AOS 571 Midterm Test

25 October 2007

Open book. No collaboration.
All 3 problems have equal weight.

1. Steady-state diffusion with a source

In one dimension, the advection-diffusion equation in steady state and with a source is

$$u\frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x}\left(D\frac{\partial\theta}{\partial x}\right) + S(x, t)\,. \tag{1}$$

Assume that the diffusivity D is constant and $u = 0$ (no advection). Suppose that S simply relaxes the tracer θ back to a specified time-independent distribution $\theta_S(x)$ over a timescale τ_S. That is, $S = -\tau_S^{-1}(\theta - \theta_S)$.

a) What is the length scale L associated with the external parameters D and τ_S?

b) Show that if $\theta_S = 0$ away from $x = 0$, the only symmetrical nontrivial solution that is bounded at infinity is

$$\theta = \begin{cases} \hat{\theta}\exp(x/L)\,, & x \le 0 \\ \hat{\theta}\exp(-x/L)\,, & x \ge 0 \end{cases} \tag{2}$$

c) Now suppose that

$$\theta_S = \begin{cases} \theta_0\,, & |x| \le L_S \\ 0\,, & |x| > L_S \end{cases}. \tag{3}$$

where θ_0 is a constant. If $L_S \ll L$, then the source is sufficiently isolated to allow us to use (2) as the solution everywhere. In this case, find the amplitude $\hat{\theta}$ of the tracer response [hint: integrate (1) from $x = -\infty$ to ∞ and don't forget that $D\partial\theta/\partial x$ is discontinuous at $x = 0$].

2. Isentropic atmosphere

a) Show that an isentropic (constant θ) atmosphere at rest has a constant temperature lapse rate and therefore a finite depth. The atmosphere is an ideal gas.

b) Suppose that a resting atmospheric column has two isentropic layers, with the potential temperature jumping from θ_1 to θ_2 at the level that divides the total mass into equal parts. The θ_2 layer is above the θ_1 layer. Let the temperature and pressure at the bottom of the column be given as T_0 and p_0.

Find these functions for the height, temperature and density: $z(p)$, $T(z)$ and $\rho(z)$.

c) A section of the interface between the two layers is displaced vertically by δz. Assuming the discontinuity in θ is relatively small, show that the buoyancy of the displaced fluid is

$$b = -\frac{g}{\bar{\theta}}(\theta_2 - \theta_1)\,\mathrm{sgn}(\delta z),$$

where $\bar{\theta}$ is the average potential temperature. Describe the time dependence of the interface displacement if the initial velocity is $w = w_0$ (when $\delta z = 0$). Ignore wave radiation, heating and diffusion. How does this compare to a continuously stratified atmosphere?

3. Shallow-water waves

The one-dimensional, inviscid shallow-water equations on an f-plane (constant Coriolis parameter), linearized about a state of rest, are

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = 0$$

$$\frac{\partial h}{\partial t} = -H\frac{\partial u}{\partial x}$$

a) Find time-dependent solutions that are wavelike with wavenumber k. The solutions should be real functions.

b) Show that the linearized equations satisfy

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0,$$

where E and F are the perturbation energy $E = (u^2 + v^2)/2 + gh^2/(2H)$ and energy flux $F = guh$, respectively. Subsequently, verify the "group velocity property", that $\langle F\rangle = c_g\langle E\rangle$, where the angle brackets denote an average over phase.

c) Show that the potential vorticity is $q = \zeta - (f/H)h$, where ζ is the relative vorticity. What is the potential vorticity of the waves? Why?

END OF TEST