Chapter 3, the first on cooling, of lectures to be given at Acc School in Vanderbilt in Jan 99, by R.B.Palmer.

3 Proton Driver

Contents

3.1 Introduction to Emittance

3.1.1 phase space Φ

2D:

$$\Phi_{2D} \ = \ \int_{-\infty}^{+infty} \ dx \ dp_x$$

6D:

$$\Phi_6 \ = \ \int_{-\infty}^{+infty} \ dx \ \ dy \ dz \ dp_x \ dp_y \ dp_z$$

Louisville's Theorem states that if there is no energy loss, but allowing any e.m. fields, static or varying, Φ_6 is conserved. If 2D area is circular, or 6D volume is spherical; i.e. if there are no correlations between the dimensions:

$$\Phi_{2D} \ = \ \pi \ \hat{x} \ \hat{p_x}$$

6D:

$$\Phi_6 \ = \ \pi^3 \ \hat{x} \ \hat{y} \ \hat{z} \ \hat{p_x} \ \hat{p_y} \ \hat{p_z}$$

The above definitions define volumes with defined boundaries, but the concept can be applied to Gaussian distributions too:

$$\Phi_{2D} \ = \ \pi \ \sigma_x \ \sigma_{p_x}$$

6D:

$$\Phi_6 \ = \ \pi^3 \ \sigma_x \ \sigma_y \ \sigma_z \ \sigma_{p_x} \ \sigma_{p_y} \ \sigma_{p_z}$$

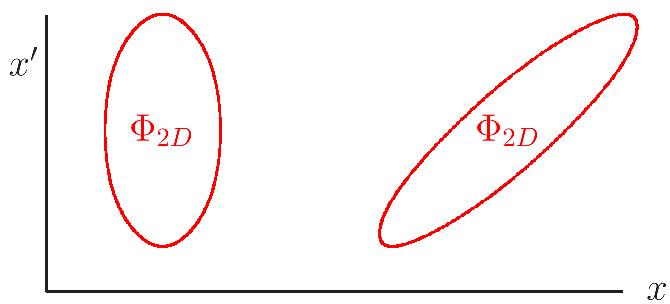
For a beam with $p_x << p_z$ and $p_y << p_z$; using "unnormalized", rms, SLAC Convention:

$$\epsilon_x \ = \ \sqrt{< x'^2 >< x^2 > - < x' \ x >^2}$$

And the same for y.

3.1.2 Phase Diagrams

It is convenient to consider phase spaces in 2 dimensions in plots known as phase diagrams. An uncorrelated distribution represented as an upright ellipse. The boundaries of the ellipse can be taken as the boundaries of volume, or, for Gaussian distributions, as a contour of equal probabilities such that the radii represent the one sigma widths of the Gaussians in the two directions represented. In either case, the area of the ellipse is the "phase space" of the distribution. If there is a first order correlation between the two variables shown, then the ellipse has a tilt, but the area will be the same.



3.1.3 r.m.s. emittance ϵ

e.g. If upright ellipse, i.e. $< x'\ x >= 0$:

$$\epsilon_x \;=\; \sigma'_x\ \sigma_x \;\approx\; \frac{\sigma_p x\ \sigma_x}{p_z}$$

Ideal acceleration increases p_z without changing p_x so ϵ_x will change, even though the phase space is conserved. It is thus convenient to define a "normalized" emittance that is conserved:

$$\epsilon_n \;=\; \beta\gamma\ \epsilon \;=\; \beta\gamma\ \ \sigma'_x\ \sigma_x$$

Note that:

$$\epsilon_n \;=\; \frac{\Phi_{zn}}{\pi\ m}$$

where m is the mass of the particle.

3.1.4 longitudinal ϵ_z and in 6D ϵ_6

Longitudinal phase space is most often quoted as $\Phi_{eV\,sec}$ in units of eV seconds. But we can define a longitudinal emittance:

$$\epsilon_{zn} = \ \beta\gamma\ \frac{\sigma_{pz}}{p_z}\sigma_z$$

$$\epsilon_{zn} \;=\; \frac{\Phi_z}{m}$$

so

$$\epsilon_6 \;=\; \epsilon_{xn}\ \epsilon_{yn}\ \epsilon_{zn}$$

$$\epsilon_6 \;=\; \frac{\Phi_6}{m^3}$$

3.1.5 Containment Probabilities P

For 2D Gaussian distribution:

Contours of equal probability are circles in $X' = x'/\sigma'_x, \quad X = x/\sigma_x$ space.

The probability P that

$$\sqrt{(x'/\sigma'_x)^2 + (x/\sigma_x)^2} \ < \ \Sigma$$

is

$$P \ = \ \int_o^\Sigma \ \Sigma \ e^{\sigma^2/2} \ d\Sigma \ = \ \left(1 - e^{-\Sigma^2/2}\right)$$

For $\Sigma = 1.0$, $P \approx 39.35\%$

For $\Sigma = 4.6$, $P \approx 90\%$

For $\Sigma = 6.0$, $P \approx 95\%$

3.1.6 Varieties of Definitions

For upright ellipses, unnormalized:

- In SLAC and CERN

$$\epsilon_x \;=\; \sigma'_x \,\sigma_x \qquad\qquad (m\ rad)$$

- Sometimes a π will be placed in the units to indicate its lack in the definition:

$$\epsilon_x \;=\; \sigma'_x \,\sigma_x \qquad\qquad (\pi\ m\ rad)$$

- But sometimes a π is included in the4 definition:

$$\epsilon_x \;=\; \pi\,\sigma'_x \,\sigma_x \qquad\qquad (m\ rad)$$

- At Fermilab (eg in Don Edwards Book) the π is included in the calculation, and a factor 6 is added. This definition defines an area that contains approximately 95 % of a Gaussian beam. The "(95%)" may or may not be included.

$$\epsilon_x(95\%) \;=\; 6\pi\,\sigma'_x \,\sigma_x \qquad\qquad (m\ rad)$$

- At BNL the π is usually not included in the calculation, but the factor 6 is added. Thus again defining an area that contains approximately 95 % of a Gaussian beam. The π is included in the dimension.

$$\epsilon_x(95\%) \;=\; 6\,\sigma'_x \,\sigma_x \qquad\qquad (\pi\ m\ rad)$$

- In this text we will use the second definition: i.e. SLAC definition, but with a π placed in the units:

$$\epsilon_x \;=\; \sigma'_x \,\sigma_x \qquad\qquad (\pi\ m\ rad)$$

This is NOT a joke.

3.2 Courant-Snyder β

For a distribution that is an upright ellipse, e.g. at Focus:

$$\beta_\perp = \frac{\sigma_x}{\sigma'_x}$$

and since

$$\epsilon = \sigma_x \sigma_\theta$$

so

$$\beta_\perp = \frac{\sigma_x^2}{\epsilon}$$

3.2.1 Definition

This last expression

$$\beta_\perp = \frac{\sigma_x^2}{\epsilon} \tag{1}$$

is the definition of β, whether the ellipse is upright or not. Note that in a non upright ellipse $\beta_x \neq \sigma_x/\sigma'_x$, but

$$\beta_x = \frac{\sigma_x}{\sigma'_x(x=0)}$$

where $\sigma'_x(x=0)$ is the rms width of the ellipse as it is cut by the x=0 axis. Note the intrinsic assymetry between x' and x in the definition of β

Using the above definition of normalized emittance definition:

$$\sigma_x = \sqrt{\epsilon_n \beta_x \frac{1}{\beta_v \gamma}} \tag{2}$$

and:

$$\sigma_{x'} \;=\; \sqrt{\frac{\epsilon_n}{\beta_x}\,\frac{1}{\beta_v\gamma}} \tag{3}$$

3.2.2 Drift

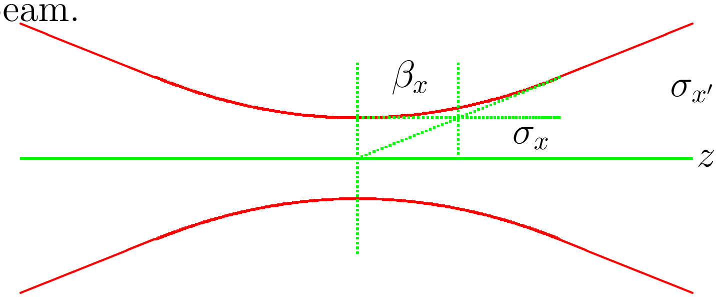
Starting from an uncorrelated (vertical ellipse) distribution, a drift by a distance z will change any x by

$$x_1 \;=\; x_o \;+\; x'\,z$$

for a Gaussian distribution, this will result in an increase of σ_x:

$$\sigma_x(1) \;=\; \sqrt{\sigma_x(o)^2 \;+\; (\sigma_{x'}\,z)^2} \;=\; \sigma_x(1)\,\sqrt{1 \;+\; \frac{z}{\beta_o}}$$

Thus we see that the beam increases in radius by $\sqrt{2}$ in a distance β_o. β_o is thus also the "depth of focus" of the beam.



3.3 Focus Channel

If smooth focusing of strength k':

$$\frac{dx'}{dz} \; - k' \; x$$

then

$$x'' \; = \; - k' \; x$$

whose solution is

$$x \; = \; \sin\left(\chi_o \; + \; \frac{z}{\beta_m}\right)$$

$$x' \; = \; \frac{1}{\beta_m} \; \cos\left(\chi_o \; + \; \frac{z}{\beta_m}\right)$$

where

$$\beta_m \; = \; \sqrt{\frac{1}{k'}}$$

3.3.1 matching

The β_m above is the matched β of a beam that would match into this channel. If a beam is introduced with this β_m then, as the beam drifts along the channel, the β remains constant. If the initial $\beta_0 \neq \beta_n$ then the β of the beam will oscillate about β_n. A subsequent lens can, however, correct such an oscillation.

In general, the focuss strength k' is a function of momentum p ($k' \propto p^2$ for solenoids and alternating quadrupoles; $k' \propto p$ for axial devices like horns, plasma lenses and lithium lenses). Thus $beta_m = f(p)$. If a beam has a mixture of momenta, but for each momentum $\beta = \beta_m(p)$ then each β will remain constant. But if the beam is mismatched, then the β's will oscillate, and each momentum component will oscillate with a different wavelengh. After a distance, the components are almost hopelessly mixed and the emittance is effectively increased. Hense the importance of matching.

3.3.2 Phase advance χ

The phase of a particle in an ensamble in x' and x is defined by

$$\tan(\chi) \;=\; \frac{x'}{x}$$

The change in χ, known as a "phase advance" is given by:

$$\Delta\chi \;=\; \int_o^z \frac{dz}{\beta(z)}$$

In the above simple case of a uniform focus channel, if the beam is matched, the the phase angle is

$$\Delta\chi = \frac{z}{\beta_m} \;=\; z\,\sqrt{k'}$$

3.3.3 Tune ν

In a ring, a"tune" ν (also refered to as Q) is defined by

$$\nu \;=\; \frac{\Delta\chi}{2\pi} \;=\; \oint \frac{\sqrt{k'}\;dz}{2\pi} \qquad (4)$$

for a ring of radius R

$$\nu \;=\; \oint \frac{dz}{2\;\pi\;\beta(z)}$$

$$\nu \;=\; \frac{R}{<\beta>} \;=\; R\;<\sqrt{k'}>$$

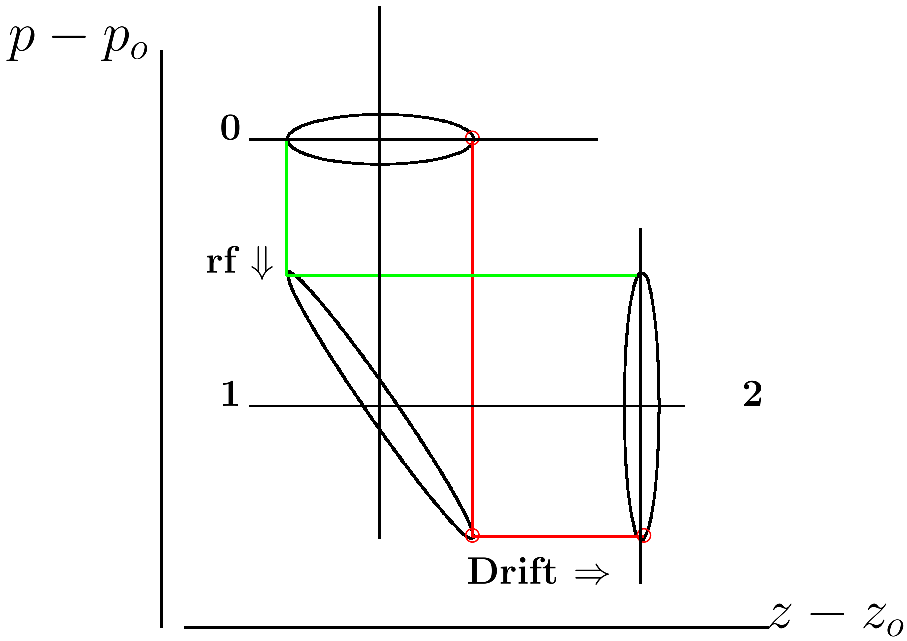
differentiating eq. 4

$$d\nu \;=\; \oint \frac{dk'}{4\pi\;\sqrt{k'}}\;dz$$

substituting back

$$d\nu \;=\; \oint \frac{\beta\;dk'}{4\pi}\;dz \qquad (5)$$

3.4 Bunch Compression



If the particles have velocity βc and $z - z_o$ is the lab distance from the bunch center, and these particles pass through an rf accelerator with maximum acceleration V_o, and if the rf phase is set to leave the bunch center alone and decelerate the front, then their change in energy ΔE will be:

$$\Delta E \;=\; -V_o \;\sin(\omega\ t) \;=\; -V_o \;\sin\left(\frac{\omega}{\beta_v\ c}\ (z - z_o)\right)$$

If $V_o << E_o$ then

$$\frac{\Delta p}{p} \;\approx\; -\frac{V_o}{E\ \beta_v^2}\ \sin\left(\frac{\omega}{\beta_v\ c}\ (z - z_o)\right)$$

If $\sigma_z << c/\omega$, so that the sin function looks linear; and the phase is such as not to accelerate the bunch center

$$\frac{\Delta p}{p} \approx -\frac{V_o\ \omega}{E\ \beta_v^3}\ (z - z_o) \tag{6}$$

This, applied to an upright elliptical partical distribution, increases the momentum spread

$$\sigma_p(1) \approx \sqrt{\sigma_p(0)^2 + \left(\frac{V_o\ \omega}{E\ \beta_v^3}\ \sigma_z\right)^2}$$

and introduces a tilt in the ellipse.

If the beam is now allowed to drift a distance ℓ, then since the higher momentum particles will travel faster than the lower momentum ones, the ellipse will be straightened. If the drift is linear then; ; and since

$$\frac{d\beta}{\beta} = \frac{1}{\gamma^2}\ \frac{dp}{p}$$

$$\Delta z \approx \frac{\Delta p}{p}\ \frac{1}{\gamma^2}\ \ell$$

Or in the more general case including bends:

$$\Delta z = \frac{\Delta p}{p}\ \alpha\ \ell \approx \frac{\Delta p}{p}\ \left(\eta + \frac{1}{\gamma^2}\right)\ \ell \tag{7}$$

where η is the momentum compaction, and α is the isochronicity.

If the appropriate distance is chosen then the ellipse can be made upright again, but the bunch length will now be less:

$$\sigma_z(2) = \sigma_z(0)\ \left(\frac{\sigma_p(0)}{\sigma_p(1)}\right)$$

3.5 Space Charge Tune Shift

3.5.1 Transverse Force on comoving Charge

For test charge at radius r, with a cylindrically symmetric charge per unit length of $Q'(r)$ inside the radius r, there are two terms.

One from the radial electric field

$$\mathcal{E}_{\rm r} = \frac{\rm Q'(r)}{2\pi\epsilon_{\rm o}{\rm r}}$$

and one from the magnetic field:

$$B_\phi = -\mu_o\, Q'(r)\, \frac{\beta_v c}{2\pi r} \;=\; -\frac{1}{\epsilon_o c^2}\, Q'(r)\, \frac{\beta_v c}{2\pi r}$$

giving

$$F_\perp = \frac{Q'(r)}{\epsilon_o}\, \frac{e}{2\pi r}\, (1 \;-\; \beta_v^2) \;=\; \frac{Q'(r)}{\epsilon_o}\, \frac{e}{2\pi r}\, \frac{1}{\gamma^2} \tag{8}$$

The charge density inside r

$$Q'(r) \;=\; Q' \int_o^r \frac{r}{\sigma_r^2}\, e^{-\frac{r^2}{2\sigma_r^2}} \;=\; Q' \left(1 \;-\; e^{-\frac{r^2}{2\sigma_r^2}}\right)$$

for $r << \sigma_r$

$$Q'(r) \;\approx\; \frac{r^2}{2\sigma^2}\, Q'$$

Thus

$$F_\perp \;=\; r\, \frac{e\, Q'}{4\pi\epsilon_o\, \gamma^2\sigma^2}$$

This force is defocussing (proportional to r and outward) and gives an additional focussing strength dk' in the x direction of:

$$dk' \;=\; -\frac{F_\perp}{p\ \beta_v c\ x} \;=\; -\frac{F_\perp}{\gamma m p\ \beta_v^2\ c^2\ x}$$

and the same for y. Noting that the clasical radius r_o is

$$r_o = \frac{e^2}{4\pi\epsilon_o mc^2}$$

and defining $N' = Q'/e$, we get

$$dk' = \frac{N'\ r_o\ 1}{\gamma^3\beta_v^2\sigma_r^2}$$

From eq. 5

$$d\nu = \oint \frac{\beta\ dk'}{4\pi}\ dz$$

so

$$d\nu = \oint \frac{\beta}{4\pi}\ \frac{N'\ r_o\ 1}{\gamma^3\beta_v^2\sigma_r^2}\ dz$$

and noting that

$$\sigma_r^2 = \frac{\epsilon_n\ \beta}{\beta_v\gamma}$$

$$d\nu = \oint \frac{\beta}{4\pi}\ \frac{N'\ r_o\ 1}{\gamma^3\beta_v^2}\ \frac{\beta_v\ \gamma}{\epsilon_n\ \beta}\ dz$$

$$d\nu = \frac{1}{4\pi}\ \frac{N'\ r_o\ 1}{\gamma^2\beta_v\ epsilon_n}\ \oint\ dz$$

$$\Delta\nu = N'\ \frac{R\ r_o}{2\gamma^2\beta_v\epsilon_n} \tag{9}$$

and in the center of a bunch of rms length σ_z, with N charges:

$$\Delta\nu = \frac{N}{\sqrt{2\pi}\ \sigma_z}\ \frac{R\ r_o}{2\gamma^2\beta_v\epsilon_n} \tag{10}$$