The Energy-Momentum Tensor of Quantum Fluctuations During Inflation

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Introduction: 1

- **WMAP** measures amplitude, spectrum, deviation from a simple power spectrum of inflationary perturbations (Peiris et al., 2003) ... what is the energy associated to these fluctuations?
- Due to the nonlinearity of Einstein eqs what is the effect on the geometry?

Outline

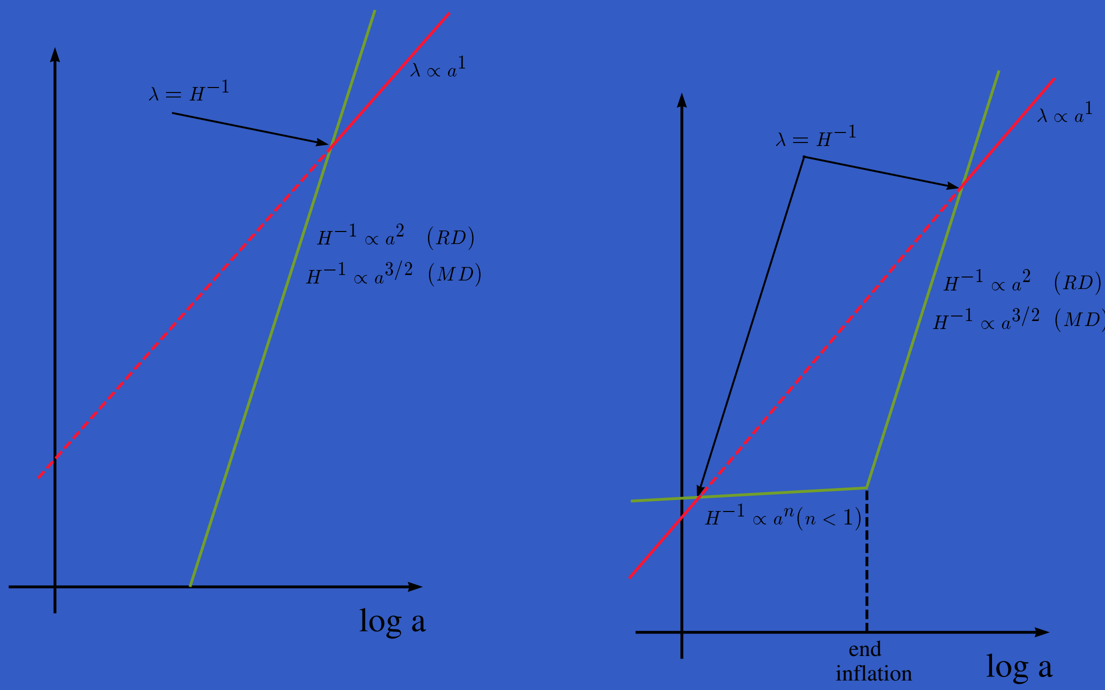
- **De Sitter**
- **Inflation (without metric perturbations)**
- **Inflation (cosmological fluctuations)**

F. F., G. Marozzi, G. P. Vacca, G. Venturi, Phys. Rev. D 65, 103521 (2001)
F. F., G. Marozzi, G. P. Vacca, G. Venturi, gr-qc/0310086 (2003).

Introduction: 2

- **Key Point of the Inflationary Idea:** the accelerated expansion does not only solve the horizon, monopole, flatness problem, but provides a scenario to develop density perturbations with a nearly scale invariant spectrum, in approximative agreement with observations.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j \qquad H = \frac{\dot{a}}{a}$$



- **De Sitter** space-time (with H constant in time) produce a scale-invariant spectrum for fluctuations. Result already known in the '70s.

De Sitter: 1

De Sitter: 2

Results from the '70-'80s for De Sitter

De Sitter space-time ($H^2 = 8\pi G\Lambda/3$):

$$ds^2 = -dt^2 + e^{2Ht}\delta_{ij}dx^i dx^j = \left(-\frac{1}{H\eta}\right)^2 \left[-d\eta^2 + \delta_{ij}dx^i dx^j\right]$$

Test fields ϕ with arbitrary mass and coupling to the curvature:

$$S = \int d^4x \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{2}\phi^2 - \frac{\xi}{2}R\phi^2\right]$$

$$\hat{\phi}(t,\mathbf{x}) = \frac{1}{(2\pi)^3}\sum_{\mathbf{k}}\left[\phi_k(t)\, e^{i\mathbf{k}\cdot\mathbf{x}}\ \hat{b}_{\mathbf{k}} + \phi_k^*(t)e^{-i\mathbf{k}\cdot\mathbf{x}}\ \hat{b}_{\mathbf{k}}^\dagger\right]$$

$$\ddot{\phi}_k + 3H\dot{\phi}_k + \left[\frac{k^2}{a^2} + m^2 + \xi R\right]\phi_k = 0$$

De Sitter: 3

Particle production in expanding space-times (Parker 1968). In the massless minimally coupled case ($\xi = m = 0$):

$$(a\phi_k)'' + \left[k^2 - \frac{a''}{a}\right](a\phi_k) = 0$$

For de Sitter $\frac{a''}{a} = \frac{2}{\eta^2}$.
Exact solutions for ϕ_k are known for arbitrary mass and couplings in terms of Bessel functions.
For $\xi = m = 0$ ϕ has a scale invariant spectrum:

$$\phi_k = \frac{e^{-ik\eta}}{a\sqrt{2k}} \to k^3 |\phi_k|^2 \propto k^0 \quad \text{when} \quad k << aH$$

What about the renormalized correlator?

De Sitter: 4

The renormalized correlator is:

$$\boxed{\langle\phi^2\rangle_{\mathrm{REN}} = \frac{\hbar}{4\pi^2} H^3 t}$$

Linde; Starobinsky; Vilenkin and Ford (1982); Allen and Folacci (1987)

Although rigorously derivated with many regularization schemes, one can use the heuristic way to compute:

$$\langle\phi^2\rangle \sim \int_{k_0}^{Ca(t)} |\phi_k|^2 k^2 dk \sim H^2 \left[\log a(t) - \log\frac{k_0}{C}\right]$$

where $a(t) = e^{Ht}$, k_0 (which can be related to the start of the De Sitter era) and $Ca(t)$ are opportune cut-offs.

The leading behaviour does not depend on C.

The phase-space of the infrared modes - which contribute to $\langle\phi^2\rangle$ - increase.
However, in the EMT for massless fields only more ultraviolet terms appear:

$$\langle T_{\mu\nu}\rangle_{\mathrm{REN}} \propto \langle\partial_\mu\phi\partial_\nu\phi\rangle_{\mathrm{REN}} \propto H^4 g_{\mu\nu}$$

and NO BACK-REACTION occurs.

The same result for $m \neq 0$, but the two point function is:

$$\boxed{\langle\phi^2\rangle_{\mathrm{REN}} = \frac{3\hbar}{8\pi^2}\frac{H^4}{m^2}}$$

EMT of Field Fluctuations during Inflation: 1

Results for Inflation without metric perturbations

Finelli, Marozzi,Vacca,Venturi, PRD 65 (2001)

Example: Inflation driven by a free scalar field with mass m. During slow-rollover H decreases linearly with time:

$$
\begin{aligned}
3H\dot{\phi} &\simeq -m^2\phi \\
H^2 &\simeq \frac{1}{6M_{\rm pl}^2}m^2\phi^2
\end{aligned}
$$

$$\boxed{\dot{H} \simeq -\frac{m^2}{3}}$$

At first order in $\epsilon = -\dot{H}/H^2$ inflaton fluctuations φ are amplified with a SCALE INVARIANT SPECTRUM even if massive (result also found numerically by Kuzmin and Tkachev, PRD 59 (1999)).

The analytic solution which agrees with adiabatic vacuum for short wavelengths is:

$$\varphi_{\mathbf{k}} = \frac{1}{a^{3/2}}\left(\frac{\pi\lambda}{2H}\right)^{1/2} H^{(1)}_{3/2}(\lambda\zeta)\,, \qquad \lambda = 1 - \frac{\dot{H}}{H^2}\,, \quad \zeta = \frac{k}{aH}$$

EMT of Field Fluctuations during Inflation: 2

In order to take care of the divergences we employ dimensional regularization (using a d dimensional space measure) plus adiabatic subtraction (we subtract the expectation value with respect to the adiabatic vacuum at fourth order).
Therefore we proceed as in the de Sitter case:

$$\begin{aligned}\langle\varphi^2\rangle_{\mathrm{REN}} &= \langle\varphi^2\rangle - \langle\varphi^2\rangle_{(4)} \\ &= \frac{\hbar}{4\pi^2}\left[\int_\ell^\infty dk\, k^2\, |\varphi_k|^2 - \int_\ell^\infty dk\, k^2\, |\varphi_k^{(4)}|^2\right]\end{aligned}$$

where $\varphi_k^{(4)}$ is the fourth adiabatic order mode function and $\ell = CH_0$ is the infrared cut-off related to the beginning of inflation.

EMT of Field Fluctuations during Inflation: 3

After lengthy calculations:

$$\boxed{\langle \varphi^2 \rangle_{\rm REN} = \frac{\hbar}{4\pi^2} H^2 \log a + \text{subleading terms}}$$

Slow-roll lasts:

$$\Delta t \sim \frac{H_0}{m^2}$$

where H_0 is the Hubble radius at the beginning of inflation.

The inflaton is massive, therefore $\langle T_{\mu\nu} \rangle_{\rm REN}$ is not more ultraviolet than $\langle \varphi^2 \rangle_{\rm REN}$.

At the end of slow-roll fluctuations saturate at ($\ln a_{|\rm MAX} \simeq 1/(2\epsilon_0)$):

$$\langle \varphi^2 \rangle_{\rm REN}^{\rm MAX} \sim \frac{3\hbar}{8\pi^2} \frac{H^2 H_0^2}{m^2}$$

and the energy and pressure stored in the fluctuations is:

$$\epsilon_{\rm REN} = \langle T_{00} \rangle_{\rm REN} \sim \frac{m^2}{2} \langle \varphi^2 \rangle_{\rm REN} \sim \frac{3\hbar}{16\pi^2} H^2 H_0^2$$

$$p_{\rm REN} = \frac{\langle T_{ii} \rangle_{\rm REN}}{a^2} \sim -\epsilon_{\rm REN}$$

EMT of Field Fluctuations during Inflation: 4

To conclude:

- Massive fluctuations may have scale infrared spectrum (or steeper) due to $\dot{H} \neq 0$.
- Particle production for fields with mass $M \leq m$ depends on the duration of inflation.
- The energy of particles with $M = m$ produced during inflation is $H^2 H_0^2$ and not H^4 as extrapolated from de Sitter calculation.
- The result obtained agrees with the Hartree-Fock description of stochastic inflation (Starobinsky and Yokoyama, PRD 50 (1994)) to lowest order in $\mathcal{O}(\epsilon)$:

$$\frac{d}{dt}\langle\varphi^2\rangle_{\rm REN} = \frac{\hbar}{4\pi^2}H^3 - 2\frac{m^2}{3H}\langle\varphi^2\rangle_{\rm REN}$$

EMT of Cosmological Fluctuations during Inflation: 1

Results for Inflation with metric perturbations

Finelli, Marozzi, Vacca, Venturi, gr-qc/0310086

- **Linear Scalar Perturbation in Uniform Curvature Gauge**

$$ds^2 = -(1+2\alpha)dt^2 - a\beta_{,i} dt dx^i + a^2 \delta_{ij} dx^i dx^j$$

- Advantage of UCG is to have the Mukhanov variable (JETP Lett 41 (1985)) as field perturbation. The Mukhanov variable is the variable used to quantize cosmological perturbations and therefore we regularize its two point function.

- In UCG the gauge invariant curvature perturbation ζ (which remains constant in time for super-Hubble scales in absence of isocurvature perturbations for an accelerated universe) is simply

$$\zeta = \frac{H}{\dot{\phi}} \varphi$$

EMT of Cosmological Fluctuations during Inflation: 2

- Einstein Constraints:

$$\begin{aligned}\frac{H}{a}\nabla^2\beta &= 8\pi G\left(\dot\phi\dot\varphi + V_\phi\varphi + 2\,V\alpha\right)\\ &= 8\pi G\,\frac{\dot\phi^2}{H}\,\dot\zeta\,,\\ \alpha_{,i} &= 4\pi G\,\frac{\dot\phi}{H}\,\varphi_{,i} \to \alpha = \epsilon\zeta \quad \text{with} \quad \epsilon = -\frac{\dot H}{H^2}\end{aligned}$$

- Eqs for field fluctuations:

$$\begin{aligned}\ddot\varphi + 3H\dot\varphi - \frac{1}{a^2}\nabla^2\varphi + V_{\phi\phi}\varphi &= 2\alpha\ddot\phi + \dot\phi\left(\dot\alpha + 6H\alpha - \frac{1}{2a}\nabla^2\beta\right)\\ &= 2\frac{\dot H}{H}\left(3H - \frac{\dot H}{H} + 2\frac{\ddot\phi}{\dot\phi}\right)\varphi\end{aligned}$$

$$\boxed{(a\varphi)'' + \left(-\nabla^2 - \frac{z''}{z}\right)(a\varphi) = 0\,, \quad z = a\frac{\dot\phi}{H}}$$

EMT of Cosmological Fluctuations during Inflation: 3

$\varphi(t, \vec{x})$ satisfy to first order in ϵ:

$$\ddot{\varphi} + 3H\dot{\varphi} + \left[-\frac{\nabla^2}{a^2} + m^2 + 6\dot{H}\right]\varphi = 0$$

We know from the theory of cosmological perturbations that the spectrum for long-wavelength curvature perturbations is RED for this single field inflationary model. We do not need only long-wavelength behaviour, but an approximation over the whole Fourier space.

Since $\dot{H} \simeq -\frac{m^2}{3}$:

$$\boxed{\ddot{\varphi}_k + 3H\dot{\varphi}_k + \left[\frac{k^2}{a^2} - m^2\right]\varphi_k = 0}$$

For large $k(> 3/2aH)$ we choose:

$$\varphi_k^{\rm UV} = \frac{1}{a^{3/2}}\left(\frac{\pi\lambda}{4H}\right)^{1/2} H_\nu^{(1)}(\lambda\zeta) \qquad \nu = \frac{3}{2} + 2\epsilon$$

For small $k(< 3/2aH)$ we choose:

$$\varphi_k^{\rm IR} = \frac{1}{a^{3/2}}\left(\frac{\pi\lambda}{4H}\right)^{1/2}\left(\frac{H(t_k)}{H(t)}\right)^2 H_{3/2}^{(1)}(\lambda\zeta)$$

where t_k is a time related to an istant in which the mode k crosses the Hubble radius.
In this case the adiabatic regularization is performed as:

$$\int_\ell^{+\infty} dk \to \int_\ell^{\frac{3}{2}\frac{aH}{\lambda}} dk + \int_{\frac{3}{2}\frac{aH}{\lambda}}^{+\infty} dk$$

$$\boxed{\langle\varphi^2\rangle_{\rm REN} = \langle\varphi^2\rangle^{IR} + \langle\varphi^2\rangle^{UV} - \langle\varphi^2\rangle_{(4)} = \frac{\hbar}{4\pi^2}H^2\left(\frac{H_0}{H}\right)^4 \log a}$$

EMT of Cosmological Fluctuations during Inflation: 4

Fluctuations saturate at a higher value than in absence of metric fluctuations ($\ln a_{|\mathrm{MAX}} \simeq 1/(2\epsilon_0)$):

$$\langle \varphi^2 \rangle_{\mathrm{REN}}^{\mathrm{MAX}} \sim \frac{\hbar}{8\pi^2} \frac{H_0^6}{m^2 H^2}$$

since the inclusion of metric fluctuations leads to a different amplitude (and spectrum) of fluctuations in the infrared.

- EMT of cosmological scalar fluctuations:

$$T_{\mu\nu}^{\mathrm{eff}} = \delta T_{\mu\nu}^{(2)} - M_{\mathrm{pl}}^2 \delta G_{\mu\nu}^{(2)}$$

The leading terms in the infrared of the energy density averaged over the adiabatic vacuum are:

$$\epsilon \sim \langle T_{00}^{\mathrm{eff}} \rangle \sim \frac{V_{\phi\phi}}{2} \langle \varphi^2 \rangle + 2 V_\phi \langle \varphi \alpha \rangle = \frac{V_{\phi\phi}}{2} \langle \varphi^2 \rangle \left(1 - 4 \frac{\epsilon}{\eta} \right)$$

where $\eta = M_{\mathrm{pl}}^2 \frac{V_{\phi\phi}}{V}$. Since $\eta \sim \epsilon$, the energy density of scalar perturbations is negative and grows in time during slow-rollover (result in agreement with previous calculations by Abramo, Brandenberger, Mukhanov (1997)).
The equation of state of the EMT of cosmological perturbations is

$$p_{\mathrm{REN}} \simeq -\epsilon_{\mathrm{REN}}$$

If inflation lasts enough e-folds the EMT of inflaton fluctuations is not negligible!

EMT of Cosmological Fluctuations during Inflation: 5

A bound on the initial scale of inflation H_0 is necessary in order to neglect the back-reaction of scalar fluctuations.

$$\frac{\epsilon_{REN}^{\mathrm{MAX}}}{\epsilon_{\mathrm{homogeneous}}} \sim 1 \quad \to \quad H_0 \sim (16\pi^2/\hbar)^{1/4} H^{2/3} M_{\mathrm{pl}}^{1/3}$$

In other words, if we want to neglect the EMT of cosmological fluctuations, the number of e-folds should have an upper bound ($10^3 < N_{\mathrm{tot}} < 10^4$).

Back-Reaction: 1

How can the Back-Reaction Problem be formulated?

- Computation of the local expansion rate $\Theta = \nabla_\mu u^\mu$ $(u_\mu u^\mu = -1)$ goes through painful second-order calculations

$$ds^2 = -(1+2\alpha+2\alpha^{(2)})dt^2 - a(\beta_{,i}+\beta_{,i}^{(2)})dt\,dx^i + a^2\left[\delta_{ij} + \frac{1}{2}\left(\partial_i\chi_j^{(2)} + \partial_j\chi_i^{(2)} + h_{ij}^{(2)}\right)\right]dx^i dx^j$$

$$\phi(t,\vec{x}) = \phi(t) + \varphi(t,\vec{x}) + \varphi(t,\vec{x})^{(2)}$$

$\chi_i^{(2)}$ is the vector perturbation, $h_{ij}^{(2)}$ is the tensor perturbation, both at second order. The inclusion of these metric perturbations, together with scalar perturbations $\alpha^{(2)}$, $\beta^{(2)}$, is necessary since at second order scalar, vector and tensor perturbations are coupled.

- All second-order perturbations are driven by quadratic terms in first-order perturbations (see also works by Matarrese and coll.)
- The inclusion of vector and tensor perturbations at second order is important in order to extract the scalar part of the second order equations.

Back-Reaction: 2

Momentum constraint to second order:

$$\alpha^{(2)}_{,i} + \frac{1}{8H}\nabla^2 \dot{\chi}^2_i = 4\pi G \frac{\dot{\phi}}{H}\varphi^{(2)}_{,i} + S_i$$

The term S_i can be written as $\partial_i s + v_i$ (with v_i divergenceless), leading to the scalar part:

$$\boxed{\alpha^{(2)} = 4\pi G \frac{\dot{\phi}}{H}\varphi^{(2)} + 4\pi G\epsilon\varphi^2 + 2\alpha^2 - \frac{1}{8}|\nabla\beta|^2 + \frac{1}{\nabla^2}\left[\frac{4\pi G}{H}\nabla\cdot(\dot{\varphi}\nabla\varphi) + \frac{1}{4aH}\left(\alpha^{,kj}\beta^{,kj} - \alpha^{,k}_{,k}\beta^{,j}_{,j}\right)\right]}$$

By using Einstein eqs to second order one obtains for scalar field perturbations:

$$\boxed{(a\varphi^{(2)})'' + \left(-\nabla^2 - \frac{z''}{z}\right)a\varphi^{(2)} = a^2 f(\varphi^2, \alpha\varphi)}$$

- **The Local Expansion Rate averaged over the quantum vacuum:**

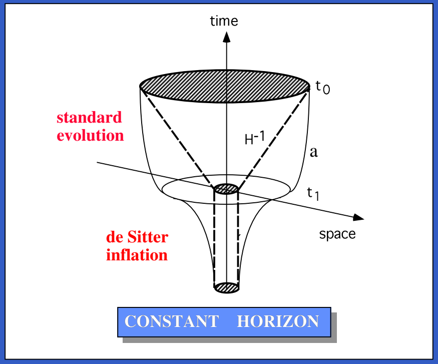
$$\langle\Theta\rangle = \Theta^{(0)} + \langle\Theta^{(2)}\rangle \simeq 3H\left(1 + \frac{3}{2}\langle\alpha^2\rangle - \langle\alpha^{(2)}\rangle\right) \simeq 3H\left(1 - \epsilon\frac{\langle\varphi^2\rangle}{4M^2_{\rm pl}}^{\rm MAX}\right)$$

- Result of the same order of magnitude of the estimate based on the comparison of the EMT!

The expansion rate Θ is slown down in average by second-order effects

Common Concerns

- **Mean Field Theory**
- **Average over Particle Horizon Scale, and not over the Hubble Scale during Inflation**



Different procedure with respect to Stochastic Inflation (Vilenkin (1983), Starobinsky (1984), Linde et al. (1988)), but maybe there is a link.

- **Are we averaging over black holes?**

No. At $\zeta_0 \sim 10^3 \zeta_{\rm obs}$. The metric perturbation α is suppressed by ϵ with respect to ζ, so it is even smaller. Therefore metric fluctuations are still small while average back-reaction can be large.

Results and Conclusions

- Particle Production in De Sitter $\neq$ Particle Production in Chaotic Inflation
- Particle Production of Light Particles depends on the duration of Inflation. For $M = m$ in $m^2\phi^2$ inflationary model:

$$\epsilon_{light\,\mathrm{REN}} \sim H^2 H_0^2\,,\mathrm{NOT} \sim H^4$$

- The Inflaton in Chaotic Inflation is light enough!
- By taking self-consistently into account metric perturbations, at first order gauge invariant field fluctuations get a negative effective mass and at second order the EMT of cosmological perturbations carry negative energy density and grows in time during slow-rollover.
- By taking into account second-order perturbations, the expansion rate is in average slown down by the EMT of cosmological perturbations, in agreement with the previous result.
- By approaching the back-reaction problem we have provided the strategy to solve the Einstein equations to second order for single field inflation.