partial z_{j}} \delta_{z_{j}}\right) \\
& +\mu\left[\left(\frac{9}{2} \frac{x_{k}}{r_{k}^{5}}-\frac{15}{2} \frac{x_{k}^{3}}{r_{k}^{7}}\right) \delta x_{k}^{2}+\left(3 \frac{y_{k}}{r_{k}^{5}}-15 \frac{x_{k}^{2} y_{k}}{r_{k}^{7}}\right) \delta x_{k} \delta y_{k}\right. \\
& \left.+\left(\frac{3}{2} \frac{x_{k}}{r_{k}^{5}}-\frac{15}{2} \frac{x_{k} y_{k}^{2}}{r_{k}^{7}}\right) \delta y_{k}^{2}+\frac{3}{2} \frac{x_{k}}{r_{k}^{5}} \delta z_{k}^{2}\right] \\
G_{2 y}= & \frac{\partial^{2} R_{0}}{\partial y_{k} \partial x_{k}} \delta x_{k}+\frac{\partial^{2} R_{0}}{\partial y_{k}^{2}} \delta y_{k}+\frac{\partial^{2} R_{0}}{\partial y_{k} \partial z_{k}} \delta z_{k} \\
& +\sum_{j=1}^{n}\left(\frac{\partial^{2} R_{0}}{\partial y_{k} \partial x_{j}} \delta x_{j}+\frac{\partial^{2} R_{0}}{\partial y_{k} \partial y_{j}} \delta y_{j}+\frac{\partial^{2} R_{0}}{\partial y_{k} \partial z_{j}} \delta z_{j}\right) \\
& +\mu\left(\frac{3}{2} \frac{y_{k}}{r_{k}^{5}}-\frac{15}{2} \frac{x_{k}^{2} y_{k}}{r_{k}^{7}}\right) \delta x_{k}^{2}+\left(3 \frac{x_{k}}{r_{k}^{5}}-15 \frac{x_{k} y_{k}^{2}}{r_{k}^{7}}\right) \delta x_{k} \delta y_{k}
\end{aligned}
$$

$$
\begin{align*}
& \left.+\left(\frac{9}{2} \frac{y_{k}}{r_{k}^{5}}-\frac{15}{2} \frac{y_{k}^{3}}{r_{k}^{7}}\right) \delta y_{k}^{2}+\frac{3}{2} \frac{y_{k}}{r_{k}^{5}} \delta z_{k}^{2}\right] \\
G_{2 z}= & \frac{\partial^{2} R_{0}}{\partial z_{k} \partial x_{k}} \delta x_{k}+\frac{\partial^{2} R_{0}}{\partial z_{k} \partial y_{k}} \delta y_{k}+\frac{\partial^{2} R_{0}}{\partial z_{k}^{2}} \delta z_{k} \\
& +\sum_{j=1}^{n}\left(\frac{\partial^{2} R_{0}}{\partial z_{k} \partial x_{j}} \delta x_{j}+\frac{\partial^{2} R_{0}}{\partial z_{k} \partial y_{j}} \delta y_{j}+\frac{\partial^{2} R_{0}}{\partial z_{k} \partial z_{j}} \delta z_{j}\right) \\
& +\mu\left(3 \frac{x_{k}}{r_{k}^{5}} \delta x_{k} \delta z_{k}+3 \frac{y_{k}}{r_{k}^{5}} \delta y_{k} \delta z_{k}\right) . \tag{4}
\end{align*}
$$

In equations (4), we have
$\begin{aligned} x_{j}, y_{j}, z_{j}= & \text { the rectangular coordinates of the disturbing planet } j, \\ & \text { with its unperturbed orbit assumed at epoch; }\end{aligned}$
$\mathrm{x}_{\mathrm{k}}, \mathrm{y}_{\mathrm{k}}, \mathrm{z}_{\mathrm{k}}=$ the rectangular coordinates of the disturbed planet k , with its unperturbed orbit assumed at epoch. Note that

$$
x_{k}=x_{0}, y_{k}=y_{0}, z_{k}=0 ;
$$

$\delta x_{k}, \delta y_{k}, \delta z_{k}=$ the perturbations in the rectangular coordinates of the disturbed planet $k$;
$\delta x_{j}, \delta y_{j}, \delta z_{j}=$ the perturbations in the rectangular coordinates of the disturbing planet $j$;
$\mu=k^{2}\left(1+m_{k}\right)$, where $k$ is the gaussian constant and $m_{k}$ is the mass of the disturbed planet $k$;
$R_{0}=$ the well-known disturbing function of the different disturbing planets on planet $k$, given by

$$
\begin{equation*}
R_{0}=k^{2} \sum_{j \neq k} m_{j}\left(\frac{1}{\Delta_{k j}}-\frac{x_{k} x_{j}+y_{k} y_{j}+z_{k} z_{j}}{r_{j}^{3}}\right) \tag{5}
\end{equation*}
$$

where $\Delta_{k j}$ is the mutual distance of planets $k$ and $j$, and $r_{j}$ is the heliocentric distance of planet $j$.

We note that $\sum_{j \neq k}$ represents the sum over all the disturbing planets $j$. For example, if we consider the theory of Mars, then we have $k=4$, and $j$ will take the numbers $1,2,3,5,6,7,8$, corresponding to the effects of Mercury, Venus, Earth, Jupiter, Saturn, Uranus, and Neptune. In this report, we shall exclude the effect of Pluto.

For the perturbations $\delta x_{k}, \delta y_{k}, \delta z_{k}$ and $\delta x_{j}, \delta y_{j}, \delta z_{j}$, we shall consider the values derived from the first-order theory. The $\delta x_{k}, \delta y_{k}, \delta z_{k}$ are composed of different parts, owing to the perturbations of the different disturbing planets.

If we let $F_{1 k}(j k), F_{2 k}(j k), F_{3 k}(j k)$ be, respectively, the first-order perturbations in $\delta x_{k}, \delta y_{k}, \delta z_{k}$ due to the disturbing planet $j$, then,

$$
\begin{align*}
& \delta x_{k}=\sum_{j \neq k} F_{1 k}(j k) \\
& \delta y_{k}=\sum_{j \neq k} F_{2 k}(j k) \\
& \delta z_{k}=\sum_{j \neq k} F_{3 k}(j k) \tag{6}
\end{align*}
$$

Similarly,

$$
\delta x_{j}=\sum_{i \neq j} F_{l j}^{(i j)}
$$

$$
\begin{align*}
& \delta y_{j}=\sum_{i \neq j} F_{2 j}(\mathrm{i} j), \\
& \delta z_{j}=\sum_{i \neq j} F_{3 j}(\mathrm{ij}) \tag{7}
\end{align*}
$$

The coefficients of $\delta x_{k}, \delta y_{k}, \delta z_{k}$ in equations (4) can be written as follows:

$$
\begin{align*}
& \frac{\partial^{2} R_{0}}{\partial x_{k}^{2}}=\sum_{j \neq k} \phi_{x l}(j k), \quad \frac{\partial^{2} R_{0}}{\partial y_{k} \partial x_{k}}=\sum_{j \neq k} \phi_{y l}(j k), \quad \frac{\partial^{2} R_{0}}{\partial z_{k} \partial x_{k}}=\sum_{j \neq k} \phi_{z l}(j k), \\
& \frac{\partial^{2} R_{0}}{\partial x_{k} \partial y_{k}}=\sum_{j \neq k} \phi_{x 2}(j k), \quad \frac{\partial^{2} R_{0}}{\partial y_{k}^{2}}=\sum_{j \neq k} \phi_{y 2}(j k) \quad, \quad \frac{\partial^{2} R_{0}}{\partial z_{k} \partial y_{k}}=\sum_{j \neq k} \phi_{z 2}(j k), \\
& \frac{\partial^{2} R_{0}}{\partial x_{k} \partial z_{k}}=\sum_{j \neq k} \phi_{x 3}(j k), \quad \frac{\partial^{2} R_{0}}{\partial y_{k} \partial z_{k}}=\sum_{j \neq k} \phi_{y 3}(j k), \quad \frac{\partial^{2} R_{0}}{\partial z_{k}^{2}}=\sum_{j \neq k} \phi_{z 3}(j k), \tag{8}
\end{align*}
$$

where

$$
\begin{aligned}
& \phi_{x l}(j k)=k^{2} m_{j}\left[-\frac{1}{\Delta_{k j}^{3}}+\frac{3\left(x_{j}-x_{k}\right)^{2}}{\Delta_{k j}^{5}}\right], \\
& \phi_{x 2}(j k)=k^{2} m_{j} \frac{3\left(x_{j}-x_{k}\right)\left(y_{j}-y_{k}\right)}{\Delta_{k j}^{5}}, \\
& \phi_{x 3}(j k)=\frac{k^{2} m_{j} 3\left(x_{j}-x_{k}\right) z_{j}}{\Delta_{k j}^{5}}, \\
& \phi_{y 2}(j k)=k^{2} m_{j}\left[-\frac{1}{\Delta_{k j}^{3}}+\frac{3\left(y_{j}-y_{k}\right)^{2}}{\Delta_{k j}^{5}}\right],
\end{aligned}
$$

$$
\begin{align*}
& \phi_{y 3}(j k)=k^{2} m_{j} \frac{3\left(y_{j}-y_{k}\right) z_{j}}{\Delta_{k j}^{5}}, \\
& \phi_{z 3}(j k)=k^{2} m_{j}\left(-\frac{1}{\Delta_{k j}^{3}}+\frac{3 z_{j}^{2}}{\Delta_{k j}^{5}}\right), \\
& \phi_{y 1}(j k)=\phi_{x 2}(j k), \\
& \phi_{z 1}(j k)=\phi_{x 3}(j k), \\
& \phi_{z 2}(j k)=\phi_{y 3}(j k) \quad . \tag{9}
\end{align*}
$$

The coefficients of $\delta x_{j}, \delta y_{j}, \delta z_{j}$ in equations (4) can be written

$$
\begin{array}{lll}
\frac{\partial^{2} R_{0}}{\partial x_{k} \partial x_{j}}=\theta_{x l}(j k), & \frac{\partial^{2} R_{0}}{\partial y_{k} \partial x_{j}}=\theta_{y l}(j k), & \frac{\partial^{2} R_{0}}{\partial z_{k} \partial x_{j}}=\theta_{z l}(j k), \\
\frac{\partial^{2} R_{0}}{\partial x_{k} \partial y_{j}}=\theta_{x 2}(j k), & \frac{\partial^{2} R_{0}}{\partial y_{k} \partial y_{j}}=\theta_{y 2}(j k), & \frac{\partial^{2} R_{0}}{\partial z_{k} \partial y_{j}}=\theta_{z 2}(j k), \\
\frac{\partial^{2} R_{0}}{\partial x_{k} \partial z_{j}}=\theta_{x 3}(j k), & \frac{\partial^{2} R_{0}}{\partial y_{k} \partial z_{j}}=\theta_{y 3}(j k), & \frac{\partial^{2} R_{0}}{\partial z_{k} \partial z_{j}}=\theta_{z 3}(j k),
\end{array}
$$

where

$$
\begin{aligned}
& \theta_{x l}(j k)=k^{2} m_{j}\left[\left(\frac{1}{\Delta_{k j}^{3}}-\frac{1}{r_{j}^{3}}\right)+\frac{3 x_{j}^{2}}{r_{j}^{5}}-\frac{3\left(x_{j}-x_{k}\right)^{2}}{\Delta_{k j}^{5}}\right], \\
& \theta_{x 2}(j k)=k^{2} m_{j}\left[-\frac{3\left(x_{j}-x_{k}\right)\left(y_{j}-y_{k}\right)}{\Delta_{k j}^{5}}+\frac{3 x_{j} y_{j}}{r_{j}^{5}}\right],
\end{aligned}
$$

$$
\begin{align*}
& \theta_{x 3}(j k)=k^{2} m_{j}\left[-\frac{3\left(x_{j}-x_{k}\right) z_{j}}{\Delta_{k j}^{5}}+\frac{3 x_{j} z_{j}}{r_{j}^{5}}\right], \\
& \theta_{y 2}(j k)=k^{2} m_{j}\left[\left(\frac{1}{\Delta_{k j}^{3}}-\frac{1}{r_{j}^{3}}\right)-\frac{3\left(y_{j}-y_{k}\right)^{2}}{\Delta_{k j}^{5}}+\frac{3 y_{j}^{2}}{r_{j}^{5}}\right], \\
& \theta_{y 3}(j k)=k^{2} m_{j}\left[-\frac{3\left(y_{j}-y_{k}\right) z_{j}}{\Delta_{k j}^{5}}+\frac{3 y_{j} z_{j}}{r_{j}^{5}}\right], \\
& \theta_{z 3}(j k)=k^{2} m_{j}\left[\left(\frac{1}{\left.\left.\Delta_{k j}^{3}-\frac{1}{r_{j}^{3}}\right)-\frac{3 z_{j}^{2}}{\Delta_{k j}^{5}}+\frac{3 z_{j}^{2}}{r_{j}^{5}}\right],}\right.\right. \\
& \theta_{y l}(j k)=\theta_{x 2}(j k), \\
& \theta_{z l}(j k)=\theta_{x 3}(j k), \\
& \theta_{z 2}(j k)=\theta_{y 3}(j k) \tag{11}
\end{align*}
$$

Finally, the coefficients of $\left(\delta x_{k}\right)^{2},\left(\delta y_{k}\right)^{2},\left(\delta z_{k}\right)^{2}, \delta x_{k} \delta y_{k}, \delta x_{k} \delta z_{k}$, $\delta y_{k} \delta z_{k}$ in equations (4) can be rewritten as follows:

$$
\begin{aligned}
& \psi_{x 1}(k)=\mu\left(4.5 \frac{x_{k}}{r_{k}^{5}}-7.5 \frac{x_{k}^{3}}{r_{k}^{7}}\right), \\
& \psi_{x 2}(k)=\mu\left(3 \frac{y_{k}}{r_{k}^{5}}-15 \frac{x_{k}^{2} y_{k}}{r_{k}^{7}}\right)
\end{aligned}
$$

$$
\begin{align*}
& \psi_{x 3}(k)=\mu\left(1.5 \frac{x_{k}}{r_{k}^{5}}-7.5 \frac{x_{k} y_{k}^{2}}{r_{k}^{7}}\right), \\
& \psi_{\mathrm{x} 4}(\mathrm{k})=\mu 1.5 \frac{\mathrm{x}_{\mathrm{k}}}{\mathrm{r}_{\mathrm{k}}}, \\
& \psi_{y l}(k)=\mu\left(1.5 \frac{y_{k}}{r_{k}^{5}}-7.5 \frac{x_{k}^{2} y_{k}}{r_{k}^{7}}\right), \\
& \psi_{y 2}(k)=\mu\left(3 \frac{x_{k}}{r_{k}^{5}}-15 \frac{x_{k_{k}}^{2}}{\mathrm{r}_{\mathrm{k}}^{7}}\right), \\
& \psi_{\mathrm{y} 3}(\mathrm{k})=\mu\left(4.5 \frac{\mathrm{y}_{\mathrm{k}}}{\mathrm{r}_{\mathrm{k}}}-7.5 \frac{\mathrm{y}_{\mathrm{k}}^{3}}{\mathrm{r}_{\mathrm{k}}^{7}}\right), \\
& \psi_{\mathrm{y} 4}(\mathrm{k})=\mu 1.5 \frac{\mathrm{y}_{\mathrm{k}}}{\mathrm{r}_{\mathrm{k}}^{5}}, \\
& \psi_{z l}(\mathrm{k})=\mu 3 \frac{\mathrm{x}_{\mathrm{k}}}{\mathrm{r}_{\mathrm{k}}^{5}}, \\
& \psi_{z 2}(k)=\mu 3 \frac{\mathrm{y}_{\mathrm{k}}}{\mathrm{r}_{\mathrm{k}}^{5}} . \tag{12}
\end{align*}
$$

With the above definitions of the various coefficients in equations (4), we have the following:

$$
\begin{aligned}
& G_{2 x}=\sum_{j \neq k} \phi_{x 1}(j k) \sum_{j \neq k} F_{1 k}(j k)+\sum_{j \neq k} \phi_{x 2}(j k) \sum_{j \neq k} F_{2 k}(j k) \\
& +\sum_{j \neq k} \phi_{x 3}{ }^{(j k)} \sum_{j \neq k} F_{3 k}{ }^{(j k)} \\
& +\sum_{j \neq k}\left[\theta_{x 1}{ }^{(j k)} \sum_{i \neq j} F_{1 j}{ }^{(i j)}+\theta_{x 2}(j k) \sum_{i \neq j} F_{\left.2 j^{(i j}\right)}+\theta_{x 3}(j k) \sum_{i \neq j} F_{3 j}{ }^{(i j)}\right] \\
& +\psi_{x 1}(k)\left[\sum_{j \neq k} F_{1 k}(j k)\right]^{2}+\psi_{x 2}(k)\left[\sum_{j \neq k} F_{1 k}(j k)\right]\left[\sum_{j \neq k} F_{2 k}(j k)\right] \\
& +\psi_{x 3}(k)\left[\sum_{j \neq k} F_{2 k}(j k)\right]^{2}+\psi_{x 4}(k)\left[\sum_{j \neq k} F_{3 k}(j k)\right]^{2} . \\
& G_{2 y}=\sum_{j \neq k} \phi_{y 1}(j k) \sum_{j \neq k} F_{1 k}(j k)+\sum_{j \neq k} \phi_{y 2}{ }^{(j k)} \sum_{j \neq k} F_{2 k}(j k) \\
& +\sum_{j \neq k} \phi_{y 3}{ }^{(j k)} \sum_{j \neq k} F_{3 k}{ }^{(j k)} \\
& +\sum_{j \neq k}\left[\theta_{y 1}(\mathrm{jk}) \sum_{\mathrm{i} \neq \mathrm{j}} \mathrm{~F}_{1 \mathrm{j}}(\mathrm{ij})+\theta_{\mathrm{y} 2}(\mathrm{jk}) \sum_{\mathrm{i} \neq \mathrm{j}} \mathrm{~F}_{2 \mathrm{j}}(\mathrm{ij})+\theta_{\mathrm{y}}(\mathrm{jk}) \sum_{\mathrm{i} \neq \mathrm{j}} \mathrm{~F}_{3 \mathrm{j}}(\mathrm{ij})\right]
\end{aligned}
$$

$$
\begin{align*}
& +\psi_{y l}(k)\left[\sum_{j \neq k} F_{1 k}(j k)\right]^{2}+\psi_{y 2}(k)\left[\sum_{j \neq k} F_{1 k}(j k)\right]\left[\sum_{j \neq k} F_{2 k}(j k)\right] \\
& +\psi_{y 3}(k)\left[\sum_{j \neq k} F_{2 k}(j k)\right]^{2}+\psi_{y 4}(k)\left[\sum_{j \neq k} F_{3 k}(j k)\right]^{2}, \\
& G_{2 z}=\sum_{j \neq k} \phi_{z l}(j k) \sum_{j \neq k} F_{l k}(j k)+\sum_{j \neq k} \phi_{z 2}(j k) \sum_{j \neq k} F_{2 k}(j k) \\
& +\sum_{j \neq k} \phi_{z 3}(j k) \sum_{j \neq k} F_{3 k}(j k) \\
& +\sum_{j \neq k}\left[\theta_{z l}(j k) \sum_{i \neq j} F_{1 j}(i j)+\theta_{z 2}(j k) \sum_{i \neq j} F_{2 j}(i j)+\theta_{z 3}(j k) \sum_{i \neq j} F_{3 j}(i j)\right] \\
& +\psi_{z 1}(k)\left[\sum_{j \neq k} F_{1 k}(j k)\right]\left[\sum_{j \neq k} F_{3 k}(j k)\right] \\
& +\psi_{z 2}(k)\left[\sum_{j \neq k} F_{2 k}(j k)\right]\left[\sum_{j \neq k} F_{3 k}^{(j k)}\right] \tag{13}
\end{align*}
$$

Let us now look more closely at the different terms defining the quantities $G_{2 x}, G_{2 y}, G_{2 z}$. The $\phi$ and $\theta$ terms can be represented by double Fourier series in the mean anomalies $\ell_{k}, \ell_{j}$ of the disturbed planet $k$ and the disturbing planet $j$. These series can be obtained by computing the special numerical values of $\phi$ and $\theta$ for different combinations of equidistant values of the mean anomalies $\ell_{k}, \ell_{j}$. These special values are then subjected to double-harmonic analysis.

The $\psi$ terms can be represented by Fourier series in one argument, the mean anomaly $\ell_{k}$ of the disturbed planet $k$. These series can be obtained by computing the special numerical values of $\psi$ for different equidistant values of the mean anomaly $\ell_{k}$ and then subjecting these values to single-harmonic analysis.

In other words, by expressing the $\phi, \theta$, and $\psi$ terms as Fourier series in the mean anomalies, we can avoid analytical expansions. Only doubleand single-harmonic-analysis techniques can be applied. This is what we have done in the present work. In fact, a general computer program can be constructed to have as output the Fourier representations of the different $\phi, \theta$, and $\psi$ terms for any given values of $j, k$.

The terms $F_{1 k}(j k), F_{2 k}(j k), F_{3 k}(j k)$ have already been obtained in the first-order theory. It should be remembered that these perturbations in rectangular coordinates are composed of two parts: the periodic and the secular. The periodic part is represented as double Fourier series in the mean anomalies $\ell_{j}, \ell_{k}$, and the secular part by the product of the time $t$ (measured from the given epoch) and a single Fourier series in the mean anomaly $\ell_{k}$. Let the periodic part be denoted by $f_{l k}(j k), f_{2 k}(j k), f_{3 k}(j k)$, and the secular part, by $t S_{1 j k}(k), t S_{2 j k}(k), t S_{3 j k}(k)$. Hence,

$$
\begin{equation*}
F_{i k}(j k)=f_{i k}(j k)+t S_{i j k}(k) \tag{14}
\end{equation*}
$$

where $i=1,2,3$.

Let us consider the part $\sum_{j \neq k} \phi_{x l}(j k)$. From the above remarks, this part is represented by the summation of different Fourier series, and each series is represented in the mean anomalies $\ell_{j}$ and $\ell_{k}$. For example, if we are considering the theory of Mars, we have $k=4$, and $\sum_{j \neq k} \phi_{x l}(j k)$ will be composed of the sum of seven Fourier series: the first series in $\ell_{1}, \ell_{4}$, the mean anomalies of Mercury and Mars; the second series in $\ell_{2}, \ell_{4}$, the mean anomalies of Venus and Mars; and so on. Similarly, the part $\sum_{j \neq k} F_{l k}(j k)$ is composed of the sum of different Fourier series, and each series ${ }^{j \neq k}$
expanded in the mean anomalies $\ell_{j}, \ell_{k}$. In addition to these Fourier series, this part contains a term $t$ multiplied by a Fourier series in single argument $\ell_{k}$, the mean anomaly of the disturbed planet. In fact, for $i=1,2,3$,

$$
\begin{equation*}
\sum_{j \neq k} F_{i k}(j k)=\sum_{j \neq k} f_{i k}(j k)+t S_{i k}(k) \tag{15}
\end{equation*}
$$

where

$$
S_{i k}(k)=\sum_{j \neq k} S_{i j k}(k)
$$

We note that $t S_{1 k}(k), \mathrm{t}_{2 k}(k), t S_{3 k}(k)$ are the total secular perturbations in rectangular coordinates of all the disturbing planets on planet $k$.

Similar considerations apply for the different parts of equations (13). Hence, $G_{2 x}, G_{2 y}, G_{2 z}$ can be represented by the following equations:

$$
\begin{aligned}
G_{2 x}= & G_{2 x k t t}(k) \cdot t^{2}+\sum_{j \neq k}\left[G_{2 x j k t}(j k) \cdot t+G_{2 x j k}(j k)\right] \\
& +\sum_{j} \sum_{m} G_{2 x m j k}(m j k), \\
G_{2 y}= & G_{2 y k t t}(k) \cdot t^{2}+\sum_{j \neq k}\left[G_{2 y j k t}(j k) \cdot t+G_{2 y j k}(j k)\right]
\end{aligned}
$$

$$
+\sum_{j} \sum_{m} G_{2 y m j k}(m j k)
$$

$$
\begin{align*}
G_{2 z}= & G_{2 z k t t}(k) \cdot t^{2}+\sum_{j \neq k}\left[G_{2 z j k t}(j k) \cdot t+G_{2 z j k}(j k)\right] \\
& +\sum_{j} \sum_{m} G_{2 z m j k}(m j k) \tag{16}
\end{align*}
$$

where $G(k), G(j k)$, and $G(m j k)$, appearing on the right-hand side of equations (16), denote, respectively, Fourier series in one argument, the mean anomaly $\ell_{k}$; in two arguments, the mean anomalies $\ell_{j}$, $\ell_{k}$; and in three arguments, the mean anomalies $\ell_{m}, \ell_{j}, \ell_{k}$. The double summation $\sum_{j} \sum_{m}$ means that $m$ and $j$ take the values corresponding to all the disturbing ${ }^{j}$ planets, excluding $m=j$ and avoiding double counting.

Following are the expressions for the different $G(m j k)$ in equations (16):

$$
\begin{aligned}
G_{2 q m j k}(m j k)= & \sum_{i=1}^{3}\left[\phi_{q i}(j k) f_{i k}(m k)+\phi_{q i}(m k) f_{i k}(j k)+\theta_{q i}(j k) f_{i j}(m j)\right. \\
& \left.+\theta_{q i}(m k) f_{i m}(j m)\right]+2 \psi_{q 1}(k) f_{1 k}(j k) f_{1 k}(m k) \\
& +\psi_{q 2}(k)\left[f_{1 k}(j k) f_{2 k}(m k)+f_{1 k}(m k) f_{2 k}(j k)\right] \\
& +2 \psi_{q 3}(k) f_{2 k}(j k) f_{2 k}(m k)+2 \psi_{q 4}(k) f_{3 k}(j k) f_{3 k}(m k),
\end{aligned}
$$

where $q=x, y$, and

$$
\begin{aligned}
G_{2 z m j k}(m j k)= & \sum_{i=1}^{3}\left[\phi_{z i}(j k) f_{i k}(m k)+\phi_{z i}(m k) f_{i k}(j k)\right. \\
& \left.+\theta_{z i}(j k) f_{i j}(m j)+\theta_{z i}(m k) f_{i m}(j m)\right]
\end{aligned}
$$

$$
\begin{align*}
& +\psi_{z l}(k)\left[f_{l k}(j k) f_{3 k}(m k)+f_{l k}(m k) f_{3 k}(j k)\right] \\
& +\psi_{z 2}(k)\left[f_{2 k}(j k) f_{3 k}(m k)+f_{2 k}(m k) f_{3 k}(j k)\right] \tag{17}
\end{align*}
$$

The terms $G(j k)$ (not multiplied by $t$ ) in equations (16) are expressed as follows:

$$
\begin{aligned}
G_{2 q j k}(j k)= & \sum_{i=1}^{3}\left[\phi_{q i}(j k) f_{i k}(j k)+\theta_{q i}(j k) f_{i j}(k j)\right]+\psi_{q 1}(k) f_{l k}^{2}(j k) \\
& +\psi_{q 2}(k) f_{l k}(j k) f_{2 k}(j k)+\psi_{q 3}(k) f_{2 k}^{2}(j k)+\psi_{q 4}(k) f_{3 k}^{2}(j k),
\end{aligned}
$$

where $q=x, y$, and

$$
\begin{align*}
G_{2 z j k}(j k)= & \sum_{i=1}^{3}\left[\phi_{z i}(j k) f_{i k}(j k)+\theta_{z i}(j k) f_{i j}(k j)\right]+\psi_{z l k} f_{l k}(j k) f_{3 k}(j k) \\
& +\psi_{z 2}(k) f_{2 k}(j k) f_{3 k}(j k) \tag{18}
\end{align*}
$$

The terms $G(j k)$ (multiplied by $t$ ) take the following forms:

$$
\begin{aligned}
G_{2 q j k t}(j k)= & \sum_{i=1}^{3}\left[\phi_{q i}(j k) S_{i k}(k)+\theta_{q i}(j k) S_{i j}(j)\right]+2 \psi_{q 1}(k) f_{1 k}(j k) S_{1 k}(k) \\
& +\psi_{q 2}(k)\left[f_{l k}(j k) S_{2 k}(k)+f_{2 k}(j k) S_{l k}(k)\right] \\
& +2 \psi_{q 3}(k) f_{2 k}(j k) S_{2 k}(k)+2 \psi_{q 4}(k) f_{3 k}(j k) S_{3 k}(k)
\end{aligned}
$$

where $q=x, y$, and

$$
\begin{align*}
G_{2 z j k t}(j k)= & \sum_{i=1}^{3}\left[\phi_{z i}(j k) S_{i k}(k)+\theta_{z i}(j k) S_{i j}(j)\right] \\
& +\psi_{z 1}(k)\left[f_{1 k}(j k) S_{3 k}(k)+f_{3 k}(j k) S_{1 k}(k)\right] \\
& +\psi_{z 2}(k)\left[f_{2 k}(j k) S_{3 k}(k)+f_{3 k}(j k) S_{2 k}(k)\right] \tag{19}
\end{align*}
$$

For the terms with coefficient $\mathrm{t}^{2}$ in equations (16), we have

$$
\begin{aligned}
G_{2 q k t t}(k)= & \psi_{q 1}(k) S_{1 k}^{2}(k)+\psi_{q 2}(k) S_{1 k}(k) S_{2 k}(k) \\
& +\psi_{q 3}(k) S_{2 k}^{2}(k)+\psi_{q 4}(k) S_{3 k}^{2}(k)
\end{aligned}
$$

where $\mathrm{q}=\mathrm{x}, \mathrm{y}$, and

$$
\begin{equation*}
G_{2 z k t t}(k)=\psi_{z 1}(k) S_{1 k}(k) S_{3 k}(k)+\psi_{z 2}(k) S_{2 k}(k) S_{3 k}(k) \tag{20}
\end{equation*}
$$

In equations (17) to (20), all the different terms on the right-hand side are expressed in Fourier series in the mean anomalies $\ell_{m}, \ell_{j}, \ell_{k}$. We have already outlined how we obtain these series. We are now in a position to evaluate the Fourier representations in mean anomalies of the functions $G_{2 q m j k}(m j k), G_{2 q j k}(j k), G_{2 q j k t}, G_{2 q k t t}$, where $q$ denotes the parameters $\mathrm{x}, \mathrm{y}$, and z .

Let us consider, for example, $G_{2 q m j k}(m j k)$, whose expressions are given in equations (17). Since we have the Fourier series for all the terms appearing in the right-hand side of equations (17), we can, by the technique of multiplying Fourier series, obtain the Fourier series representing $G_{2 q m j k}(\mathrm{mjk})$. In this case, we did not resort to triple-harmonic analysis because it would have been excessively laborious. In fact, we constructed
a general computer program that has as input the numerical values of $m$, $j$, and $k$ and that will give as output the Fourier representations of $G_{2}$ qmjk $(m j k)$ in the mean anomalies $\ell_{m}, \ell_{j}, \ell_{k}(q$ denotes the values $x, y$, and $z)$.

In computing the $F$ ourier representations of $G_{2 q j k t}(j k)$ and $G_{2 q j k}(j k)$ for $q=x, y, z$, we can use the double-harmonic-analysis approach or the multiplication-of-series approach. To compute the Fourier representations of $G_{2 q k t r}(k)$ for $q=x, y, z$, we can very conveniently use the single-harmonic-analysis technique.

In our work, we have a general program that, for given $j, k$ as input, produces as intermediate output the Fourier representations of $G_{2 q j k t}(j k)$, $G_{2 q j k}(j k)$, and $G_{2 q k t t}(k)$ for $q=x, y, z$.

## 4. THE DECOMPOSITION OF $\delta \mathrm{x}_{2}, \delta \mathrm{y}_{2}, \delta \mathrm{z}_{2}$

In the previous section, we developed the different components of the functions $G_{2 x}, G_{2 y}, G_{2 j}$. We found that these functions are generally composed of the summation of the following series:
A. Fourier series in three arguments.
B. Fourier series in two arguments.
C. Fourier series in two arguments, multiplied by the time $t$.
D. Fourier series in one argument, multiplied by $\mathrm{t}^{2}$.

By substituting the general expressions of $G_{2 x}, G_{2 y}, G_{2 z}$ in equations (3), we can see that $\delta_{x 2}, \delta_{y 2}, \delta_{z 2}$ will be composed of the following different parts, where $q$ takes the values $x, y, z$ :
A. Fourier series in three argument $\left(\ell_{m}, \ell_{j}, \ell_{k}\right)$, denoted by $\delta q_{2 m j k}(m j k)$.
B. Fourier series in two arguments $\left(\ell_{j}, \ell_{k}\right)$, denoted by $\delta q_{2 j k}(j k)$.
C. Fourier series in two arguments $\left(\ell_{j}, \ell_{k}\right)$ multiplied by the time $t$, denoted by $\delta q_{2 j k t}(j k)$.
D. Fourier series in one argument $\left(\ell_{k}\right)$ denoted by $\delta q_{2 k}(k)$.
E. Fourier series in one argument ( $\ell_{k}$ ) multiplied by $t$, denoted by $\delta q_{2 k t}(k)$.
F. Fourier series in one argument $\left(\ell_{k}\right)$ multiplied by $t^{2}$, denoted by $\delta q_{2 k t t}(k)$.
G. Fourier series in one argument $\left(\ell_{k}\right)$ multiplied by $t^{3}$, denoted by $\delta q_{2 k t t t}(k)$.
H. Fourier series in one argument $\left(\ell_{k}\right)$ multiplied by $t^{4}$, denoted by $\delta q_{2 k t t t t}(k)$.

We must remember that $\ell_{k}$ is the mean anomaly of the disturbed planet, and $\ell_{m}, \ell_{j}$ are the mean anomalies of the disturbing planets $m, j$.

In order to present more conveniently the equations defining the various parts of $\delta x_{2}, \delta y_{2}, \delta z_{2}$, let us put

$$
\begin{align*}
\sum_{\alpha, \beta} & \frac{\partial q}{\partial a} \int\left(\frac{\partial x_{0}}{\partial \beta} G_{2 x}+\frac{\partial y_{0}}{\partial \beta_{1}} G_{2 y}\right) d t \\
= & \frac{\partial q}{\partial L_{0}} \int\left(\frac{\partial x_{0}}{\partial \omega_{0}} G_{2 x}+\frac{\partial y_{0}}{\partial \omega_{0}} G_{2 y}\right) d t-\frac{\partial q}{\partial \omega_{0}} \int\left(\frac{\partial x_{0}}{\partial L_{0}} G_{2 x}+\frac{\partial y_{0}}{\partial L_{0}} G_{2 y}\right) d t \\
& +\frac{\partial q}{\partial \xi_{0}} \int\left(\frac{\partial x_{0}}{\partial \eta_{0}} G_{2 x}+\frac{\partial y_{0}}{\partial \eta_{0}} G_{2 y}\right) d t \\
& -\frac{\partial q}{\partial \eta_{0}} \int\left(\frac{\partial x_{0}}{\partial \xi_{0}} G_{2 x}+\frac{\partial y_{0}}{\partial \xi_{0}} G_{2 y}\right) d t \tag{21}
\end{align*}
$$

for $q=x, y$. With this abbreviated notation, we have

$$
\begin{aligned}
\delta q_{2 m j k}(m j k)= & \sum_{a, \beta} \frac{\partial q}{\partial a} \int\left[\frac{\partial x_{0}}{\partial \beta} G_{2 x m j k}(m j k)+\frac{\partial y_{0}}{\partial \beta} G_{2 y m j k}(m j k)\right] d t \\
& -3 \mu^{2} L_{0}^{-4} \frac{\partial q}{\partial \omega_{0}} \iint\left[\frac{\partial x_{0}}{\partial \omega_{0}} G_{2 x m j k}(m j k)+\frac{\partial y_{0}}{\partial \omega_{0}} G_{2 y m j k}(m j k)\right] d^{2},
\end{aligned}
$$

where $q=x, y$, and

$$
\begin{equation*}
\delta z_{2 m j k}(m j k)=q_{2} \int q_{1} G_{2 z m j k}(m j k) d t-q_{1} \int q_{2} G_{2 z m j k}(m j k) d t \tag{22}
\end{equation*}
$$

The integrands on the right side of equations (22) can now be developed in Fourier series. Integrating these Fourier representations, we obtain other Fourier representations of the integrals. Multiplying these Fourier representations by the Fourier series representing the coefficients $\partial \mathrm{q} / \partial a$, $-3 \mu^{2} L_{0}^{-4}\left(\partial q / \partial \omega_{0}\right), q_{2},-q_{1}$, and adding the different results, we obtain the Fourier representation $\delta x_{2 m j k}(m j k), \delta y_{2 m j k}(m j k), \delta z_{2 m j k}(m j k)$. We note that the constant coefficient in the Fourier series representing these different integrands, i. e., the coefficients of the argument 0 , will, when integrated once, give rise to a numerical coefficient multiplied by $t$; when integrated twice, it will give rise to a numerical coefficient multiplied by $\mathrm{t}^{2}$. Hence, the final representations of $\delta q_{2 m j k}(\mathrm{mjk})$ will contain, besides the purely periodic terms given by the Fourier representations in three arguments, mixed terms composed of the time $t$ multiplied by Fourier series in one argument and, in the case of $q=x, y$ only, the square of the time $\left(t^{2}\right)$ multiplied by Fourier series in one argument. These mixed terms will be added to the perturbations $\delta q_{2 k t}(k), \delta q_{2 k t t}(k)$.

A computer program has been constructed with the series $G_{2 x m j k}(m j k)$, $G_{2 y m j k}(m j k), G_{2 z m j k}(m j k)$ as input and, as output, Fourier representations $\delta x_{2 m j k}(m j k), \delta y_{2 m j k}(m j k), \delta z_{2 m j k}(m j k)$ and the corresponding mixed terms in $\delta \mathrm{x}_{2 \mathrm{kt}}(\mathrm{k}), \delta \mathrm{y}_{2 \mathrm{kt}}(\mathrm{k}), \delta \mathrm{z}_{2 \mathrm{kt}}(\mathrm{k}), \delta \mathrm{x}_{2 \mathrm{ktt}}(\mathrm{k}), \delta \mathrm{y}_{2 \mathrm{ktt}}(\mathrm{k})$.

For the evaluation of $\delta q_{2 j k}(j k), \delta q_{2 j k t}(j k)$, for $q=x, y, z$, we must recall the following relations:

$$
\begin{align*}
& \int \mathrm{tfdt}=\mathrm{t} \int \mathrm{f} d \mathrm{t}-\iint \mathrm{f} d \mathrm{t}^{2} \\
& \iint \mathrm{t} f d t^{2}=\mathrm{t} \iint \mathrm{f} d \mathrm{t}^{2}-2 \iiint \mathrm{f} d \mathrm{t}^{3}, \tag{23}
\end{align*}
$$

where $f$ is any function of time $t$. The equations defining $\delta q_{2 j k}(j k)$ will be given by

$$
\begin{aligned}
\delta q_{2 j k}(j k)= & \sum_{a, \beta} \frac{\partial q}{\partial a} \int\left[\frac{\partial x_{0}}{\partial \beta} G_{2 x j k}(j k)+\frac{\partial y_{0}}{\partial \beta} G_{2 y j k}(j k)\right] d t \\
& -3 \mu^{2} L_{0}^{-4} \frac{\partial q}{\partial \omega_{0}} \iint\left[\frac{\partial x_{0}}{\partial \omega_{0}} G_{2 x j k}(j k)+\frac{\partial y_{0}}{\partial \omega_{0}} G_{2 y j k}(j k)\right] d t^{2} \\
& -\sum_{a, \beta} \frac{\partial q}{\partial a} \iint\left[\frac{\partial x_{0}}{\partial \beta} G_{2 x j k t}(j k)+\frac{\partial y_{0}}{\partial \beta} G_{2 y j k t}(j k)\right] d t^{2} \\
& \left.+2\left(3 \mu^{2} L_{0}^{-4} \frac{\partial q}{\partial \omega_{0}}\right) \iiint\left[\frac{\partial x_{0}}{\partial \omega_{0}} G_{2 x j k t^{(j k}}\right)+\frac{\partial y_{0}}{\partial \omega_{0}} G_{2 y j k t}(j k)\right] d t^{3},
\end{aligned}
$$

where $q=x, y$, and

$$
\begin{align*}
\delta z_{2 j k}(j k)= & q_{2} \int q_{1} G_{2 z j k}(j k) d t-q_{1} \int q_{2} G_{2 z j k}(j k) d t \\
& -q_{2} \iint q_{1} G_{2 z j k t}(j k) d t^{2}+q_{1} \iint q_{2} G_{2 z j k t}(j k) d t^{2} . \tag{24}
\end{align*}
$$

Through the multiplication-of-series approach or the double-harmonicanalysis technique, we can develop the Fourier representations of all the integrands appearing in the above equations and then evaluate the Fourier representations of $\delta q_{2 j k}(j k)$ for $q=x, y, z$. We note again that the constant terms in the various harmonic representations of the above integrands will give rise to mixed terms with coefficients $t$ and $t^{2}$ in the expressions for $\delta q_{2 j k}(j k)$, for $q=x, y, z$. Mixed terms with coefficient $t^{3}$ will also appear in the cases for $q=x$, $y$ because of the presence of triple integrals. These various mixed terms appearing in $\delta q_{2 j k}(\mathrm{jk})$ will be included in the perturbations $\delta q_{2 k t}(k), \delta q_{2 k t t}(k)$, and $\delta q_{2 k t t t}(k)$ for $q=x, y, z$.

The equations defining $\delta \mathrm{q}_{2 \mathrm{jkt}}(\mathrm{jk})$ will be given by

$$
\begin{aligned}
\delta q_{2 j k t}(j k)= & \sum_{a, \beta} \frac{\partial q}{\partial a} \int\left[\frac{\partial x_{0}}{\partial \beta} G_{2 x j k t}(j k)+\frac{\partial y_{0}}{\partial \beta} G_{2 y j k t}(j k)\right] d t \\
& -3 \mu^{2} L_{0}^{-4} \frac{\partial q}{\partial \omega_{0}} \iint\left[\frac{\partial x_{0}}{\partial \omega_{0}} G_{2 x j k t}(j k)+\frac{\partial y_{0}}{\partial \omega_{0}} G_{2 y j k t}(j k)\right] \quad d t^{2},
\end{aligned}
$$

where $q=x, y$, and

$$
\begin{equation*}
\delta z_{2 j k t}(j k)=q_{2} \int q_{1} G_{2 z j k t}(j k) d t-q_{1} \int q_{2} G_{2 z j k t}(j k) d t \tag{25}
\end{equation*}
$$

Again, through the double-harmonic-analysis technique or the multiplication-of-series approach, we can get the harmonic representations of $\delta q_{2 j k t}(j k)$ for $q=x, y, z$. Also, we expect mixed terms with coefficient $t$ in $\delta q_{2 j k t}(j k)$ for $q=x, y, z$ and with coefficient $t^{2}$ in the case of $q=x, y$. Since $\delta q_{2 j k t}(j k)$ is already multiplied by $t$, these mixed terms will have coefficients $t^{2}$ and $t^{3}$. As before, the se mixed terms will be included in the perturbations $\delta q_{2 k t t}(\mathrm{k}), \delta q_{2 k t t t}(\mathrm{k})$.

Finally, for the evaluation of $\delta q_{2 k t}(k), \delta q_{2 k t t}(k), \delta q_{2 k t t t}(k)$ for $q=x, y, z$ and $\delta \mathrm{q}_{2 k t t t}(\mathrm{k})$ for $\mathrm{q}=\mathrm{x}, \mathrm{y}$, we must recall the following relations:

$$
\begin{align*}
& \int t^{2} f d t=t^{2} \int f d t-2 t \iint f d t^{2}+2 \iiint f d t^{3} \\
& \iint t^{2} f d t=t^{2} \iint f d t^{2}-4 t \iiint f d t^{3}+6 \iiint \int f d t^{4} \tag{26}
\end{align*}
$$

where $f$ is any function of time $t$.

We mentioned earlier the contributions to $\delta q_{2 k t}(k), \delta q_{2 k t t}(k), \delta q_{2 k t t t}(k)$ obtained while we were deriving expressions for $\delta q_{2 m j k}(m j k), \delta q_{2 j k}(j k)$, $\delta q_{2 j k t}(j k)$. In addition to these contributions, we have the following:

$$
\begin{aligned}
\delta q_{2 k t}(k)= & -2 \sum_{a, \beta} \frac{\partial q}{\partial a} \iint\left[\frac{\partial x_{0}}{\partial \beta} G_{2 x k t t}(k)+\frac{\partial y_{0}}{\partial \beta} G_{2 y k t t}(k)\right] d t^{2} \\
& +4\left(3 \mu^{2} L_{0}^{-4} \frac{\partial q}{\partial \omega_{0}}\right) \iiint\left[\frac{\partial x_{0}}{\partial \omega_{0}} G_{2 x k t t}(k)+\frac{\partial y_{0}}{\partial \omega_{0}} G_{2 y k t t}\right] d t^{3},
\end{aligned}
$$

where $q=x, y$;

$$
\begin{align*}
\delta z_{2 k t}(k)= & -2 q_{2} \iint q_{1} G_{2 z k t t}(k) d t^{2}+2 q_{1} \iint q_{2} G_{2 z k t t}(k) d t^{2},  \tag{27}\\
\delta q_{2 k t t}(k)= & \sum_{a, \beta} \frac{\partial q}{\partial a} \int\left[\frac{\partial x_{0}}{\partial \beta} G_{2 x k t t}(k)+\frac{\partial y_{0}}{\partial \beta} G_{2 y k t t}(k)\right] d t \\
& -3 \mu^{2} L_{0}^{-4} \frac{\partial q}{\partial \omega_{0}} \iint\left[\frac{\partial x_{0}}{\partial \omega_{0}} G_{2 x k t t^{\prime}}(k)+\frac{\partial y_{0}}{\partial \omega_{0}} G_{2 y k t t}(k)\right] d t^{2},
\end{align*}
$$

where $q=x, y$; and

$$
\begin{equation*}
\delta z_{2 k t t}(k)=q_{2} \int q_{1} G_{2 z k t t}(k) d t-q_{1} \int q_{2} G_{2 z k t t}(k) d t \tag{28}
\end{equation*}
$$

We note that mixed terms with coefficients $\mathrm{t}^{3}$ and $\mathrm{t}^{4}$ will appear from the expressions of $\delta \mathrm{q}_{2 \mathrm{kt}}(\mathrm{k}), \delta \mathrm{q}_{2 \mathrm{ktt}}(\mathrm{k})$ given in equations (27) and (28). These terms can be added to those defining $\delta q_{2 k t t t}(k), \delta q_{2 k t t t t}(k)$. Terms that are purely periodic and are expressed in Fourier series in one argument $\ell_{k}$ will appear and are given by $\delta q_{2 k}(k)$, where

$$
\begin{aligned}
\delta \mathrm{q}_{2 k}(\mathrm{k})= & 2 \sum \frac{\partial \mathrm{q}}{\partial \mathrm{a}} \iiint \int\left[\frac{\partial \mathrm{x}_{0}}{\partial \beta} \mathrm{G}_{2 \mathrm{xktt}}(\mathrm{k})+\frac{\partial \mathrm{y}_{0}}{\partial \beta} \cdot G_{2 y k t t}(\mathrm{k})\right] \mathrm{dt}^{3} \\
& -6\left(3 \mu^{2} L_{0}^{-4} \frac{\partial \mathrm{q}}{\partial \omega_{0}}\right) \iiint \int\left[\frac{\partial \mathrm{x}_{0}}{\partial \omega_{0}} G_{2 x k t t}(\mathrm{k})+\frac{\partial \mathrm{y}_{0}}{\partial \omega_{0}} G_{2 y k t t}(\mathrm{k})\right] \mathrm{dt}^{4},
\end{aligned}
$$

for $q=x, y$, and

$$
\begin{equation*}
\delta z_{2 k}(k)=2 q_{2} \iiint q_{1} G_{2 z k t t}(k) d t^{3}-2 q_{1} \iiint q_{2} G_{2 z k t t}(k) d t^{3} \tag{29}
\end{equation*}
$$

Again, the terms may give rise to mixed terms with coefficients $t^{3}$ and $t^{4}$. These will be added to $\delta \mathrm{q}_{2 \mathrm{kttt}}(\mathrm{k})$ for $\mathrm{q}=\mathrm{x}, \mathrm{y}, \mathrm{z}$ and to $\delta \mathrm{q}_{2 \mathrm{ktttt}}(\mathrm{k})$ for $\mathrm{q}=\mathrm{x}, \mathrm{y}$.

A computer program THEORY 2 has been constructed to compute $\delta q_{2 j k}(j k), \delta q_{2 j k t}(j k), \delta q_{2 k}(k), \delta q_{2 k t}(k), \ldots, \delta q_{2 k t t t t}(k)$. The input of this program is $\mathrm{j}, \mathrm{k}$. The final output is the Fourier representations of these perturbations. In this program, we followed the double- and single-harmonicanalysis methods; we did not apply the multiplication-of-series technique.

## 5. NUMERICAL APPLICATION

In the previous section, we outlined the method followed for computing the second-order perturbations in $\delta x, \delta y$, and $\delta z$. We have two main computer programs. By use of the harmonic-analysis approach, program THEORY 2 computes the periodic and secular perturbations expressed in Fourier series in the two mean anomalies $\ell_{j}, \ell_{k}$ of the disturbing and the disturbed planets and also in one mean anomaly $\ell_{k}$ of the disturbed planet.

The second main program computes the periodic perturbations expressed in Fourier series in the three mean anomalies: $\ell_{m}, \ell_{j}$, the mean anomalies of the disturbing planets, and $\ell_{k}$, the mean anomaly of the disturbed planet.

We used the multiplication-of-series approach, which required carrying the multiplication to a certain tolerance. This tolerance is taken to be directly proportional to the divisor when we compute the integrand that will be integrated once. For the case of the integrand that will be integrated twice, we take the tolerance to be directly proportional to the square of the divisor (the constant of proportionality is $10^{-13}$ ). This variable tolerance device will assure us that there has been no loss of any significant digits owing to the small divisor.

The author will soon publish the details of these two main programs and the different subroutines associated with them.

## 6. NUMERICAL RESULTS

In this section, we present the results of the computation of the secondorder perturbation of Mars containing the masses of Jupiter and Saturn; according to the notation given previously, we give the results of $\delta q_{2 m j k}$ $(q=x, y, z)$, where $m=6, j=5$, and $k=4$. The other results will be given in another paper.

Tables 1, 2, and 3 give the periodic part of the Fourier series representation of $\delta \mathrm{x}_{2 \mathrm{mjk}}, \delta \mathrm{y}_{2 \mathrm{mjk}}$, and $\delta \mathrm{z}_{2 \mathrm{mjk}}$. The mixed terms arising from the evaluation of these perturbations are given in Tables 4,5, and 6. The coefficients in these mixed terms are computed up to a tolerance of $10^{-19}$. As a check, it would be interesting to compare the series we obtained for these perturbations with the results obtained from numerical integration.

The above series representations (periodic and mixed) are simply the analytical solution of the set of differential equations (2), where $G_{2 x}, G_{2 y}$, and $G_{2 z}$ are replaced by $G_{2 x m j k}, G_{2 y m j k}$, and $G_{2 z m j k}$. That set of differential equations has been solved numerically, using Cowell's method of numerical integration. In applying this method, the tenth difference has been neglected and the interval of integrations is taken to be 10 days. The initial values of the numerical integration are chosen such that $\delta x, \delta y, \delta z$ at $\mathrm{t}=90$ days and $\mathrm{t}=100$ days are given by the analytical solutions of $\delta \mathrm{x}, \delta \mathrm{y}, \delta \mathrm{z}$.

The evaluation of $G_{2 x m j k}, G_{2 y m j k}, G_{2 z m j k}$ in the numerical integration of the differential equation was carried out by use of the original definition of these G's as given by equations (17). The integration has been carried out up to $t=40,000$ days.

When we compare the results of the numerical integration with the analytical representation, deviation is found between the two. The deviations found in the comparison of the perturbations in $x$ and $y$ are periodic in character, with the amplitude increasing with time. The amplitude reaches $5 \times 10^{-7}$ around $t=20,000$. The deviation found in the comparison of the perturbation in $z$ is again periodic, with smaller amplitude. The amplitude reaches $1 \times 10^{-9}$. The disagreement between the numerical solution and the analytical representation of the perturbation in $x, y$ is very alarming.

However, we must expect a satisfactory agreement if the starting values used in initiating the numerical integration are given to a great accuracy. These starting values have been obtained, as mentioned earlier, from the analytical solution. In obtaining the analytical solution, we carried the evaluation of the different integrals involved up to a tole rance of $10^{-13}$; i.e., terms with absolute values less than $10^{-13}$ have been neglected. These terms may add up, causing the accuracy of the evaluation of the integral to be more than $10^{-13}$. We must remember, also, that these integrals must be multiplied by the partial derivatives $\partial \mathrm{x}_{0} / \partial \omega_{0}, \partial \mathrm{y}_{0} / \partial \omega_{0}, \partial \mathrm{x}_{0} / \partial \mathrm{L}_{0}, \partial \mathrm{y}_{0} / \partial \mathrm{L}_{0}, \ldots$ The coefficients of the harmonic representation of these partial derivatives amount to $10^{2}$. Thus, the accuracy of the evaluation of the periodic representation of the perturbation in $x, y$, and $z$ may amount to $10^{-11}$ or even $10^{-10}$.

Numerical integration of the differential equations defining the secondorder perturbation is very sensitive to the starting values, which we have just found may be in error to within $10^{-10}$ to $10^{-11}$. To meet that situation,* we can apply differential corrections to the starting values, such that the deviation between the numerical integration and the analytical solution is minimum, in the least-squares sense.

[^1]We have applied differential corrections where our equation of condition corresponded to deviations at $t=200,1800,3400, \ldots, 19,400$ days. The results of this follow:

$$
\begin{aligned}
\text { At } t & =90 \text { days, } \\
\Delta(\delta x) & =-8.6198240974 \times 10^{-11} \\
\Delta(\delta y) & =-1.3774726475 \times 10^{-10} \\
\Delta(\delta z) & =+3.2748239038 \times 10^{-10} \\
\text { At } t & =100 \text { days, } \\
\Delta(\delta x) & =-3.6212757640 \times 10^{-11} \\
\Delta(\delta y) & =-2.4084833490 \times 10^{-10} \\
\Delta(\delta z) & =+3.5234609123 \times 10^{-10}
\end{aligned}
$$

When we apply these corrections to the starting values of the numerical integration, the agreement between the analytical solution and the numerical integration improves appreciably. The deviation, after the integration is carried to 40,000 days, never exceeds $4 \times 10^{-10}$ in $x, y$ and $1 \times 10^{-9}$ in $z$, an excellent agreement indeed. This comparison is shown in Table 7.




8.
0000000000000000000000000000000000000000000000000000000000







 
-

:

n
$\therefore \quad \mathrm{mmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmm}$




$\odot$ ..... 








0000000000000000000000000000000000000000000000000000000000




$\bigcirc \mathrm{mmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmm}$





















©























の $\mathrm{r} \boldsymbol{\sim}$


| $\underset{\sim}{\text { ¢ }}$ |  <br>  <br>  |
| :---: | :---: |
| $\stackrel{8}{8}$ |  <br>  |
| $\pm$ |  |
| un |  <br>  |
| 0 |  |
| 宕 |  |
| 8 |  |
| $\pm$ |  |
| in |  <br>  |
| $\bigcirc$ |  |
| . |  |
| $\stackrel{08}{80}$ |  |
| + |  |
| n |  |
| $\infty$ |  |
| . 4 |  <br>  |
| \% |  |
| + |  |
| ก |  <br>  |
| $\bigcirc$ |  |

















```
g
```






```
O NNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNN
```





































Table l (Cont.)



－












シ





＊onmininmanin



－H tinertimurarn©－ㅇ9ㅇ9ㅇ9ㅇํ으응

品 NNMOMMNONO


© inmingironncio
＊ormogれすMNッO
ヘ
－$\quad \mathrm{mmmmmmmmmm}$










0000000000000000000000000000000000000000000000000000000000000000000000
N





$\bigcirc \mathrm{mmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmm}$

品



$\odot$








|  |  <br> ค ค0000000000000000000000000000000000000000000000000000000000000000 <br> 保势雲 |
| :---: | :---: |
|  |  |
|  |  |




















- 00000000000000000000000000000000000000000000000000000000000




$\therefore \quad \mathrm{mmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmm}$













































- NNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNN

[^2]

















```
O NNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNN
```

N
0
0
0
0
0

|  | $5$ |  |
| :---: | :---: | :---: |
|  | \％ |  |
|  | － |  |
|  | $\sim$ |  |
|  | $\bigcirc$ |  |
|  | 砢 |  |
|  | 0 |  |
| － | $\pm$ |  |
| ¢ | in |  |
| $\cup$ | $\bigcirc$ |  |
| N |  |  |
| ＋ | 䂞 |  |
|  | 8 |  <br>  <br> －minnminNo <br> 1才ず <br>  <br> －！ <br>  <br> जnincrons <br>  |
|  | $\stackrel{+}{4}$ |  |
|  | n |  |
|  | $\bigcirc$ |  |
|  | $\stackrel{5}{5}$ |  |
|  | $\stackrel{\square}{8}$ |  |
|  |  |  |
|  |  |  |
|  | $\bigcirc$ |  |






5









A $\quad$ ONo








-















N



















© $\quad \mathrm{mmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmm}$
둧쿠

＊セッププi
－ロ゚ロ゚ニロッ



$\stackrel{\square}{8} \boldsymbol{m o n n n m}$
＊9ipirity
n ooraong
－innoniniminu

がローローディ
＊トwooges

－ mmmmmnm















```
O NNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNNMMMMMMMMMMMMMMMMMM
```














|  |  |  |
| :---: | :---: | :---: |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |












-


















omNNNNNNNNNNNNNNNMNNNNNNNNNNNNNNNNNNNNNNNNNNNMN

| Table 4. Fourier representation of $\delta x_{2}$ (mixed par T measured from epoch in days. The coefficients are in units of $10^{-19}$. The argument is $\mathrm{k}_{\ell}, \ell_{4}$ mean anomaly of Mars. |  |  |
| :---: | :---: | :---: |
| k | cos | $\sin$ |
| 1 | $0 \mathrm{~T}^{2}$ | $-26 \mathrm{~T}^{2}$ |
| 2 | $0 \mathrm{~T}^{2}$ | $-2 \mathrm{~T}^{2}$ |
| 0 | 313774 T | 0 T |
| 1 | 10849 T | -78999480 T |
| 2 | -102779 T | -8840577 T |
| 3 | -14383 T | -977485 T |
| 4 | -1786 T | -110676 T |
| 5 | -217 T | -12768T |
| 6 | -26 T | -1493 T |
| 7 | -3 T | -176 T |
| 8 | 0 T | -21 T |
| 9 | 0 T | -3 T |

Table 5. Fourier representation of $\delta y_{2}$ (mixed part) T measured from epoch in days. The coefficients are in units of $10^{-19}$. The argument is $\mathrm{k} \mathrm{\ell}_{4}, \ell_{4}$ mean anomaly of Mars.

| $k$ | $\cos$ | $\sin$ |
| :--- | ---: | ---: |
| 1 | $26 \mathrm{~T}^{2}$ | $0 \mathrm{~T}^{2}$ |
| 2 | $2 \mathrm{~T}^{2}$ | $0 \mathrm{~T}^{2}$ |
| 0 | 4512938 T | 0 T |
| 1 | 78686060 T | 20653 T |
| 2 | 8823289 T | -102322 T |
| 3 | 976183 T | -14351 T |
| 4 | 110563 T | -1784 T |
| 5 | 12758 T | -216 T |
| 6 | 1492 T | -26 T |
| 7 | 176 T | -3 T |
| 8 | 21 T | 0 T |
| 9 | 3 T | 0 T |

Table 6. Fourier representation of $\delta z_{2}$ (mixed part) $T$ measured from epoch in days. The coefficients are in units of $10^{-19}$.
The argument is $\mathrm{kl}_{4}, \ell_{4}$ mean anomaly of Mars.

| $k$ | $\cos$ | $\sin$ |
| :--- | ---: | ---: |
| 0 | 658698 T | 0 T |
| 1 | -4692988 T | 873448 T |
| 2 | -218295 T | 40658 T |
| 3 | -15234 T | 2838 T |
| 4 | -1260 T | 235 T |
| 5 | -115 T | 21 T |
| 6 | -11 T | 2 T |
| 7 | -1 T | 0 T |


| $\stackrel{t}{\text { (days) }}$ |  | $\Delta x_{2}$ | $\Delta y_{2}$ | $\Delta z_{2}$ |  |  | $\mathrm{x}_{2}$ | $\Delta y_{2}$ | $\Delta z_{2}$ |  | $\Delta x_{2}$ | $\Delta y_{2}$ | $\Delta z_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  | $\stackrel{\mathrm{t}}{\text { (days) }}$ |  |  |  |  | $\begin{gathered} t \\ \text { (days) } \end{gathered}$ |  |  |  |









的







Table 7 (Cont.)

Table 7 (Cont.)

| $\sqrt[N]{4}^{N}$ |  snulu <br>  <br>  rejirouinn |
| :---: | :---: |
| $\mathrm{N}_{4}^{N}$ | ио!ฺe. 8) snuyu <br>  мо!дел 8әли! <br>  |
| $x_{4}^{N}$ | иощехяәวи! snutur <br>  <br>  <br>  |
|  | - |
| $\underbrace{n}$ | иопุеляәии! snulu теэ! <br>  [еวาxวunn |
| $\frac{\sim}{4}$ |  впитuи <br>  ио!มе.88эาи! <br>  |
| $\tilde{x}^{N}$ | noinerse7tu snutu теэплүей <br>  <br>  |
|  | + |
| $\mathbb{N}^{N}$ |  snuilu โеорикеuy <br>  feotiaminn |
| $\stackrel{N}{4}_{N}$ | wotex вnutur โеวпиィеич บоที่าจวาน jejtraunn |
| ${\underset{d}{x}}^{N}$ |  |
|  | + |


Table 7 (Cont.)

|  |  | $\Delta \mathrm{x}_{2}$ | $\Delta y_{2}$ | $\Delta z_{2}$ |  | $\Delta x_{2}$ | $\Delta y_{2}$ | $\Delta z_{2}$ |  | $\Delta \mathrm{x}_{2}$ | $\Delta y_{2}$ | $\Delta \dot{z}_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\stackrel{t}{(\text { days })}$ |  |  |  |  | $\stackrel{\mathrm{t}}{\text { (days) }}$ |  |  |  | $\begin{gathered} \mathrm{t} \\ (\text { days }) \end{gathered}$ |  |  |  |


Table 7 (Cont.)

|  |  <br>  |
| :---: | :---: |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |
| 离 |  |
|  |  |
|  |  |
|  |  |
| - $\frac{\text { 零 }}{}$ |  <br>  |

## 7. ACKNOWLEDGMENTS

I wish to acknowledge the continuous interest and encouragement given by Drs. F. L. Whipple and C. A. Lundquist. I also express my gratitude to Dr. G. Clemence for his valuable advice and his numerous illuminating discussions. I wish to extend my thanks to Drs. P. Musen, R. Broucke, and M. Davies for clarifying and constructive discussions. My thanks also to Mrs. Yun-Ying Fang, from the SAO's programing department, who programed the multiplication-of-series program used in this work. I thank Drs. A. Allison and B. Marsden for discussing various aspects of the problem with me.

## REFERENCES

BROUWER, D., and CLEMENCE, G. M.
1961. Methods of Celestial Mechanics. Academic Press, New York, $598 \mathrm{pp} .(2 \mathrm{nd}$ printing, 1965 ).
HAMID, S. E.
1968. First-order planetary theory, perturbations in rectangular coordinates, Hansen's variables, longitude, and distance. Smithsonian Astrophys. Obs. Spec. Rep. No. 285, 35 pp.

## BIOGRAPHICAL NOTE

[^3]
## NO TICE

This series of Special Reports was instituted under the supervision of Dr. F. L. Whipple, Director of the Astrophysical Observatory of the Smithsonian Institution, shortly after the launching of the first artificial earth satellite on October 4, 1957. Contributions come from the Staff of the Observatory.

First issued to ensure the immediate dissemination of data for satellite tracking, the reports have continued to provide a rapid distribution of catalogs of satellite observations, orbital information, and preliminary results of data analyses prior to formal publication in the appropriate journals. The Reports are also used extensively for the rapid publication of preliminary or special results in other fields of astrophysics.

The Reports are regularly distributed to all institutions participating in the U.S. space research program and to individual scientists who request them from the Publications Division, Distribution Section, Smithsonian Astrophysical Observatory, Cambridge, Massachusetts 02138.


[^0]:    This work was supported in part by grant NGR 09-015-002 from the National Aeronautics and Space Administration.

[^1]:    *The author owes this idea to Prof. G. M. Clemence.

[^2]:    
    
    
    
    

    |  |  |  |  |  |  |
    | :---: | :---: | :---: | :---: | :---: | :---: |
    |  |  |  |  |  |  |
    |  |  |  |  |  |  |
    |  |  |  |  |  |  |
    |  |  |  |  |  |  |
    |  |  |  |  |  |  |
    |  |  |  |  |  |  |
    |  |  |  |  |  |  |
    |  |  |  |  |  |  |

    
    
    
    
    N
    Table
    
    
    
    
    
    咢
    
    :
    
    
    
    

[^3]:    SALAH E. HAMID received his B.S.C. from Cairo University in 1944 and his Ph. D. in astronomy from Harvard in 1950.

    Dr. Hamid was an assistant professor of astronomy on the Faculty of Sciences, Cairo University, and has also held the position of director of the Operation Research Center of the National Planning Institute of Cairo, the U. A. R. He joined SAO in 1961 as a celestial mechanician.

