Replacing the Axioms for Connecting Lines and Intersection Points by Two Single Axioms [1]

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1 Introduction

In 1995, Jan von Plato introduced the axiomatization of constructive apartness geometry [1], which makes possible the automation of constructive geometry theorem proving. von Plato's axiomatization includes 14 axioms. Using our natural deduction automated theorem prover ANDP [2], we proved that 4 axioms of von Plato's 14 axioms can be replaced by 2 new axioms. As von Plato has noted [3], the new axioms are "more stylish in avoiding negations."

We also used ANDP to prove the equivalence for intuitionistic logic of the two axiomatizations.

2 von Plato's Axioms

von Plato's axioms can be organized in four groups. We use & and | to stand for $\wedge$ and $\vee$, respectively, and A for universal quantifier.

Axiom group 1: Apartness axioms for distinct points, distinct lines, and convergent lines

```
1.1 (Ax)~DIPT x x
1.2  (Ax)~DILN x x
1.3  (Ax)~CON x x
1.4 (Ax)(Ay)(Az)[DIPT x y -> [DIPT x z | DIPT y z]]
1.5  (Ax)(Ay)(Az)[DILN x y -> [DILN x z | DILN y z]]
1.6  (AX)(Ay)(Az)[CON  x y ->[CON  x z  | CON y z]]
```

Axiom group 2: Axioms for connecting lines and intersection points

```
2.1 (Ax)(Ay)[DIPT x y -> ~APT x [ln x y]]
2.2 (Ax)(Ay)[DIPT x y -> ~APT y [ln x y]]
2.3 (Ax)(Ay)[CON x y -> ~APT [pt x y] x]]
2.4 (Ax)(Ay)[CON x y -> ~APT [pt x y] y]]
```

Axiom group 3: Constructive uniqueness axiom for lines and points

[1] This article appears in the *Association for Automated Reasoning Newsletter* No. 37 (August 1997), http://www.mcs.anl.gov/AAR/37aar.html.

```
Axiom 3 (Ax)(Ay)(Au) (Av)[DIPT x y & DILN u v
     -> [[APT x u | APT x v] | [APT y u | APT y v]]]
```

Axiom group 4: Compatibility of equality with apartness and convergence

```
4.1  (Ax)(Ay)(Az)[APT x y ->[ DIPT x z | APT z y]]
4.2  (Ax)(Ay)(Az)[APT x y ->[ DILN y z | APT x z]]
4.3  (Ax)(Ay)(Az)[CON x y ->[ DILN y z | CON x z]]
```

$DIPT\ x\ y$ means that x and y are distinct points.

$DILN\ x\ y$ means that x and y are distinct lines.

$CON\ x\ y$ means that x and y are convergent lines.

$APT\ x\ y$ means that point x is apart from line y.

$ln\ x\ y$ is the connecting line of points x and y.

$pt\ x\ y$ is the intersection point of line x and line y.

3 A New Axiomatization

We use the following Axiom 2a* to replace von Plato's Axioms 2.1 and 2.2.

```
Axiom 2a*: (Ax)(Ay)(Az)[DIPT x y -> [APT z [ln x y]
           -> DIPT z x & DIPT z y]]
```

Axiom 2a* means that if x and y are distinct points, and if point z is apart from the line connecting points x and y, then z and x and z and y are distinct.

We use the following Axiom 2b* to replace von Plato's Axioms 2.3 and 2.4.

```
Axiom 2b*: (Ax)(Ay)(Az)[CON x y -> [[APT z x |  APT z y]
           -> DIPT z [pt x y]]]
```

Axiom 2b* means that if x and y are convergent lines, and if point z is apart from x or y, then points z and [pt xy] are distinct.

Thus, we obtain a new axiomatization of the apartness geometry. The new axiomatization consists of the following 12 axioms:

```
Axiom group 1: von Plato's axiom group 1
Axiom group 2: Axioms 2a* and 2b*
Axiom group 3: von Plato's Axiom 3
Axiom group 4: von Plato's axiom group 4
```

3.1 Proof of the Relative Equivalence for Intuitionistic Logic

We can prove the relative equivalence <=> when replacing von Plato's Axioms 2.1 and 2.2 by Axiom 2a*. Specifically, we need to prove that von Plato's axiomatization <=> von Plato's axiom groups 1, 3, and 4 and Axioms 2.3, 2.4, and Axiom 2a*. For direction =>, we need only to infer Axiom $2a*$. The proof of Axiom 2a* is in Appendix 1. For direction <=, we need only to infer Axiom 2.1 and 2.2, respectively. The proofs are omitted.

The time used and lengths of proofs are as follows.

Axiom 2a*	207 seconds	27 steps
Axiom 2.1	10 seconds	16 steps
Axiom 2.2	2 seconds	16 steps

We can also prove the relative equivalence when replacing von Plato's Axioms 2.3 and 2.4 by Axiom 2b*. We need to prove that von Plato's axiomatization <=> von Plato's axiom groups 1, 3, and 4, Axioms 2.1 and 2.2, and Axiom 2b*. For direction =>, we need only to infer Axiom 2b*. The proof of Axiom 2b* is in Appendix 2. For direction <=, we need only to infer Axiom 2.3 and 2.4, respectively. The proofs are omitted.

The time used and lengths of proofs are follows.

Axiom 2b*	70 seconds	31 steps
Axiom 2.3	2 seconds	16 steps
Axiom 2.4	2 seconds	16 steps

We use only the rules for intuitionistic logic to get the proofs.

3.2 Equivalence of von Plato's Axiomatization and the New Axiomatization for Intuitionistic Logic

We now use ANDP to prove the equivalence of von Plato's axiomatization and the new axiomatization.

The time and the lengths of proofs used to infer Axioms 2a* and 2b* from von Plato's axiomatization are the same as above.

The new axiomatization => von Plato's axiomatization. von Plato's Axioms 2.1, 2.2, 2.3, and 2.4 are inferred from the new axiomatization by using ANDP. The time and the proof lengths are as follows.

Axiom 2.1	10 seconds	16 steps
Axiom 2.2	2 seconds	16 steps
Axiom 2.3	2 seconds	16 steps
Axiom 2.4	2 seconds	16 steps

The proofs of von Plato's Axioms 2.1, 2.2, 2.3, and 2.4 from the new axiomatization are omitted.

Note that we use only the rules for intuitionistic logic to get the proofs.

3.3 Remarks

Recently, we found another formula used to replace von Plato's Axioms 2.1 and 2.2:

```
(AX)(Ay)(Az)[DIPT x y -> [[APT x z | APT y z] -> DILN z [ln x y]]]
```

We can prove the relative equivalence for intuitionistic logic. ANDP required 18 seconds to infer the above formula from von Plato's axiomatization. The proof has 31 steps. ANDP spent 2 seconds to infer Axioms 2.1 and 2.2 from the axiomatization obtained replacing Axioms 2.1 and 2.2 by the formula above, respectively. The proof of each axiom is 16 steps.

Axioms 2.3 and 2.4 can be replaced by another formula:

```
(Ax)(Ay)(Az)[CON x y -> [APT [pt x y] z -> DILN x z & DILN y z]].
```

We can also prove the relative equivalence for intuitionistic logic. ANDP spent 163 seconds to infer the above formula from von Plato's axiomatization. The proof has 39 steps. ANDP spent 25 seconds to infer Axiom 2.3 and 2 seconds for Axiom 2.4. The proofs of each is 16 steps.

4 Conjecture

For classical logic, a shorter formula can equivalently replace von Plato's Axiom 3.

Proof. We use the following Axiom 3* to replace von Plato's Axiom 3.

```
Axiom 3*: (Ax)(Ay)(Az)[DIPT x y -> [ DILN z [ln x y]
          -> APT x z | APT y z]]
```

Axiom 3* means that if x and y are distinct points, and if line z and the line connecting points x and y are distinct, then point x is apart from line z or point y is apart from line z.

When replacing Axiom 3 by Axiom 3$*$, we can use ANDP to prove the equivalence for classical logic. For direction =>, we need to infer Axiom 3$*$ from von Plato's axiomatization. For direction <=, we need to infer von Plato's Axiom 3 from von Plato's axiom group 1, 2, 4 and Axiom 3*. The proof of von Plato's Axiom 3 from von Plato's axiom groups 1, 2, and 4 and Axiom 3* is omitted. The proof in natural deduction style has 43 steps. The proof of Axiom 3$*$ from von Plato's axiomatization is omitted. The proof has 32 steps.

The time used and lengths of proofs are listed as follows.

Axiom 3	70 seconds	43 steps
Axiom 3*	6 seconds	32 steps

The rule to infer $p \vee q$ from $\sim p \rightarrow q$ is used to get the proofs of Axioms 3 and 3*. We note that this rule does not hold for intuitionistic logic. We do not wish to limit the use of some rules, however. We therefore simply conjecture that von Plato's Axiom 3 can be equivalently replaced by Axiom 3* for intuitionistic logic.

Note: The conjecture recently was proved to be right. The constructive proofs were given by hand.

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Appendix 1

ANDP found a 27-step proof in 207 seconds on a SPARCstation 10.

```
1. von Plato's axiomatization                      ASSUMED-PREMISE
2.   (Ax)(Ay)[DIPT x y -> ~APT x [ln x y]]                  SIMP 1
3.   (Ax)(Ay)[DIPT x y -> ~APT y [ln x y]]                  SIMP 1
4.   (Ax)(Ay)(Az)[APT x y -> DIPT x z | APT z y]            SIMP 1
5.   DIPT v1 v2                                    ASSUMED-PREMISE
6.   APT v3 [ln v1 v2]                             ASSUMED-PREMISE
7.   (Ay)(Az)[APT v3 y -> DIPT v3 z | APT z y]        US (v3 x) 4
8.   (Ay)[DIPT v1 y -> ~APT y [ln v1 y]]              US (v1 x) 3
9.   (Ay)[DIPT v1 y -> ~APT v1 [ln v1 y]]             US (v1 x) 2
10.  DIPT v1 v2 -> ~APT v2 [ln v1 v2]                 US (v2 y) 8
11.  ~APT v2 [ln v1 v2]                                    MP 10 5
12.  DIPT v1 v2 -> ~APT v1 [ln v1 v2]                 US (v2 y) 9
13.  ~APT v1 [ln v1 v2]                                    MP 12 5
14.  (Az)[APT v3 [ln v1 v2] -> DIPT v3 z | APT z [ln v1 v2]]
                                                US ([ln v1 v2] y) 7
15.  APT v3 [ln v1 v2] -> DIPT v3 v2 | APT v2 [ln v1 v2]
                                                     US (v2 z) 14
16.  DIPT v3 v2 | APT v2 [ln v1 v2]                        MP 15 6
17.  DIPT v3 v2                                          RDS 16 11
18.  APT v3 [ln v1 v2] -> DIPT v3 v1 | APT v1 [ln v1 v2]
                                                     US (v1 z) 14
19.  DIPT v3 v1 | APT v1 [ln v1 v2]                        MP 18 6
20.  DIPT v3 v1                                          RDS 19 13
21.  DIPT v3 v2                                            SAME 17
22.  DIPT v3 v1                                            SAME 20
23.  DIPT v3 v1 & DIPT v3 v2                            CONJ 21 22
24.  APT v3 [ln v1 v2] -> DIPT v3 v1 & DIPT v3 v2            CP 23
25.  DIPT v1 v2 -> [APT v3 [ln v1 v2]
```

```
 -> DIPT v3 v1 & DIPT v3 v2]                          CP 24
26.  (Ax)(Ay)(Az)[DIPT x y -> [APT z [ln x y]
 -> DIPT z x & DIPT z y]]                             UG 25
27. von Plato's axiomatization
     ->(Ax)(Ay)(Az)[DIPT x y ->[APT z [ln x y]
                    -> DIPT z x & DIPT z y]]          CP 26
```

Appendix 2

ANDP found a 31-step proof in 70 seconds on a SPARCstation 10.

```
1.   von Plato's axiomatization              ASSUMED-PREMISE
2.   (Ax)(Ay)[CON x y -> ~APT [pt x y] x]             SIMP 1
3.   (Ax)(Ay)[CON x y -> ~APT [pt x y] y]             SIMP 1
4.   (Ax)(Ay)(Az)[APT x y -> DIPT x z | APT z y]      SIMP 1
5.   CON v1 v2                               ASSUMED-PREMISE
6.   APT v3 v1 | APT v3 v2                   ASSUMED-PREMISE
7.   (Ay)[CON v1 y -> ~APT [pt v1 y] y]         US (v1 x) 3
8.   (Ay)[CON v1 y -> ~APT [pt v1 y] v1]        US (v1 x) 2
9.   CON v1 v2 -> ~APT [pt v1 v2] v2            US (v2 y) 7
10.  ~APT [pt v1 v2] v2                               MP 9 5
11.  CON v1 v2 -> ~APT [pt v1 v2] v1            US (v2 y) 8
12.  ~APT [pt v1 v2] v1                              MP 11 5
13.  APT v3 v1                                       CASE2 6
14.  APT v3 v2                                       CASE1 6
15.  (Ay)(Az)[APT v3 y -> DIPT v3 z | APT z y]   US (v3 x) 4
16.  (Az)[APT v3 v2 -> DIPT v3 z | APT z v2]    US (v2 y) 15
17.  APT v3 v2 -> DIPT v3 [pt v1 v2] | APT [pt v1 v2] v2
                                        US ([pt v1 v2] z) 16
18.  DIPT v3 [pt v1 v2] | APT [pt v1 v2] v2        MP 17 14
19.  DIPT v3 [pt v1 v2]                           RDS 18 10
20.  (Ay)(Az)[APT v3 y -> DIPT v3 z | APT z y]   US (v3 x) 4
21.  (Az)[APT v3 v1 -> DIPT v3 z | APT z v1]    US (v1 y) 20
22.  APT v3 v1 -> DIPT v3 [pt v1 v2] | APT [pt v1 v2] v1
                                        US ([pt v1 v2] z) 21
23.  DIPT v3 [pt v1 v2] | APT [pt v1 v2] v1        MP 22 13
24.  DIPT v3 [pt v1 v2]                           RDS 23 12
25.  DIPT v3 [pt v1 v2]                              SAME 19
26.  DIPT v3 [pt v1 v2]                              SAME 24
27.  DIPT v3 [pt v1 v2]                       CASES 6 26 25
28.  APT v3 v1 | APT v3 v2 -> DIPT v3 [pt v1 v2]       CP 27
29.  CON v1 v2 -> [APT v3 v1 | APT v3 v2
                  -> DIPT v3 [pt v1 v2]]               CP 28
```

```
30.  (Ax)(Ay)(Az)[CON x y -> [APT z x | APT z y
 -> DIPT z [pt x y]]]                                UG 29
31.  von Plato's axiomatization
     -> (Ax)(Ay)(Az)[CON x y -> [[APT z x | APT z y]
                    -> DIPT z [pt x y]]]             CP 30
```

References

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