Topological Analysis of Chaotic Orbits: Hyperion

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Abstract

Recent well-sampled ground-based observations of the Saturnian satellite Hyperion strongly suggest that it is exhibiting chaotic behavior. We present a method that allows the analysis of fairly small data sets consisting of nightly observations carried out over a few years. With such an analysis, the skeletal features of the organizers of the dynamics can be uncovered. We analyze a numerically generated data-set, based on a model for Hyperion extensively studied in the literature, corresponding to nightly observations of the satellite. We show that even with such a small data set, the embedded unstable periodic orbits can be extracted, and that these can be responsible for "almost periodic behavior" lasting a substantial fraction of the observing run.

Subject Headings: Analytical Methods, Celestial Mechanics, Planets: Satellites, Rotation

1 Introduction

Nonlinear dynamics and chaos have been very active fields of research in applied mathematics and physics in the last ten years. A century ago, the study of nonlinear dynamics was strongly tied to celestial mechanics; in fact it was the search for a solution to the gravitational three-body problem that motivated Poincaré to develop the first qualitative studies in this area (Poincaré 1892; Holmes 1990). Interest in this problem was linked to predicting observables within the solar system such as the perturbation on some bodies due to the non-negligible mass of Jupiter. Thus it is ironic that there exists today a gap between the recently developed theoretical dynamical methods and observational astronomy. This is largely due to the natural restriction on the length of an astronomical data set. The presence of universalities in bifurcations has caught the attention of many physicists, biologists and others (Bai-Lin 1984), but it is not possible (or desirable) to make the necessary parameter changes in such a study of the dynamical evolution of celestial bodies to uncover similar bifurcations. Recently, additional methods have been proposed to address the class of problems which contains celestial mechanical problems, i.e., systems in which all information must be gleaned from only one point in parameter space in which the system may behave chaotically. The most widespread approach to this problem is the study of the metric properties of the chaotic solution such as the calculation of Lyapunov characteristic exponents and fractal dimensions (Lichtenberg & Lieberman

1982). Unfortunately, this method usually involves the analysis of large amounts of data, making the comparison impossible between information obtained numerically and observationally.

In our solar system there is a satellite which appears to be undergoing chaotic rotation, the Saturnian satellite Hyperion. The rotational dynamics of this highly aspherical body was theoretically predicted to be chaotic by Wisdom, Peale & Mignard (1984). Sound evidence supporting this claim was presented recently by Klavetter (1989a,b), who performed well sampled, precise ground-based observations covering 3.5 orbital periods and found no periodic fit to the data. In fact, Klavetter invested a great deal of CPU time fitting a *chaotic* rotation curve to his observations. The properties of Hyperion adopted by Klavetter are listed in Table 1.

In this paper we apply a method recently proposed (Mindlin *et al.* 1990; Tufillaro *et al.* 1990) for analyzing a fairly small chaotic data set to a simplified model for the rotational behavior of Hyperion. The method seeks to identify closed orbits of low period, and to determine how these orbits are organized among themselves. This organization provides information about the mechanisms responsible for generating chaotic dynamics. The paper is organized as follows. In section 2 we review the model studied by Wisdom *et al.* (1984) . In section 3 we show how to extract the unstable periodic orbits, to which the trajectory stays near, from a chaotic data set. We apply this method to a numerical data set, generated using Klavetter's measurements of the body parameters, with various initial conditions. The length of our numerical data set corresponds to approximately 2.6 years of nightly observations. In section 4 we discuss our results and in section 5 we state our conclusions.

2 The Model

The present study is based on a model that consists of an orbiting body in the shape of a triaxial ellipsoid parametrized by the axes of inertia, $C \geq B \geq A$, spinning only about its largest axis of inertia which is assumed to be oriented perpendicular to the orbital plane (Fig. 1). This is a reduction of the full 6-dimensional system for rotational motion about all three principal axes. The reduced model corresponds to a tidally-evolved satellite. The characteristics that we study in the two-dimensional model are present as well in the full 6-dimensional system. In the spirit of Wisdom, Peale, Mignard (1984) and Klavetter (1989b) we use the two-dimensional dynamical system (θ, θ') as a numerical laboratory. The above reduction yields the dynamical equation for the rotational angular variable θ,

$$\theta'' + \frac{\omega^2}{2r^3} sin2(\theta - f) = 0, \tag{1}$$

where f is the true anomaly of the satellite, the prime indicates the time derivitive, and ω^2 is a measure of the deviation of the body from sphericity, i.e.,

$$\omega^2 = 3\frac{(B-A)}{C}. \tag{2}$$

The time dependence of r, present for all eccentricities greater that zero, essentially provides a periodic forcing to the system at each periastron passage.

Note that in the case of circular orbits r is no longer a function of time and the equation of motion can be written as

$$\theta'' + \frac{\omega^2}{a^3} sin(\theta) = 0, \tag{3}$$

which is an *integrable* equation, namely that for the motion of the simple pendulum. In this integrable limit, three qualitatively different states of motion are present for this system, parametrized by the total energy. For high energy the state of motion is rotational. For low energy the system will present librations about θ=0. The separatrices between these qualitatively different kinds of motion are known as the homoclinic orbits that connect (θ, θ')=(π,0) with itself. They correspond to a clockwise (counterclockwise) rotation of the pendulum, of infinite period. It should be noted that these orbits are highly degenerate structures; every point on either of them belongs to both the stable and unstable manifolds of the fixed point (π,0).

The system (1) differs from the integrable limit by the presence of the driving term, which acts to complicate the dynamics considerably. Note that (1) contains no dissipative terms, hence the system is Hamiltonian. It is routine in the study of Hamiltonian systems to consider a Poincaré section (alternatively called a "surface of section"), obtained by plotting the phase space variables θ and θ' at each periastron passage (i.e.f=0). From the patterns described by these points, we are able to determine in which regions of phase space regular islands dominate, as well as which regions are dominated by chaotic motion—referred to as "chaotic seas." The Poincaré section for Hyperion is shown in Fig. 2a; it consists of 22200 points (200 points for 111 initial conditions). To obtain these points we integrated the equations (Goldreich 1965)

$$\begin{aligned} \theta' &= \phi \\ \phi' &= \frac{-\omega^2}{2r^3} sin2(\theta - f) \\ f' &= \frac{(1 + ecos(f))^2}{(1 - e^2)^{\frac{3}{2}}}. \end{aligned} \tag{4}$$

We take the asphericity parameter ω to be 0.89 in keeping with the results quoted by Klavetter(1989b). The main feature of Fig. 2a is the large chaotic sea that separates islands of regular motion. However, it must be noted that there exist unstable periodic orbits in the apparently chaotic region; in fact regular and chaotic motions are known to coexist on all scales in the phase space of Hamilotinian systems (Dana, 1990). Moreover, this characteristic causes chaotic orbits to resemble a collection of almost periodic segments, each acting like (a piece of) a regular trajectory (Dana 1990) . We use this characteristic of a chaotic data set in our method described below.

We point out that a useful Poincaré section for Hyperion can not be generated using observational data because of the limited number of data points obtained even in a long-baseline observing program of 949 nightly observations, i.e., 2.6 years of nightly observations. In addition, an observer is restricted to only one point in the space of initial conditions, (i.e., the point that corresponds to that particular trajectory being observed) whereas numerous unique initial conditions must be explored when generating a meaningful Poincaré section. In Fig. 2b we show a hypothetical attempt to display almost 2.6 years worth of "observational" data as a Poincaré map. With so few points corresponding to near-periastron passage (45 points) it is impossible to make out any but the slightest resemblence to Fig. 2a. It is for this reason that metric techniques are not appropriate when dealing with the typical astronomical data set. Fortunately, the method of close returns allows for the investigation of shorter time series, most often with more meaningful results than the metric approach.

3 The Search for Close Returns

Numerous unstable periodic orbits exist in regions of phase space where chaotic motion takes place. This is true in general for dynamical systems; in particular, for Hamiltonian systems the unstable periodic orbits are dense.

As the point describing the state of a system evolves through phase space, it eventually enters the neighborhood of some unstable orbit of low period. At this point, it will be attracted to that orbit along the orbit's stable direction and repelled from the orbit along its unstable direction. The point will move away from the unstable periodic orbit after a certain time depending on a) how near to the orbit it is in the unstable direction, and b) how strong the repulsion is in the unstable direction (measured by the Lyapunov exponent, λ). If it is not too strongly repelled, and the orbit is of sufficiently low period (P), then it may return to its initial neighborhood one or more times before evolving to other regions of phase space. Such close returns are more likely for shorter periods and smaller Lyapunov exponents. The product λP is a useful indicator that the system will evolve in the neighborhood of an unstable periodic orbit for a long time; the smaller the product, the more likely a close return near that orbit will be seen in the data.

The search for close returns can be carried out easily in a discretely sampled scalar or vector data set, $\mathbf{x}(\mathrm{i}) = (x_1(i), x_2(i), ..., x_n(i)), i = 1, 2, ...$ This is most conveniently done by computing the distance between $\mathbf{x}$(i) and $\mathbf{x}$(i+p), $d[\mathbf{x}(\mathrm{i})\ \text{-}\mathbf{x}(\mathrm{i+p})]$, where i locates the vector in the time series, p identifies the lag, or separation between two points being compared, in terms of the time sample rate, and d is any reasonable metric. This is visually implemented by color coding the (i, p^{th}) pixel black if $d[\mathbf{x}(\mathrm{i})\ \text{-}\mathbf{x}(\mathrm{i+p})] \leq \epsilon$, white otherwise. The threshold ϵ is typically taken to be in the range $10^{\pm 1}$ percent of the maximum $d[\mathbf{x}(\mathrm{i})\ \text{-}\mathbf{x}(\mathrm{i+p})]$. In this visual representation, close returns appear as nearly horizontal line segments which are easily seen by inspection

($c.f.$, Fig. 3).

For the present Hyperion study, the vector $\mathbf{x}(\mathrm{i}) = (\theta(i), \theta'(i))$, where i=1,2,...,950. To obtain $(\theta(i), \theta'(i))$, the equations of motion (4) were integrated with a 4^{th} order Runge-Kutta procedure. The step-size was reduced until convergence was achieved. The sampling rate corresponded to one night intervals. One orbital period corresponds to approximately 21 points. We recorded 950 points ($\approx$ 45 orbital periods) for each of the numerical experiments carried out, four of which are discussed in the following section. Our data sets simulate observations of $\approx$ 2.6 years. While this may seem like an enormous data set to observers accustomed to relatively short observing runs, it is staggeringly small to workers in the field of nonlinear dynamics who employ metric techniques.

4 Discussion

As discussed in the introduction and in section 3, chaotic solutions behave like a series of different (parts of) regular episodes. These give rise to close returns in the chaotic time series. This means that, for a while, the chaotic solution behaves like a nearly periodic orbit.

We perform four numerical experiments to simulate observational data of the rotation state of Hyperion. The physical parameters are kept constant (see Table 1); the initial conditions (θ, θ') are chosen to be in different regions of phase space. The corresponding images which contain the close return information are shown in Figs. 3a-d. Figs. 3a,b,and c were generated from initial conditions lying within the large chaotic region while Fig. 3d was generated from an initial condition belonging to a regular region of phase space (c.f. Fig. 2a).

In each of the four numerical experiments, close returns of low periodicity are evident for a substantial amount of time. These results are summarized in the Tables 2, 3 and 4 for the images of Figs. 3a, b, and c. Since the experiment corresponding to Fig. 3d was run using initial conditions within a regular region, close returns are regularly spaced and endure for the entire 800 days. The return after 10.5 days clearly indicates a 2:1 spin-orbit resonance. This is qualitatively different from the results obtained in any of the chaotic experiments, where several regular orbits are visited, and for shorter amounts of time.

The close return information for Fig.3a is displayed in Table 2. The initial conditions for this run are $\theta = 1.0$ radians, $\theta' = 0.5$ rotations per orbital period and $f = 0.0$. Table 2 lists the periods of the most obvious close returns along with their duration and their starting (i) position in the image, (i.e., the first close return is of period 3 and begins 125 units from the left edge of Fig. 3a). Also included in the table is a "comments" row in which we list the duration in months of the longer lived close return episodes. The very long period 4 close return in Fig. 3a lasts approximately 8.5 months, hence an observer confined to this span of time might conclude a nearly periodic rotation state in an inherently chaotic evolution. In Figs. 4a and 4b we plot two separate period 4 close return orbits occurring during this 8.5 month interval. In these figures, the pentagons correspond to nightly "observations", the dotted lines to the full numerical trajectories from which we extracted the nightly points. The similarity of these two trajectories is obvious, as well as the fact that the orbits do not exactly close on themselves in four orbital periods, but come very close (hence the term "close return"). Figures 3b and 3c are similar to 3a; their quantitative aspects are listed in the two remaining tables. In all of the images the close return features become weaker and of a shorter duration as the time increases (i.e., as we go down the y-axis). This is due to the fact that the close return behavior we are studying occurs for only short times. Thus the study of close returns in a chaotic data set is ideally suited to observational data files of a reasonable length; in the particular case of Hyperion where the period of the forcing is on the order of 21 days a data file corresponding to 2.6 years was adequate to observe a reasonable number of close returns.

To date, the most ambitious observational study of Hyperion is that performed by Klavetter, who collected 63 points covering approximately 3.5 orbital periods of the satellite. His attempts to fit these data to periodic rotation ranging between 2 hours and 30 days all exhibited large deviations. Klavetter further analysed the lightcurve by finding the best numerical fit to the initial conditions of the full 6-dimensional nonlinear system. This involved a tremendous amount of computer time resulting in a well determined point in the 6-dimensional phase space. It is certainly worthwhile pinpointing the region of phase space in which the dynamics of Hyperion live. With a longer data set of the same sampling rate, the close return analysis can be performed, yielding the unstable periodic orbits underlying the chaotic rotation in addition to the region of phase space in which the dynamics occurs.

We note that this method is in no way restricted to the study of the rotation state of celestial bodies. In fact, numerous astronomical data sets exist for which a better understanding of the time evolution is desired. These include climate records, sunspot number, cataclysmic variable light curves, as well as observations of perhaps "multiply-periodic" pulsating variables, i.e., δ Scuti stars.

5 Conclusions

We have presented a method for analyzing fairly small data sets on the order of 2.6 years of nightly observations of the satellite Hyperion. This method introduces two additional observables in an observational data set, namely the periods of the major close returns and the duration of these features. The power of this method is that with a relatively small number of points it yields information about the unstable periodic orbits embedded within a chaotic time series. These orbits are not only the "skeletons" of the chaotic solution, they also allow us to test different theoretical models against the observational evidence by studying the spectrum of orbits present in the chaotic data sets. The study of the unstable periodic orbits as a tool for the modelling of dynamical systems has been shown to be successful when applied to dissipative systems (Mindlin *et al.* 1991). Our results indicate the applicability of these techniques to small data sets arising in Hamiltonian systems as well.

We have used four numerical examples of well sampled observations which all show the ubiquitous nature of close return episodes for the 2+1 dimensional reduced model of Hyperion. It should be mentioned that close returns in the full 6-dimensional system imply close returns in any lower dimensional subset of the phase space, since the 6-dimensional vector describing the phase space "distance" of two points is minimized only when all of its components are likewise minimized. We also note that a two-dimensional close return image of the type shown in Fig. 3 will always result from this analysis, for a system of arbitrary dimension, since the image is the comparison of a scalar distance between two points in an n-dimensional phase space.

We foresee two independent directions for the continuation of this work. The first is the analysis of the full 6-dimensional problem with the techniques outlined in this work. As mentioned previously, the method of close returns for finding unstable periodic orbits will be successful and will yield insight into the nearly periodic states of the 6-dimensional motion that take place during the chaotic evolution.

Another useful direction is a topological analysis of the spin-orbit coupling problem for a generic non-spherical satellite well-approximated by an ellipsoid, which is a 2+1 dimensional Hamiltonian system. In this case, the unstable orbits are closed curves embedded in a three dimensional phase space, and will therefore be knotted in a particular way. The topological description of these knots gives insight into the mechanism underlying the transition to chaos in the dynamical system under study. To date, this method has never been implemented in the study of Hamiltonian systems although it has proven fruitful in the study of dissipative systems (Mindlin *et al.* 1991). The close return analysis has the further advantage that it clearly distinguishes between chaotic and stochastic processes, the former giving rise to horizontal segments, the latter to a nearly uniform distribution of accidental close returns. It may be fruitful to apply these techniques to astrophysical data sets for which the periodicity as well as the underlying dynamics are not as yet determined.

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8 Figure Captions

Figure 1. Schematic diagram showing the important angular variables in the spin-orbit coupling of Hyperion. The ellipsoid is assumed to be tidally evolved such that the two remaining dynamical variables are θ and θ'. The periodicity of f provides the forcing for the system.

Figure 2. Poincaré section for the 2 dimensional model of Hyperion. a) This section was generated by collecting at each periastron passage θ and θ' at 200 passages for each of 111 initial conditions. Note the large chaotic zone for a body such as Hyperion with large asphericity. b) This section is the result of plotting 40 hypothetical observations at periastron passage. Only the slightest resemblence to Fig. 2a can be detected from this plot.

Figure 3. Close return maps for four separate initial conditions of the hyperion system. The body parameters are fixed and the initial value of f is taken as zero in each experiment. The initial conditions are a) θ=1.0,θ'=0.5; b) θ=0.7,θ'=1.0; c) θ=0.5,θ'=0.5; and d) $theta = 2.0, \theta' = 2.0$. The horizontal axis is the position of a point in the data file (i) that is compared with another point in the data file separated by p nights, with p plotted vertically downward. If the phase space distance is less than 10 percent of the maximum we plot a black point at (i, p).

Figure 4. Two period 4 orbit segments corresponding to the long lived close return in Figure 3a. The pentagons denote data points corresponding to nightly observations.

The dotted line shows the full integration. These segments do not quite close up.

$Parameter$	$Value$	$Reference$
$Body\, Axes\, a\, b\, c$	$185 \times 140 \times 113\ \pm 10km$	$Thomas\, and\, Veverka\, (1985)$
$Semimajor\, axis$	$24.55\, R_{Saturn}$	$Woltjer\, (1928)$
$Orbital\, Period$	$21.277\, days$	$Woltjer\, (1928)$
$Eccentricity$	0.1042	$Woltjer\, (1928)$

Table 1: From Klavetter (1989b)

$Period\,of\,Close\,Return$	3	7	4	3	4	2
$Duration\,(Days)$	37	154	66	66	254	57
$Starting\,Position$	125	215	221	309	367	737
$Comments$		$\approx 5\,m$			$\approx 8.5\,m$	

Table 2: See Fig. 3.a

$Period\,of\,Close\,Return$	4	4	3	2
$Duration\,(Days)$	40	94	56	55
$Starting\,Position$	1	235	264	680
$Comments$		$\approx 3\,m$		

Table 3: See Fig. 3.b

$Period\,of\,Close\,Return$	4	3	4	2	6	3
$Duration\,(Days)$	111	49	66	99	40	25
$Starting\,Position$	14	247	406	449	587	627
$Comments$	$\approx 3.5\,m$		$\approx 3.5\,m$			

Table 4: See Fig. 3.c