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A Shorter and Intuitive Axiom to Replace the Third Axiom of Compatibility of Equality with Apartness and Convergence

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1 Introduction

In the 1920s, A. Heyting attempted to axiomatize constructive geometry [1]. Recently, von Plato used different concepts to axiomatize constructive geometry [2]. Specifically, he used point apartness instead of point equality, line apartness instead of line equality, apartness of a point from a line instead incidence of a point with a line, and convergence of two lines instead of parallelism of two lines. Hence, he calls the geometry *apartness geometry* rather than incidence geometry, as it is known in classical terminology. von Plato's axiomatization of apartness geometry consists of 14 axioms (see *AAR Newsletter 37*, August 1997). It makes possible the automation of *constructive* geometry theorem proving.

In [3] we reported results obtained in constructive apartness geometry using our natural deduction automated prover ANDP [4]. In particular, we showed that von Plato's axioms for connecting lines can be replaced by one single axiom; that his axioms for intersection points can be replaced by a single axiom; and that his axiom for constructive uniqueness for lines and points can be replaced by a shorter axiom. We note that people have long been interested in replacing more axioms by fewer axioms, especially by shorter axioms (see, for example, [5–7]).

In this article we report the fact that a shorter and more intuitive formula can be used to constructively replace von Plato's Axiom 4.3, the third axiom of compatibility of equality with apartness and convergence. This result was obtained after many computer experiments using our prover ANDP.

2 Description of ANDP

There are many rules in a natural deduction system adapted from Gentzen system. The rules UG (universal generalization) and EG (existential generalization) are used to introduce quantifiers.

The rules US (universal specialization) and ES (existential specialization) are used to eliminate quantifiers. A natural deduction system including the rules UG, EG, US, and ES and rules for connectives is complete.

In our natural deduction prover ANDP, we have two algorithms to handle quantifiers: one is for introducing quantifiers, the other for eliminating quantifiers. If two formulas can become equal after applying the rules for introducing quantifiers, then our algorithm can find the series of operations UG and EG and decide which occurrences of terms should be universally or existentially generalized [4]. Similarly, if two formulas can become equal after applying the rules for eliminating quantifiers, then our algorithm can find the series of operations US and ES and decide which terms should be used to replace the bound variables without using Skolem functions [4].

3 Experiments

In this section we present results from computer experiments made on a SPARCstation-10 at SUNY-Albany.

3.1 Shortening the Third Axiom

The following formula is the third axiom of compatibility of equality with apartness and convergence. We call it the third axiom, for short.

```
(Ax)(Ay)(Az)[CON x y ->[ DILN y z | CON x z]]
```

It means that for all lines x, y, and z, if lines x y are convergent, then lines y z are different or lines x z are convergent.

We use the following formula to replace the axiom above.

```
(Ax)(Ay)[CON x y -> DILN x y]
```

It means that if lines x y are convergent, they are different.

Clearly, the formula is shorter and more intuitive than the third axiom.

Theorem. *The third axiom of compatibility of equality with apartness and convergence can be replaced constructively by the new formula.*

Proof. Let the new axiomatization be obtained from von Plato's axiomatization for apartness geometry by replacing the third axiom by the new formula above. Let us prove that new axiomatization $\Longleftrightarrow$ von Plato's axiomatization constructively.

For $\Longleftarrow$: It is enough to infer the new formula above from von Plato's axiomatization. It is just Theorem 3.2 in [2].

For $\Longrightarrow$: It is enough to infer the third axiom from the new axiomatization. The mechanical proof of the third axiom from 14 axioms of the new axiomatization as premises is given in Appendix 1. The proof, which is classical, has 34 steps and required 10 CPU-seconds on a SPARCstation-10.

From the proof it is not hard to see that only four axioms of the new axiomatization are used to infer the third axiom. The four axioms are 1.2, 1.3, 1.6, and the new formula.

Motivated by this success, we then removed some of the four axioms to minimize the premises. After several experiments, we obtained a mechanical proof from two axioms as premises. The two axioms are 1.6 and the new formula. The proof, which also is classical, is given in Appendix 2. The proof has 19 steps and required 2 seconds on a SPARCstation 10.

It is easy to transform the classical proof into a constructive proof. We give it as follows.

It is enough to prove $CONlm \rightarrow [DILNmn \vee CONln]$. Assume $CONlm$. Let's prove that $DILNmn \vee CONln$. After eliminating quantifiers, it follows from Axiom 1.6 that $CONlm \rightarrow [CONln \vee CONmn]$. By assumption, we get $CONln \vee CONmn$. If $CONln$, then the conclusion. Assume $CONmn$. From the new formula, we get $CONmn \rightarrow DILNmn$. By the assumption $CONmn$, then $DILNmn$, then the conclusion. Q.E.D.

3.2 Experiments

The following are experiments we did recently to shorten the axioms for apartness geometry.

1. Use the symmetry of *DIPT* to replace Axiom 1.4.

2. Use the symmetry of *DILN* to replace Axiom 1.5.

3. Use the symmetry of *CON* to replace Axiom 1.6.

4. Use the new formula above to replace any one of 14 axioms of von Plato's axiomatization. For example, replace Axiom 1.5 or 1.6.

After many experiments we proved the theorem above.

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Appendix 1

In this appendix we show how one can, from the 14 axioms of the new axiomatization, infer the third axiom of compatibility of equality with apartness and convergence.

The proof has 34 steps. The time consumed on SPARCstation 10 to get the proof is 10 seconds. Only Axioms 1.2, 1.3, 1.6, and the new axiom are used to get the proof.

Solution

```
1.   14 axioms of new axiomatization for apartness geometry
                                                    ASSUMED-PREMISE
2.   (Ax)~DILN x x                                    Ax 1.2 SIMP 1
3.   (Ax)~CON x x                                     Ax 1.3 SIMP 1
4.   (Ax)(Ay)(Az)[CON x y -> CON x z | CON y z]  Ax1.6 SIMP 1
5.   (Ax)(Ay)[CON x y -> DILN x y]                    New Ax SIMP 1
6.   CON v1 v2                                      ASSUMED-PREMISE
7.   ~DILN v2 v3                                    ASSUMED-PREMISE
8.   (Ay)[CON v1 y -> DILN v1 y]                        US (v1 x) 5
9.   (Ay)(Az)[CON v1 y -> CON v1 z | CON y z]    US (v1 x) 4
10.  (Ay)[CON v2 y -> DILN v2 y]                        US (v2 x) 5
11.  CON v1 v1 -> DILN v1 v1                            US (v1 y) 8
12.  ~CON v1 v1                                            CASE2 11
13.  DILN v1 v1                                            CASE1 11
14.  ~CON v1 v1                                         US (v1 x) 3
15.  (Ay)(Az)[CON v2 y -> CON v2 z | CON y z]    US (v2 x) 4
16.  (Az)[CON v1 v2 -> CON v1 z | CON v2 z]      US (v2 y) 9
17.  CON v2 v3 -> DILN v2 v3                           US (v3 y) 10
18.  ~CON v2 v3                                              MT 17 7
19.  CON v1 v2 -> CON v1 v1 | CON v2 v1               US (v1 z) 16
20.  CON v1 v1 | CON v2 v1                                   MP 19 6
21.  CON v2 v1                                              LDS 20 14
22.  (Az)[CON v2 v1 -> CON v2 z | CON v1 z]     US (v1 y) 15
23.  ~DILN v1 v1                                        US (v1 x) 2
24.  CON v2 v1 -> CON v2 v3 | CON v1 v3               US (v3 z) 22
25.  CON v2 v3 | CON v1 v3                                  MP 24 21
26.  CON v1 v3                                             LDS 25 18
27.  CON v1 v3                               ~-ELIMINATION 13 23
28.  CON v1 v3                                               SAME 26
29.  CON v1 v3                                     CASES 11 28 27
30.  ~DILN v2 v3 -> CON v1 v3                                  CP 29
31.  DILN v2 v3 | CON v1 v3                                   IMP 30
32.  CON v1 v2 -> DILN v2 v3 | CON v1 v3                       CP 31
33.  (Ax)(Ay)(Az)[CON x y -> DILN y z | CON x z]        UG 32
34.  14 axioms of new axiomatization for apartness geometry
 -> (Ax)(Ay)(Az)[CON x y -> DILN y z | CON x z]          CP 33
```

Appendix 2

From Axiom 1.6 and the new axiom, we infer the third axiom of compatibility of equality with apartness and convergence.

The proof has 19 steps, and the time consumed on SPARCstation-10 is 2 seconds.

```
Solution


1.   (Ax)(Ay)(Az)[CON x y -> CON x z | CON y z]
     & (Ax)(Ay)[CON x y -> DILN x y]           ASSUMED-PREMISE
2.   (Ax)(Ay)(Az)[CON x y -> CON x z | CON y z]       SIMP 1
3.   (Ax)(Ay)[CON x y -> DILN x y]                    SIMP 1
4.   CON v1 v2                                ASSUMED-PREMISE
5.   ~DILN v2 v3                              ASSUMED-PREMISE
6.   (Ay)(Az)[CON v1 y -> CON v1 z | CON y z]    US (v1 x) 2
7.   (Ay)[CON v2 y -> DILN v2 y]                 US (v2 x) 3
8.   (Az)[CON v1 v2 -> CON v1 z | CON v2 z]      US (v2 y) 6
9.   CON v2 v3 -> DILN v2 v3                     US (v3 y) 7
10.  ~CON v2 v3                                       MT 9 5
11.  CON v1 v2 -> CON v1 v3 | CON v2 v3          US (v3 z) 8
12.  CON v1 v3 | CON v2 v3                          MP 11 4
13.  CON v1 v3                                    RDS 12 10
14.  CON v1 v3                                      SAME 13
15.  ~DILN v2 v3 -> CON v1 v3                         CP 14
16.  DILN v2 v3 | CON v1 v3                          IMP 15
17.  CON v1 v2 -> DILN v2 v3 | CON v1 v3              CP 16
18.  (Ax)(Ay)(Az)[CON x y -> DILN y z | CON x z]      UG 17
19.  (Ax)(Ay)(Az)[CON x y -> CON x z | CON y z]
     & (Ax)(Ay)[CON x y -> DILN x y]
     -> (Ax)(Ay)(Az)[CON x y -> DILN y z | CON x z]     CP 18
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