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TENSOR ANALYSIS AND RIGID BODY KINEMATICS

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INTRODUCTION

The topic of rigid body kinematics is an ancient one and many treatments of the subject are available. Although these treatments differ in detail, they generally have two common factors. First of all they rely heavily upon Cartesian orthogonal coordinates for the analyses. This reliance upon Cartesian coordinates is no real handicap since in practice one invariably assumes the space to be Euclidean. However, it should be possible to recover all the usual results for a Euclidean space covered by Cartesian orthogonal coordinates as a special case of a more general analysis valid in a Riemannian space covered by an arbitrary coordinate system. The second common factor which characterizes previous treatments was that they assumed an intuitive knowledge of rigid body motions. That is, they operationally defined rather than deduced, the nature of rigid body motions. Thus, for example, Whittaker [1] defines translations as those displacements for which the final and initial positions of all points of a rigid body can be connected by parallel straight lines while rotations about a line are those displacements which leave unchanged the coordinates of points that lie along the straight line.

With the operational definition of translations and the use of Cartesian orthogonal coordinates the analysis of translational motion becomes trivial and detailed mathematical treatment is reserved for rotational motion. Currently

[redacted]

the most popular approach to the study of rotations is through the mathematical machinery of orthogonal transformations between Cartesian orthogonal coordinates. See for example the treatments given by Goldstein [2] and Mayer [3]. Orthogonal transformations leave the form and the value of the expression for distance between two points invariant and thus they qualify as mapping of rigid body motions. Synge and Schild [4] discuss rotations in an n-dimensional Euclidean space covered by a Cartesian orthogonal coordinate system. Their formulation is based on Cartesian tensors and represents a generalization of the three-dimensional case.

In this paper tensor analysis will be used to discuss rigid body kinematics and a knowledge of tensor analysis equivalent to that obtainable from the references [4], [5], and [6] will be presumed. The general discussion will be valid in an n-dimensional Riemannian space; however, illustrations of the general results will be for a Euclidean space. The discussion will be based upon the fact that distances between neighboring points of a rigid body are invariant during the motion.

The use of tensor analysis in the discussion of rigid body kinematics has the expected advantage that the results are presented in a form suitable for use in any coordinate system. However, other benefits accrue from the use of tensor analysis. First, it makes possible a general discussion of moving coordinate systems and the components of the time derivatives of vectors and tensors with respect to these coordinates. Second, it makes possible a discussion of those constants of the motion known as the Cartesian components of the linear momentum and angular momentum of a closed system, and shows that these are properly regarded as a set of scalars rather than components of vectors.

We conclude the introduction with a brief word regarding the notation to be employed in subsequent sections of the paper. We will use the summation convention in the form "a repeated index in a subscript and a superscript implies a summation over the index." This form of the summation convention does not imply summation when indices are repeated only as subscripts or only as superscripts. Arbitrary coordinates will be denoted by x^k while y^k will be used exclusively for Cartesian orthogonal coordinates. The Christoffel symbol of the second kind will be written as $\{^{\,i}_{jk}\}$ while the covariant derivative will be symbolized by ∇_j.

RIGID BODY MOTIONS

The following two sections will consider the topic of rigid body motions in a very general way. In the first section we will consider what conditions must be placed on an arbitrary motion in order that it qualify as the motion of a rigid body. The second section will then identify certain types of these motions with rotations and translations. This identification will then immediately lead to the separation of arbitrary, infinitesimal rigid body motions into rotations and translations.

Conditions for Rigid Body Motions

Consider an n-dimensional space covered by some coordinate systems with coordinates $x^1, x^2, x^3, \cdot \cdot \cdot \cdot x^n$. Assume that there is an essentially continuous distributions of particles in this space and examine the displacements which these particles undergo. Displacements are considered to a mapping M of the space onto itself such that the particle with initial coordinates x^k_o is taken continuously to a position where its coordinates are x^k as shown in fig. 1.

$$M: \qquad x^k = x^k(x_o^j, t) \tag{1}$$

The variable t is some continuous parameter of the mapping such that at $t = t_o$, the mapping reduces to the identity mapping.

$$x^k(x_o, t_o) = x_o^k \tag{2}$$

Two points x_o^k and $x_o^k + dx_o^k$ which are neighboring points at $t = t_o$ will be mapped into x^k and $x^k + dx^k$, respectively. The relationship of the displacements dx^k and dx_o^k is obtained from the mapping M since

$$x^k + dx^k \equiv x^k(x_o^j + dx_o^j, t) = x^k(x_o^j, t) + \frac{\partial x^k}{\partial x_o^j} dx_o^j .$$

and therefore

$$dx^k = \frac{\partial x^k}{\partial x_o^j} dx_o^j \tag{3}$$

If the positive definite metric tensor of the space is $g_{ij}(x)$, then the square of the lengths of the two displacements are

$$\begin{aligned} (ds_o)^2 &= \underset{o}{g}_{ij}\, dx_o^i\, dx_o^j \\ (ds)^2 &= g_{ij}\, dx^i\, dx^j \end{aligned} \tag{4}$$

where $\underset{o}{g}_{ij}$ is the metric tensor evaluated at x_o^k. Combining (3) and (4) we can write for the change in the square of the lengths

$$(ds)^2 - (ds_o)^2 = 2\eta_{ij}\, dx_o^i\, dx_o^j \tag{5}$$

where

$$\eta_{ij}(x_o, t) \equiv \frac{1}{2}\left(\frac{\partial x^k}{\partial x_o^i} \frac{\partial x^l}{\partial x_o^j} g_{kl} - \underset{o}{g}_{ij}\right)$$

is the Lagrangean strain tensor [5]. The displacements dx_o^k are arbitrary, and the condition that distances remain invariant for rigid body displacements is equivalent to the requirement that the strain tensor vanish. This can be written as

$$\underset{o}{g}_{ij} = \frac{\partial x^k}{\partial x_o^i}\frac{\partial x^l}{\partial x_o^j} g_{kl} \tag{6}$$

It should be noted that (6) has the appearance of the transformation rule for the covariant components of a second order tensor, however, we are considering here a mapping of the space onto itself and not a coordinate transformation. That is, (6) is to be interpreted as a condition on the mapping (1) assuming both $\underset{o}{g}_{ij}$ and g_{ij} to be known rather than as an expression defining $\underset{o}{g}_{ij}$ in terms of g_{ij} as is the case when discussing coordinate transformations.

The condition (6) will not be satisfied by some arbitrary mapping M. To make further progress we will consider only infinitesimal displacements, that is, displacements which take place when the parameter t changes from t_o to $t_o + dt$. In this situation the mapping differs only infinitesimally from the identity mapping. From expansion of the mapping (1) about $t = t_o$ we obtain

$$M_1: \qquad x^k = x_o^k + \underset{o}{V}^k\, dt \tag{7}$$

where

$$\underset{o}{V}^k(x_o, t_o) \equiv \left.\frac{\partial x^k}{\partial t}\right|_{t=t_o}$$

If the parameter t is the time, then $\underset{o}{V}^k$ would be the velocity field at x_o^k and t_o. With the aid of the infinitesimal mapping (7) we can now evaluate the partial derivatives appearing in (6) in the form

$$\frac{\partial x^k}{\partial x_o^i} = \delta_i^k + \frac{\partial \underset{o}{V}^k}{\partial x_o^i}\, dt \tag{8}$$

Further by a Taylor series expansion we find for infinitesimal displacements

$$g_{kl} = \underset{o}{g}_{kl} + \left.\frac{\partial g_{kl}}{\partial x^m}\right|_{x=x_o} (x^m - x_o^m) \tag{9}$$

Since the covariant derivative of the metric tensor vanishes

$$\nabla_m g_{kl} \equiv \frac{\partial g_{kl}}{\partial x^m} - \left\{ {\alpha \atop km} \right\} g_{\alpha l} - \left\{ {\alpha \atop lm} \right\} g_{k\alpha} = 0 \tag{10}$$

it is possible to express the partial derivative of the metric tensor in terms of itself and the Christoffel symbol of the second kind. Substituting equations (7) to (10) into (6) one obtains

$$\underset{o}{g}_{ij} = \left(\delta_i^k + \frac{\partial \underset{o}{V}^k}{\partial x_o^i} dt\right)\left(\delta_j^l + \frac{\partial \underset{o}{V}^l}{\partial x_o^j} dt\right)\left[\underset{o}{g}_{kl} + \left(\left\{ {\alpha \atop km} \right\}_o \underset{o}{g}_{\alpha l} + \left\{ {\alpha \atop lm} \right\}_o \underset{o}{g}_{k\alpha}\right)\underset{o}{V}^m dt\right]$$

To first order in dt this becomes

$$\left[\underset{o}{g}_{il}\left(\frac{\partial \underset{o}{V}^l}{\partial x_o^j} + \left\{ {l \atop jm} \right\}_o \underset{o}{V}^m\right) + \left(\frac{\partial \underset{o}{V}^k}{\partial x_o^i} + \left\{ {k \atop im} \right\}_o \underset{o}{V}^m\right)\underset{o}{g}_{kj}\right] dt = 0 \tag{11}$$

Since the choice of the point x_o^k and the time t_o is arbitrary the subscript "o" can be dropped from equation (11). The terms in the parenthesis will be recognized as the covariant derivatives of the vector field V^k and since the covariant derivative of the metric tensor vanishes it can be used to lower the index on V^k. Thus we obtain finally the condition which must be satisfied by the vector field V^k in order that the infinitesimal mapping (7) correspond to a rigid body displacement

$$\nabla_j V_i + \nabla_i V_j = 0 \tag{12}$$

This can only be true if V^k is a solution of the equation

$$\nabla_l V_k = \Omega_{lk} \tag{13}$$

where $\Omega_{lk}(x,t)$ are the components of a skew-symmetric second order tensor field. The tensor Ω_{lk} is just the vorticity tensor of fluid mechanics [4], [6]

$$\nabla_l V_k - \nabla_k V_l = 2\Omega_{lk}$$

If the vector field V_k is to be obtained as a solution of the equation (13) for a known Ω_{lk}, then the choice of Ω_{lk} is not completely arbitrary, but must satisfy some conditions. To establish these conditions it will be necessary to use a generalized Stokes' theorem [4] valid in an n-dimensional Riemannian space.

$$\int_{R_M} \frac{\partial T_{k_1 k_2 \cdots k_{M-1}}}{\partial x^{k_M}} \, d\tau_{(M)}^{k_1 k_2 \cdots k_M} = \int_{R_M} \nabla_{k_M} T_{k_1 k_2 \cdots k_{M-1}} \, d\tau_{(M)}^{k_1 \cdots k_M}$$

$$= \int_{R_{M-1}} T_{k_1 k_2 \cdots k_{M-1}} \, d\tau_{(M-1)}^{k_1 \cdots k_{M-1}} \qquad (14)$$

In (14) $T_{k_1 k_2 \cdots k_{M-1}}$ is any tensor of rank $(M - 1)$ where $M \leq n$; $d\tau_{(M)}^{k_1 k_2 \cdots k_M}$ is the extension of an infinitesimal M-cell and is a tensor of rank M, skew symmetric in all its indices, while R_M is an oriented finite region of an M-dimensional subspace of the n-dimensional space which is bounded by the closed $(M - 1)$-space R_{M-1}. The first equality in (14) follows from the fact that the extension is skew-symmetric in all its indices while the partial derivative and the covariant derivative differ by terms which are symmetric in two indices. We will now show that the integral of $\nabla_l V_k$ over a closed subspace R_2 must vanish. This integral can be written as a sum of two integrals each over an open subspace (see fig. 2)

$$\int_{R_2} \nabla_l V_k \, d\tau^{kl}_{(2)} = \int_{D_2} \nabla_l V_k \, d\tau^{kl}_{(2)} + \int_{D_2'} \nabla_l V_k \, d\tau^{kl}_{(2)}$$

where D_2 and D_2' combine to give R_2. By Stokes' theorem both of these integrals can be replaced by line integrals over their bounding curves D_1 and D_1'

$$\int_{R_2} \nabla_l V_k \, d\tau^{kl}_{(2)} = \int_{D_1} V_k \, d\tau^{k}_{(1)} + \int_{D_1'} V_k \, d\tau^{k}_{(1)} \qquad (15)$$

The extension $d\tau^k_{(1)}$ is just dx^k and thus the two integrals are line integrals around closed curves. The orientation of D_2 and D_2' must be such as to agree with the orientation assigned to R_2. This, however, requires that the 1-cell along D_1 have the opposite orientation of the 1-cell along D_1'. Thus the right hand side of (15) is a sum of two line integrals each taken along the same curve but in opposite directions and hence cancel and therefore the right hand side of (15) is zero. But using Stokes' theorem on (13) we can write

$$\int_{R_2} \nabla_l V_k \, d\tau^{kl}_{(2)} = \int_{R_2} \Omega_{lk} \, d\tau^{kl}_{(2)} = \int_{R_3} \nabla_i \Omega_{lk} \, d\tau^{kli}_{(3)}$$

By a cycle permitation of indices in the last integral and using the skew-symmetry of $d\tau^{kli}_{(3)}$ we finally obtain

$$0 = \int_{R_2} \nabla_l V_k \, d\tau^{kl}_{(2)} = \frac{1}{3} \int_{R_3} (\nabla_i \Omega_{lk} + \nabla_l \Omega_{ki} + \nabla_k \Omega_{il}) d\tau^{kli}_{(3)} \qquad (16)$$

Clearly the integral will vanish if we require

$$\nabla_i \Omega_{lk} + \nabla_l \Omega_{ki} + \nabla_k \Omega_{il} = 0 \tag{17}$$

This then supplies us with the desired condition on the tensor Ω_{lk}. As was previously pointed out it is permissible to interchange covariant derivatives and ordinary derivatives in the integrand without invalidating Stokes' theorem. Thus (17) can be replaced by the equivalent requirement

$$\frac{\partial\Omega_{lk}}{\partial x^i} + \frac{\partial\Omega_{ki}}{\partial x^l} + \frac{\partial\Omega_{il}}{\partial x^k} = 0 \tag{18}$$

Equation (18) can also be obtained directly from (17) by utilizing the definition of the covariant derivative.

The conditions (17) or (18) are known as the integrability conditions [6] for the set of equations (13), and are precisely the requirements which must be satisfied if one is to be able to solve equations (13). For a three-dimensional space these integrability conditions can be written in a more recognizable form. This is possible because in 3-space all conditions (18) are identically satisfied except

$$\frac{\partial\Omega_{12}}{\partial x^3} + \frac{\partial\Omega_{31}}{\partial x^2} + \frac{\partial\Omega_{23}}{\partial x^1} = 0 \tag{19}$$

Further one can always associate with a skew-symmetric tensor Ω_{lk} an oriented vector Ω^i defined by

$$\Omega^i \equiv \frac{1}{2}\, \epsilon^{ijk}\Omega_{jk} \tag{20}$$

where ϵ^{ijk} is the oriented third order tensor defined in terms of the permitation symbol e^{ijk} and the determinant of the metric tensor g as

$$\epsilon^{ijk} \equiv \frac{e^{ijk}}{\sqrt{g}} \tag{21}$$

The expression (20) can be solved for Ω_{lm} by multiplying by the covariant components $\epsilon_{lmi} = \sqrt{g}\, e_{lmi}$ and using the expansion in terms of the Kronecker tensor δ^i_j

$$\epsilon^{jki}\epsilon_{lmi} = \delta^j_l\delta^k_m - \delta^j_m\delta^k_l$$

to obtain

$$\Omega_{lm} = \epsilon_{lmi}\Omega^i \tag{22}$$

Using (22) the integrability condition (19) can be written in terms of the oriented vector Ω^i as

$$\frac{\partial(\sqrt{g}\,\Omega^3)}{\partial x^3} + \frac{\partial(\sqrt{g}\,\Omega^2)}{\partial x^2} + \frac{\partial(\sqrt{g}\,\Omega^1)}{\partial x^1} = \sqrt{g}\,\nabla_k\,\Omega^k = 0 \tag{23}$$

Thus in 3-space if the determinant g of the metric tensor does not vanish then the integrability conditions reduce to the requirement that the oriented tensor Ω^k have a vanishing divergence.

It should be noted that our initial approach to rigid body motion was Lagrangian in nature, as typified by the mapping (1), and further that we defined the vector field V^k in a Lagrangian manner. However in passing to the infinitesimal case we succeeded in changing our point of view to Eulerian. This is analogous to the situation in fluid mechanics where the velocity field is defined by a consideration of the motion of individual fluid particles, and thereafter the Eulerian approach is adopted. The Lagrangian viewpoint can be regained by solving (13) for $V_k(x,t)$ and then integrating the equations

$$\frac{dx^l}{dt} = V^l(x,t) = g^{lk}V_k \tag{24}$$

to obtain the corresponding mapping (1). The initial coordinates x_o^k are introduced as initial values in the solution of (24). We will carry out this procedure, in a subsequent section, for the simple case of uniform rotation and translation in a Euclidean 3-space.

Definitions of Rotations and Translations

The vector field V^k, which defines a rigid body displacement, must satisfy the inhomogeneous partial differential equations (13) where the choice of the skew-symmetric tensor Ω_{lk} is arbitrary except that it must satisfy the integrability conditions (17). Now to any particular solution of (13) we can always add a solution of the homogeneous problem obtained by setting Ω_{lk} equal to zero. We thus conclude that any solution to equations (13) can be written as

$$V_k = U_k + u_k \tag{25a}$$

where

$$\nabla_l U_k = \Omega_{lk} \tag{25b}$$

and

$$\nabla_l u_k = 0 \tag{25c}$$

We will identify U_k as corresponding to infinitesimal rotations while u_k will be identified with infinitesimal translations. The identification is motivated by the fact that u_k satisfies the condition for a parallel vector field [6]. If the identifications of U_k and u_k are accepted, then (25) is the infinitesimal version of Chasles' theorem [2], which refers to the decomposition of a general displacement into rotations and translations.

ROTATIONS IN A EUCLIDEAN SPACE

In the following three sections we will specialize our general results to a Euclidean space. In particular it will be shown that this specialization leads to the usual relationships for rigid body rotations in a three-dimensional Euclidean space.

Analytical Expressions for the Vector Field V_k

Since we have assumed the space to be Euclidean we can introduce a set of Cartesian orthogonal coordinates y^k. In this coordinate system the metric tensor is just the Kronecker delta

$$g_{ij}(y) = \delta_{ij}$$

and all distinction between covariant and contravariant components vanishes. However raised and lowered indices will be retained to permit retention of the summation convention. Also in this coordinate system the Christoffel symbols vanish identically and thus covariant differentiation and ordinary differentiation become identical.

To distinguish the components of the rotation tensor Ω_{lk} in some arbitrary coordinate system from its components in a Cartesian orthogonal coordinate system, the latter will be denoted by ω_{lk}. With this convention the equations (13) and the integrability conditions (18) become

$$\frac{\partial V_k}{\partial y^l} = \omega_{lk} \tag{26}$$

$$\frac{\partial \omega_{lk}}{\partial y^i} + \frac{\partial \omega_{ki}}{\partial y^l} + \frac{\partial \omega_{il}}{\partial y^k} = 0 \tag{27}$$

In general it is to be expected that V_k will be a function of all the coordinates and the parameter t which is disregarded for the moment.

$$V_k = V_k(y^1, y^2, \cdots y^n) \tag{28}$$

However from (26) with $(l = k)$ we have by virtue of the skew-symmetry of ω_{lk}

$$\frac{\partial V_k}{\partial y^k} = 0 \qquad (k \text{ not summed}) \tag{29}$$

Thus V_k cannot be a function of y^k. And thus its derivative cannot be a function of y^k

$$\frac{\partial V_k}{\partial y^l} \equiv f_{lk}(y^i, i \neq k) = \omega_{lk} \tag{30}$$

This implies that ω_{lk} cannot be taken as a function of y^k. Interchanging the roles of l and k in (30) shows that ω_{kl} cannot be a function of y^l. However ω_{lk} and ω_{kl} differ only in sign and thus ω_{kl} can be neither a function of y^k nor y^l.

$$\omega_{lk} = \omega_{lk}(y^i, i \neq l, k) \tag{31}$$

The integrability conditions (27) impose additional constraints on the functional form of ω_{lk}; in fact, these conditions are such that ω_{lk} cannot depend upon any coordinates. To see this let us focus attention on the component ω_{12} by writing out equations (27) in detail for the cases $l = 1$, $k = 2$, $i = 3, 4, 5 \cdots n$

$$- \frac{\partial\omega_{12}}{\partial y^3} = \frac{\partial\omega_{23}}{\partial y^1} + \frac{\partial\omega_{31}}{\partial y^2} = 0$$

$$- \frac{\partial\omega_{12}}{\partial y^4} = \frac{\partial\omega_{24}}{\partial y^1} + \frac{\partial\omega_{41}}{\partial y^2} = 0 \qquad (32a,\ b,\ c,\ \cdots\ )$$

$$- \frac{\partial\omega_{12}}{\partial y^5} = \frac{\partial\omega_{25}}{\partial y^1} + \frac{\partial\omega_{51}}{\partial y^2} = 0$$

. . . .

From (31) it can be seen that ω_{12} and all its derivatives may contain only the variables $y^3, y^4, \cdots y^n$. However, by virtue of (31), the first term on the right hand side of (32a) can be a function of $y^1, y^4 y^5 \cdots y^n$ while the next term may contain the variables $y^2, y^4, y^5 \cdots y^n$. Thus the left hand side contains the variable y^3 which does not appear on the right while conversely the variables y^1 and y^2 appear on the right but not on the left. If (32a) is to be valid for an arbitrary point in the space then ω_{12} may at most be linear in the variable y^3 while ω_{23} must be linear in y^1 and ω_{31} linear in y^2. Similarly (32b) implies that ω_{12} must be linear in y^4 while (32c) implies linearity in y^5. Considering the remaining equations of (32) shows that ω_{12} must be a linear function of the coordinates if it is to contain them at all. Considering other components of ω_{lk} in like fashion we reach the conclusion that ω_{lk} can be, at most, a linear function of the coordinate variables. Alternately this can be expressed by saying that V_k can be no more than quadratic in the coordinate variables. Therefore the most general form of V_k can be written as

$$V_k = \frac{1}{2} A_{klj} y^l y^j + B_{kl} y^l + C_k \qquad (33)$$

where A_{klj}, B_{kl} and C_k are independent of the coordinate. We point out at this juncture that the use of indices on A, B and C is not to be construed as implying some tensor character to these quantities but is only a convenient notation to indicate summations. With no loss in generality we can assume A_{klj} to be symmetric in the last two indices

$$A_{klj} = A_{kjl} \tag{34}$$

Differentiation of (33) gives

$$\omega_{ik} = \frac{\partial V_k}{\partial y^i} = A_{kil}y^l + B_{ki} \tag{35}$$

From the skew-symmetry of ω_{ik} we have

$$\omega_{ik} + \omega_{ki} = (A_{kil} + A_{ikl})y^l + B_{ki} + B_{ik} = 0$$

If this is to be satisfied for an arbitrary choice of y^l then we must have

$$A_{ikl} = -A_{kil} \tag{36}$$

$$B_{ik} = -B_{ki} \tag{37}$$

Equations (34) and (36) give two conditions on the A_{ikl}, while a third can be obtained from the integrability conditions (27). Using (35) the integrability conditions can be written as

$$A_{kli} + A_{ikl} + A_{lik} = 0 \tag{38}$$

However from the symmetry properties (34) and (36) of the A_{kli} we have

$$A_{kli} = -A_{lki} = -A_{lik}$$

Combining this with (38) we find that

$$A_{ikl} = 0$$

Thus the tensor ω_{ik} cannot be a function of the coordinates but can only depend upon the parameter t

$$\omega_{ik} = \omega_{ik}(t) \tag{39}$$

In some coordinate system which is not Cartesian this will not in general be true since we have from the usual transformation rules

$$\Omega_{kl}(x,t) = \frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^l}\,\omega_{ij}(t) \tag{40}$$

Since the A_{ikl} vanish we have from (35)

$$\omega_{ik} = B_{ki} \tag{41}$$

And therefore (33) can be written as

$$V_k = -\omega_{kl}y^l + C_k \tag{42}$$

The derivative, with respect to the coordinate variables, of the first term of (42) gives ω_{lk} while the derivative of the second term vanishes and thus we see by comparison with (25b,c) that the first term is a particular integral of (26) while the second is a solution to the corresponding homogeneous problem. Therefore we have the identification

$$U_k = -\omega_{kl}y^l \tag{43}$$

$$u_k = C_k \tag{44}$$

Using (22) in Cartesian orthogonal coordinates in a three-dimensional space we can write the rotational field (43) in the more familiar form

$$U_k = -e_{klj}\omega^j y^l = e_{kjl}\omega^j y^l \tag{45}$$

And therefore

$$V_k = U_k + u_k = e_{kjl}\omega^j y^l + C_k \tag{46}$$

In the notation of vector analysis (45) is just

$$\underset{\sim}{U} = \underset{\sim}{\omega} \times \underset{\sim}{r}$$

And thus we can write (46) in the form

$$\underset{\sim}{V} = \underset{\approx}{\omega} \times \underset{\sim}{r} + \underset{\sim}{C} \tag{47}$$

It should be noted that although y^l can be considered a vector for the group of centered affine transformations, it is not a vector for some more general transformation. In the latter case either (43) or (45) are only to be regarded as representing the dependence of the vector field U_k upon the Cartesian coordinates y^l. The components in some other coordinate system must be obtained by the usual transformation rules for a vector.

We conclude this section by showing that in a Euclidean 3-space the rotational part of the motion is always perpendicular to the vector Ω^i, that is, $U_k\Omega^k = 0$. Working in Cartesian coordinates we have from (45)

$$U_k\omega^k = e_{kjl}\omega^j y^l \omega^k = -e_{jkl}\omega^k y^l \omega^j = -U_j\omega^j$$

And therefore

$$U_k\omega^k = 0$$

Since this is a scalar equation it is valid for all coordinate systems in the Euclidean 3-space. Thus the vector field Ω^k can be thought of as determining an axis of rotation in a Euclidean 3-space.

Particle Trajectories in a Euclidean 3-Space

It was previously indicated that the mapping (1) could be regained by an integration of equation (24). This integration can be readily accomplished in a Euclidean 3-space covered by Cartesian coordinates in the simple case where the vector field V_k is independent of the parameter t. This will be true if the rotation tensor ω_{lk} and the vector C_k are independent of t. Instead of working with the tensor ω_{lk} we will use the associated oriented vector ω^i. Thus we write for V_i

$$V_i = e_{i l k}\omega^l y^k + C_i \tag{48}$$

The system of coupled first order differential equations (24) now becomes

$$\delta_{ik}\frac{dy^k}{dt} = e_{i l k}\omega^l y^k + C_i \tag{49}$$

We want the solution of this system of equations subject to the initial conditions

$$y^k(0) = \bar{y}^k$$

The system of equations (49) is most easily integrated by the method of Laplace transformations [7]. The transform of y^k will be written as

$$L\{y^k\} = \int_0^\infty e^{-pt} y^k(t)dt = Y^k(p)$$

Taking the transform of (49) we obtain

$$(p\delta_{ik} + e_{ikl}\omega^l)Y^k = \frac{C_i}{p} + \delta_{il}\bar{y}^l \tag{50}$$

where we used the relations

$$L\left\{\frac{dy^k}{dt}\right\} = pY^k(p) - \bar{y}^k$$

$$L\{1\} = \frac{1}{p}$$

The system of equations (50) is a set of inhomogeneous linear equations for the three unknowns Y^k $(k = 1,2,3)$ which can be solved if the coefficient matrix

$$A_{ik} = p\delta_{ik} + e_{ikl}\omega^l \tag{51}$$

can be inverted. If we denote the inverse by A^{li} it can be shown by direct inversion that

$$A^{li} = \frac{1}{p(p^2 + \omega^2)} [p^2\delta^{li} - pe^{lij}\omega_j + \omega^l\omega^i] \tag{52}$$

where we have written ω^2 for $\omega_i\omega^i$. By direct multiplication it can be verified that

$$A^{li}A_{ik} = \delta^l_k$$

Multiplying (50) by A^{li} we obtain

$$Y^l(p) = \frac{1}{p(p^2 + \omega^2)} \left[p^2\bar{y}^l + p(C^l + e^{lji}\omega_j\delta_{ik}\bar{y}^k) + (\omega^l\omega_k\bar{y}^k + e^{lji}\omega_jC_i) + \frac{1}{p}\,\omega^l\omega^iC_i\right] \tag{53}$$

Performing the multiplication and using the partial fraction expansion

$$\frac{1}{p^2(p^2 + \omega^2)} = \frac{1}{\omega^2p^2} - \frac{1}{\omega^2[p^2 + \omega^2]}$$

we finally obtain

$$Y^l(p) = \frac{p}{(p^2 + \omega^2)}\,\bar{y}^l + \frac{1}{(p^2 + \omega^2)}\left[C^l + e^{lji}\omega_j\delta_{ik}\bar{y}^k - \frac{\omega^l\omega^iC_i}{\omega^2}\right]$$
$$+ \frac{1}{p(p^2 + \omega^2)}\left[\omega^l\omega_k\bar{y}^k + e^{lji}\omega_jC_i\right] + \frac{1}{p^2}\,\frac{\omega^l\omega^iC_i}{\omega^2} \tag{54}$$

Taking the inverse transform of (54) using

$$L^{-1}\left\{\frac{p}{p^2 + \omega^2}\right\} = \cos \omega t$$

$$L^{-1}\left\{\frac{1}{p^2 + \omega^2}\right\} = \frac{\sin \omega t}{\omega}$$

$$L^{-1}\left\{\frac{1}{p(p^2+\omega^2)}\right\} = \frac{1-\cos\omega t}{\omega^2}$$

$$L^{-1}\left\{\frac{1}{p^2}\right\} = t$$

and defining a unit vector λ^l by

$$\lambda^l \equiv \frac{\omega^l}{\sqrt{\omega_k\omega^k}} = \frac{\omega^l}{\omega}$$

there results the expression

$$y^l = R^l_k\bar{y}^k + B^l_kC^k \tag{55}$$

where

$$B^l_k(\omega t) = \frac{1}{\omega}\left[(\delta^l_k - \lambda^l\lambda_k)\sin\omega t - e^{lji}\lambda_j\delta_{ik}\cos\omega t + \lambda^l\lambda_k\omega t + e^{lji}\lambda_j\delta_{ik}\right] \tag{56}$$

and

$$R^l_k(\omega t) = \frac{dB^l_k}{dt} = (\delta^l_k - \lambda^l\lambda_k)\cos\omega t + e^{lji}\lambda_j\delta_{ik}\sin\omega t + \lambda^l\lambda_k \tag{57}$$

The equations (55) correspond to a particular mapping of the form (1) where our initial coordinates $\bar{y}^k$ are used in place of the former notation for initial coordinates x^k_o. These are the trajectories of the particles of a rigid body when the motion of the rigid body is characterized by a constant ω^k and C^k.

Orthogonal Rotation Matrices

We are now in a position to show that an orthogonal matrix can be used to describe a pure rotational displacement. For a pure rotation the constants C_k in (55) must be taken as zero. Thus it becomes apparent that the matrix R^l_k can be used to completely characterize the rotation. It is therefore appropriate to call this a rotation matrix. In (57) ωt appears as the argument of the trigonometric functions sine and cosine. Therefore it seems reasonable to associate ωt with the angle of rotation θ about an axis specified by the vector $\omega^k = \omega\lambda^k$. The magnitude of the vector ω^k is to be associated with $d\theta/dt$. The matrix R^l_k is identical to the rotation matrix given by Mayer [3] as his equation (5.12). This same matric appears in equation (7) of the note by Grubin [8]. His equation (7) corresponds directly to our equation (55) with $C_k = 0$.

The rotation matrix is an orthogonal matrix, that is, it satisfies the relations

$$R^i_j \delta^{jk} R^l_k = \delta^{il}$$

$$R^i_j \delta_{il} R^l_k = \delta_{jk}$$

These relations can be verified with some algebraic tedium. Thus one can easily solve (55) for the $\bar{y}^k$ in terms of y^l as

$$\bar{y}^i = \mathcal{R}^i_l y^l - \mathcal{R}^i_j B^j_l C^l$$

where

$$\mathcal{R}^i_l \equiv \delta^{ij} R^k_j \delta_{kl} = (\delta^i_l - \lambda^i\lambda_l)\cos\theta - e^{isk}\lambda_s\delta_{kl}\sin\theta + \lambda^i\lambda_l \tag{58}$$

is the inverse matrix (in this case also the transpose) of the matrix R^l_k.

$$\mathcal{R}^i_l R^l_k = \delta^i_k = R^i_l \mathcal{R}^l_k \tag{59}$$

For the particular case of a rotation about the y^3 axis $\lambda^k = (0,0,1)$, the matrix $\mathcal{R}^i_l$ takes the familiar form

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where the upper index labels the rows while the lower labels the columns.

MOVING COORDINATE SYSTEMS

In discussing rigid body dynamics it is often convenient to work in a coordinate system rigidly attached to, and moving with, the rigid body [2], [9]. We will now consider how coordinate transformations between frames in relative motion differ from coordinate transformations between frames with no relative motion.

The Relationship Between a Mapping and a Coordinate Transformation

An integration of equations (24) subject to the initial conditions $x^k(t = 0) = \bar{x}^k$ gives us a set of equations of the form

$$x^k = x^k(\bar{x}^l, t) \tag{60}$$

This integration was explicitly carried through for a particularly simple case in a previous section. Because of the way in which (60) was obtained it should be considered as a mapping of the type (1). However it can also be regarded as a coordinate transformation between an x-frame and an $\bar{x}$-frame moving relative to one another with a rigid body motion. Since the $\bar{x}^l$ are fixed numbers independent of the time t, it is reasonable to regard the $\bar{x}$-frame as a coordinate system attached to a rigid body and moving relative to a "stationary" x-frame. For present purposes there is no need to assume that the x-frame is an inertial frame. This question enters only when one considers questions of dynamics rather than kinematics. The coordinate transformation point of view will be adopted in the next section where we will place particular emphasis on the transformation properties of the intrinsic time derivative of vectors and tensor.

Transformation of the Intrinsic Time Derivative

The components of an arbitrary vector or tensor transform in the usual way under the transformation (1) provided that the vector or tensor under consideration is not obtained from another vector or tensor by intrinsic differentiation. Thus the contravariant components $\overline{G}^{ij}$ of some arbitrary second order tensor in the $\overline{x}$-frame are related to the components G^{kl} in the x-frame as

$$\overline{G}^{ij} = \frac{\partial \overline{x}^i}{\partial x^k} \frac{\partial \overline{x}^j}{\partial x^l} G^{kl} \tag{61}$$

The only thing to observe here is that even though the G^{kl} may possibly not be explicit functions of time, the components in the $\overline{x}$-frame, namely $\overline{G}^{ij}$, will in general depend explicitly on time t since the partial derivatives $\partial x^i/\partial \overline{x}^k$ are explicit functions of time through the relations (60).

The situation becomes somewhat more complicated if we consider the intrinsic derivative of G^{kl} which is defined to be

$$\frac{\delta G^{kl}}{\delta t} \equiv \frac{dG^{kl}}{dt} + {}_{x}\begin{Bmatrix} l \\ \alpha\beta \end{Bmatrix} G^{k\beta} \frac{dx^\alpha}{dt} + {}_{x}\begin{Bmatrix} k \\ \alpha\beta \end{Bmatrix} G^{\beta l} \frac{dx^\alpha}{dt} \tag{62}$$

where the prefix on the Christoffel symbols indicates that they are evaluated in the x-frame. This extra index on the Christoffel symbols is necessary here since we will be dealing with expressions involving Christoffel symbols in two different frames. The definition (62) of the intrinsic derivative is valid in all coordinate systems related to each other by a coordinate transformation which is independent of t. However for transformations of the form (60) it is necessary to alter the definition of the intrinsic derivative in a moving coordinate system if $\delta G^{kl}/\delta t$ are to be components of a tensor for such transformations. We now seek to establish the appropriate definition for the intrinsic derivative in a moving frame, that is, we wish to establish an expression for $\overline{\frac{\delta G^{ij}}{\delta t}}$ which is defined by

$$\overline{\frac{\delta G^{ij}}{\delta t}} = \frac{\partial \overline{x}^i}{\partial x^k} \frac{\partial \overline{x}^j}{\partial x^l} \frac{\delta G^{kl}}{\delta t} \tag{63}$$

or equivalently

$$\frac{\delta G^{kl}}{\delta t} = \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial x^l}{\partial \overline{x}^j} \overline{\frac{\delta G^{ij}}{\delta t}} \tag{64}$$

We take as our starting point the expression relating G^{kl} to $\overline{G}^{ij}$ namely

$$G^{kl} = \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial x^l}{\partial \overline{x}^j} \overline{G}^{ij} \tag{65}$$

Differentiation with respect to time t gives

$$\frac{dG^{kl}}{dt} = \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial x^l}{\partial \overline{x}^j} \frac{d\overline{G}^{ij}}{dt} + \frac{\partial x^k}{\partial \overline{x}^i} \overline{G}^{ij} \frac{d}{dt}\left(\frac{\partial x^l}{\partial \overline{x}^j}\right) + \frac{\partial x^l}{\partial \overline{x}^j} \overline{G}^{ij} \frac{d}{dt}\left(\frac{\partial x^k}{\partial \overline{x}^i}\right) \tag{66}$$

The time derivative of the partial derivative of x^k with respect to $\overline{x}^l$ can be written as

$$\frac{d}{dt}\left(\frac{\partial x^k}{\partial \overline{x}^l}\right) = \frac{\partial^2 x^k}{\partial t\, \partial \overline{x}^l} + \frac{\partial^2 x^k}{\partial \overline{x}^j\, \partial \overline{x}^l} \frac{d\overline{x}^j}{dt} \tag{67}$$

If the order of differentiation is changed in the first term on the right we have for this term

$$\frac{\partial^2 x^k}{\partial t\, \partial \overline{x}^l} = \frac{\partial}{\partial \overline{x}^l}\left(\frac{\partial x^k}{\partial t}\right) = \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial}{\partial x^i}\left(\frac{\partial x^k}{\partial t}\right) = \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial V^k}{\partial x^i}$$

where we used the definition of V^k (7) in the last step. The second partial derivatives appearing in the second term of (67) can be expressed in terms of Christoffel symbols and first partial derivatives [5].

$$\frac{\partial^2 x^k}{\partial \bar{x}^j \, \partial \bar{x}^l} = {}_{\bar{x}}\begin{Bmatrix} \alpha \\ j l \end{Bmatrix} \frac{\partial x^k}{\partial \bar{x}^\alpha} - {}_{x}\begin{Bmatrix} k \\ \alpha\beta \end{Bmatrix} \frac{\partial x^\alpha}{\partial \bar{x}^j} \frac{\partial x^\beta}{\partial \bar{x}^l}$$

Therefore (67) can be written as

$$\frac{d}{dt}\left(\frac{\partial x^k}{\partial \bar{x}^l}\right) = \frac{\partial x^i}{\partial \bar{x}^l} \frac{\partial V^k}{\partial x^i} + {}_{\bar{x}}\begin{Bmatrix} \alpha \\ j l \end{Bmatrix} \frac{\partial x^k}{\partial \bar{x}^\alpha} \frac{d\bar{x}^j}{dt} - {}_{x}\begin{Bmatrix} k \\ \alpha\beta \end{Bmatrix} \frac{\partial x^\beta}{\partial \bar{x}^l} \frac{\partial x^\alpha}{\partial \bar{x}^j} \frac{d\bar{x}^j}{dt}$$

However by a differentiation of (60) we find

$$\frac{dx^k}{dt} = \frac{\partial x^k}{\partial t} + \frac{\partial x^k}{\partial \bar{x}^l} \frac{dx^l}{dt} = V^k + \frac{\partial x^k}{\partial \bar{x}^l} \frac{dx^l}{dt} \tag{68}$$

Using this in the last term of the preceding equation and combining terms we obtain

$$\frac{d}{dt}\left(\frac{\partial x^k}{\partial \bar{x}^l}\right) = \frac{\partial x^i}{\partial \bar{x}^l} \frac{\partial V^k}{\partial x^i} + {}_{\bar{x}}\begin{Bmatrix} \alpha \\ j l \end{Bmatrix} \frac{\partial x^k}{\partial \bar{x}^\alpha} \frac{d\bar{x}^j}{dt} - {}_{x}\begin{Bmatrix} k \\ \alpha\beta \end{Bmatrix} \frac{\partial x^\beta}{\partial \bar{x}^l} \left(\frac{dx^\alpha}{dt} - V^\alpha\right)$$

$$= \frac{\partial x^i}{\partial \bar{x}^l} \left(\frac{\partial V^k}{\partial x^i} + {}_{x}\begin{Bmatrix} k \\ \alpha i \end{Bmatrix} V^\alpha\right) + {}_{\bar{x}}\begin{Bmatrix} \alpha \\ j l \end{Bmatrix} \frac{\partial x^k}{\partial \bar{x}^\alpha} \frac{d\bar{x}^j}{dt} - {}_{x}\begin{Bmatrix} k \\ \alpha\beta \end{Bmatrix} \frac{\partial x^\beta}{\partial \bar{x}^l} \frac{dx^\alpha}{dt}$$

$$= \frac{\partial x^i}{\partial \bar{x}^l} \nabla_i V^k + {}_{\bar{x}}\begin{Bmatrix} \alpha \\ j l \end{Bmatrix} \frac{\partial x^k}{\partial \bar{x}^\alpha} \frac{d\bar{x}^j}{dt} - {}_{x}\begin{Bmatrix} k \\ \alpha\beta \end{Bmatrix} \frac{\partial x^\beta}{\partial \bar{x}^l} \frac{dx^\alpha}{dt} \tag{69}$$

Making the appropriate changes of indices in (69) and substituting into the last two terms of (66) and using (65) to simplify some of the terms there results the expression

$$\frac{\delta G^{kl}}{\delta t} = \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j}\left[\frac{d\overline{G}^{ij}}{dt} + {}_{\overline{x}}\left\{{j \atop \alpha\beta}\right\}\overline{G}^{i\beta}\frac{d\overline{x}^\alpha}{dt} + {}_{\overline{x}}\left\{{i \atop \alpha\beta}\right\}\overline{G}^{\beta j}\frac{dx^\alpha}{dt}\right] + G^{kr}\,\nabla_r\,V^l + G^{rl}\,\nabla_r\,V^k \tag{70}$$

We will denote by $\delta\overline{G}^{ij}/\delta t$ the terms contained in the brackets of (70). The last two terms in (70) are both contravariant components of second order tensors and thus obey the usual rules of transformation and therefore (70) can be written as

$$\frac{\delta G^{kl}}{\delta t} = \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial \dot{x}^l}{\partial \overline{x}^j}\left[\frac{\delta\overline{G}^{ij}}{\delta t} + \overline{G}^{ir}\,\overline{\nabla_l\,V^j} + \overline{G}^{rj}\,\nabla_r\,V^i\right] \tag{71}$$

Now if the two coordinate systems are moving relative to each other as rigid bodies, then the gradient of the velocity field V_k can be replaced by the tensor Ω_{lk} (13). With this condition we can, by a comparison of (71) and (64), make the definition

$$\overline{\frac{\delta G^{ij}}{\delta t}} \equiv \frac{\delta\overline{G}^{ij}}{\delta t} + \overline{G}^{ir}\overline{\Omega}_r{}^{\cdot j} + \overline{G}^{rj}\overline{\Omega}_r{}^{\cdot i} \tag{72}$$

This definition can be extended to the contravariant components of higher order tensors and vectors either by a procedure identical to that followed above or by analogy. In either case we can write, for example, for the contravariant components of a vector

$$\overline{\frac{\delta G^i}{\delta t}} \equiv \frac{\delta\overline{G}^i}{\delta t} + \overline{G}^r\overline{\Omega}_r{}^{\cdot i} = \frac{\delta\overline{G}^i}{\delta t} + \overline{G}_r\overline{\Omega}^{ri} \tag{73}$$

The formulae corresponding to (72) and (73) for the covariant components of vectors and tensors can be established by an identical procedure. Thus for a second order tensor one begins by differentiation of

$$\overline{G}_{ij} = \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j}G_{kl}$$

This procedure gives the definition

$$\overline{\frac{\delta G_{ij}}{\delta t}} = \frac{\delta \overline{G}_{ij}}{\delta t} + \overline{G}_{ir}\overline{\Omega}^{r}_{\cdot j} + \overline{G}_{rj}\overline{\Omega}^{r}_{\cdot i} \tag{74}$$

while for vectors

$$\overline{\frac{\delta G_{i}}{\delta t}} = \frac{\delta \overline{G}_{i}}{\delta t} + \overline{G}_{r}\overline{\Omega}^{r}_{\cdot i} \tag{75}$$

The above formulae shows that only the rotational motion characterized by the tensor Ω_{lk}, affects intrinsic differentiation.

Intrinsic differentiation in a moving coordinate system satisfies the usual rules for differentiation of sums and products as can easily be verified using the definitions (72-75). Thus if we take $\overline{G}^{ij} = \overline{A}^{i}\overline{B}^{j}$

$$\overline{\frac{\delta (A^{i}B^{j})}{\delta t}} = \frac{\delta(\overline{A}^{i}\overline{B}^{j})}{\delta t} + \overline{A}^{i}\overline{B}^{r}\overline{\Omega}_{r}^{\cdot j} + \overline{A}^{r}\overline{B}^{j}\overline{\Omega}_{r}^{\cdot i}$$

But

$$\frac{\delta(\overline{A}^{i}\overline{B}^{j})}{\delta t} = \overline{A}^{i}\,\frac{\delta \overline{B}^{j}}{\delta t} + \overline{B}^{j}\,\frac{\delta \overline{A}^{i}}{\delta t}$$

and therefore

$$\overline{\frac{\delta (A^{i}B^{j})}{\delta t}} = \overline{A}^{i}\left[\frac{\delta \overline{B}^{j}}{\delta t} + \overline{B}^{r}\overline{\Omega}_{r}^{\cdot j}\right] + \overline{B}^{j}\left[\frac{\delta \overline{A}^{i}}{\delta t} + \overline{A}^{r}\overline{\Omega}_{r}^{\cdot i}\right]$$

$$= \overline{A}^{i}\,\overline{\frac{\delta B^{j}}{\delta t}} + \overline{B}^{j}\,\overline{\frac{\delta A^{i}}{\delta t}}$$

This section will be concluded by applying (74) to two particular tensors, the metric tensor g_{ij} and the tensor Ω_{ij}. Since the intrinsic derivative of the metric tensor vanishes in the x-frame, it also vanishes in the $\overline{x}$-frame.

$$\overline{\frac{\delta g_{ij}}{\delta t}} = 0 = \frac{\delta \overline{g}_{ij}}{\delta t} + \overline{g}_{ir}\overline{\Omega}^{r}_{\cdot j} + \overline{g}_{rj}\overline{\Omega}^{r}_{\cdot i}$$

$$= \frac{\delta \overline{g}_{ij}}{\delta t} + \overline{\Omega}_{ij} + \overline{\Omega}_{ji} = \frac{\delta \overline{g}_{ij}}{\delta t}$$

Thus

$$\overline{\frac{\delta g_{ij}}{\delta t}} = \frac{\delta \overline{g}_{ij}}{\delta t} = 0$$

This result can be used in connection with (72) to establish the expression for the intrinsic derivative of tensors with mixed components. We next apply (74) to Ω_{ij}.

$$\overline{\frac{\delta \Omega_{ij}}{\delta t}} = \frac{\delta \overline{\Omega}_{ij}}{\delta t} + \overline{\Omega}_{ir}\overline{\Omega}^{r}_{\cdot j} + \overline{\Omega}_{rj}\overline{\Omega}^{r}_{\cdot i}$$

However

$$\overline{\Omega}_{rj}\overline{\Omega}^{r}_{\cdot i} = \overline{\Omega}^{r}_{\cdot j}\overline{\Omega}_{ri} = -\overline{\Omega}^{r}_{\cdot j}\overline{\Omega}_{ir}$$

And therefore

$$\overline{\frac{\delta \Omega_{ij}}{\delta t}} = \frac{\delta \overline{\Omega}_{ij}}{\delta t}$$

APPLICATIONS

In the concluding two sections of this paper some of the general discussions presented earlier will be applied to two specific topics. The first will be a derivation of the constants of motion known as the linear and angular momenta of a closed system [10]. The second will be a derivation of Newton's equations of motion in a non-inertial frame.

Linear and Angular Momentum of a Closed System

Consider a closed system of N particles moving in a three-dimensional space covered by a coordinate system x^k. We assume that the particles are point particles possessing no internal degrees of freedom. If $\underset{\mu}{x}^k$ are the coordinates of the μ^{th} particle, then for a conservative system (all forces derivable from a potential) the Lagrangian equations of motion for the system are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k_\mu}\right) - \frac{\partial L}{\partial x^k_\mu} = 0 \tag{76}$$

where $\dot{x}^k_\mu$ has been written for dx^k_μ/dt and where the Lagrangian function is

$$L = L(x^k_\mu, \dot{x}^k_\mu)$$

We will assume that the mechanical properties of the system are unaffected by any rigid body displacement of the system. This assumption when applied to translations refers to the homogeneity of space and when applied to rotations refers to the isotropy of space [10]. At this point our discussion requires no distinction between translations and rotations, however to obtain the constants of the motion in the usual form we will make the distinction at a later stage. Now a rigid body displacement of the system is equivalent to the introduction of a new coordinate system z^k by the transformation

$$z^k = z^k(x^l, \zeta) \tag{77}$$

where ζ is a parameter of the transformation other than t. Now the form of the equations of motion (76) is unaffected by a coordinate transformation of the type (77). And thus if the mechanical properties of the system are to remain unchanged the value of the Lagrangian must not be affected by (77).

$$L(z^k, \dot{z}^k, \zeta) - L(x^k, \dot{x}^k) = 0 \tag{78}$$

Since (77) must hold for all rigid body displacements, it must hold for the infinitesimal rigid body displacements, which by analogy to (7) can be written as

$$z^k = x^k + d\zeta\, V^k(x,\zeta)$$

Differentiation of this expression gives

$$\dot{z}^k = \dot{x}^k + d\zeta \frac{dV^k}{dt}$$

By a Taylor expansion we have to first order in the infinitesimal $d\zeta$

$$L(z^k,\dot{z}^k,\zeta) - L(x^k,\dot{x}^k) = d\zeta \sum_{\lambda} \left[\frac{\partial L}{\partial \underset{\lambda}{x}^k} V^k(\underset{\lambda}{x},\zeta) + \frac{\partial L}{\partial \underset{\lambda}{\dot{x}}^k} \frac{dV^k}{dt} (\underset{\lambda}{x},\zeta) \right]$$

Substituting the equations of motion (76) in the first term and combining with the condition (78) we obtain

$$\frac{d}{dt}\left[\sum_{\lambda} V^k(\underset{\lambda}{x},\zeta) \frac{\partial L}{\partial \underset{\lambda}{\dot{x}}^k} \right] = 0$$

And thus we have the constant of the motion

$$\sum_{\lambda} V^k(x,\zeta) \frac{\partial L}{\partial \underset{\lambda}{\dot{x}}^k} = \text{const} \tag{79}$$

Now $\partial L/\partial \underset{\lambda}{\dot{x}}^k$ is the momentum canonically conjugate to the coordinate $\underset{\lambda}{x}^k$ and is a vector since

$$\underset{\lambda}{p}_k \equiv \frac{\partial L}{\partial \underset{\lambda}{\dot{x}}^k} = \underset{\lambda}{m} \underset{\lambda}{g}_{ki} \underset{\lambda}{\dot{x}}^i \tag{80}$$

Thus the constants of the motion represented by (79) are scalars. Since (79) must be true for all rigid body motions it must be true separately for infinitesimal rotations and translations. Thus separately

$$\sum_{\lambda} u^k(\underset{\lambda}{x},\zeta)\underset{\lambda}{p}_k = \sum_{\lambda} u_k(\underset{\lambda}{x},\zeta)\underset{\lambda}{p}^k = \text{const} \tag{81}$$

$$\sum_{\lambda} U^k(\underset{\lambda}{x},\zeta)\underset{\lambda}{p}_k = \sum_{\lambda} U_k(\underset{\lambda}{x},\zeta)\underset{\lambda}{p}^k = \text{const} \tag{82}$$

Equations (81) and (82) appear to supply us with an infinite number of constants of the motion however in reality there are only six. This is because in a three-dimensional linear vector space there can only be three linearly independent translations $\underset{s}{u}_k$ (s = 1,2,3) and three linearly independent rotations $\underset{s}{U}_k$ (s = 1,2,3). To display the constants of the motion in the usual form we return to a Euclidean space covered by Cartesian orthogonal coordinates which give for the canonical momenta (from (80))

$$\underset{\lambda}{p}^k = \underset{\lambda}{p}_k = \underset{\lambda}{m}\underset{\lambda}{\dot{y}}^k \tag{83}$$

For the translational displacements we take unit vectors along the coordinate lines thus

$$\underset{1}{u}_k = (1,0,0)$$

$$\underset{2}{u}_k = (0,1,0)$$

$$\underset{3}{u}_k = (0,0,1)$$

With this choice for $\underset{s}{u}_k$ we have from (81) and (83) three constants of the motion

$$\underset{s}{P} \equiv \sum_{\lambda} \underset{s}{u}_k \underset{\lambda}{p}^k = \sum_{\lambda} \underset{\lambda}{m\dot{y}}^s \tag{84}$$

For the rotational motion we choose as linearly independent displacements, those obtained when the vectors $\underset{s}{\omega}^k$ are unit vectors along the coordinate lines

$$\underset{1}{\omega}^k = (1,0,0)$$

$$\underset{2}{\omega}^k = (0,1,0)$$

$$\underset{3}{\omega}^k = (0,0,1)$$

Combining (82) and (45) with this choice we obtain

$$\underset{s}{M} = \sum_{\lambda} \underset{s}{\ddot{U}}_k p^k = \sum_{\lambda} \underset{\lambda}{m}(e_{sik}\underset{\lambda}{y}^i \underset{\lambda}{\dot{y}}^k) \tag{85}$$

Equations (84) and (85) will be recognized as the usual definitions of linear momentum and angular momentum of a system of particles [2],[10]. As already pointed out $\underset{s}{P}$ and $\underset{s}{M}$ are scalars however the $\underset{s}{P}$ can be considered as components of a vector for all affine transformations while $\underset{s}{M}$ are vector components only for centered affine transformations.

Newton's Law of Motion in a Non-Inertial Frame

In this final section of the paper we develop the form of Newton's equations in a non-inertial coordinate system and show that for the special case of Cartesian coordinates this reduces to the usual equations. We take as our starting point Newton's equations in an inertial frame x^k

$$F_k = ma_k = m\,\frac{\delta v_k}{\delta t} = mg_{kl}\,\frac{\delta v^l}{\delta t} \tag{86}$$

where

$$\frac{\delta v^l}{\delta t} = \ddot{x}^l + \left\{ {l \atop ij} \right\} \dot{x}^i \dot{x}^j$$

In a non-inertial frame $\bar{x}^k$ this becomes

$$\bar{F}_k = m \overline{\frac{\delta v_k}{\delta t}} = m\bar{g}_{kl} \overline{\frac{\delta v^l}{\delta t}} \tag{87}$$

However from (68) we can express the velocity v^k in the form

$$v^k = \dot{x}^k = \frac{\partial x^k}{\partial \bar{x}^l} (\bar{V}^l + \dot{\bar{x}}^l) \tag{88}$$

where we have performed the substitution

$$V^k = \frac{\partial x^k}{\partial \bar{x}^l} \bar{V}^l$$

Thus from (88) we can write

$$\bar{v}^l = \bar{V}^l + \dot{\bar{x}}^l \tag{89}$$

Using the expression (73) for the form of the intrinsic derivative we have then

$$\overline{\frac{\delta v^l}{\delta t}} = \frac{\delta \bar{V}^l}{\delta t} + \frac{\delta \dot{\bar{x}}^l}{\delta t} + \bar{V}_r \bar{\Omega}^{rl} + \dot{\bar{x}}^r \bar{\Omega}_r{}^{\cdot l} \tag{90}$$

Now $\delta \dot{\bar{x}}^l / \delta t$ is just the acceleration $\bar{a}^l$ as observed in the non-inertial frame. Further the derivative $\delta \bar{V}^l / \delta t$ of the rigid body velocity field can be written as

$$\frac{\delta \bar{V}^l}{\delta t} = \frac{\partial \bar{V}^l}{\partial t} + \dot{\bar{x}}^r \, \overline{\nabla_r V^l}$$

$$= \frac{\partial \bar{V}^l}{\partial t} + \dot{\bar{x}}^r \bar{\Omega}_r{}^{\cdot l}$$

Making these substitutions into (90) we obtain the expression

$$\overline{\frac{\delta v^l}{\delta t}} = \bar{a}^l + \bar{V}_r\Omega^{rl} + 2\dot{\bar{x}}^r\Omega_r^{\cdot\,l} + \frac{\partial \bar{V}^l}{\partial t} \tag{91}$$

Upon the substitution of (91) into (87) we obtain after a slight rearrangement

$$m\bar{a}_k = \bar{F}_k + 2m\bar{\Omega}_{kr}\dot{\bar{x}}^r + m\bar{\Omega}_{kr}\bar{V}^r - m\bar{g}_{kl}\frac{\partial \bar{V}^l}{\partial t} \tag{92}$$

If the tensor $\bar{\Omega}_{kr}$ is replaced by the oriented vector $\bar{\Omega}^i$ using equation (22) then equation (92) takes the form

$$m\bar{a}_k = \bar{F}_k + 2m\bar{\epsilon}_{kri}\dot{\bar{x}}^r\bar{\Omega}^i + m\bar{\epsilon}_{kri}\bar{V}^r\bar{\Omega}^i - m\bar{g}_{kl}\frac{\partial \bar{u}^l}{\partial t} \tag{93}$$

Finally writing the rigid body motion as a sum of rotational and translational velocities by means of (25a) expressed in $\bar{x}$-frame we have

$$m\bar{a}_k = \bar{F}_k + 2m\bar{\epsilon}_{kri}\dot{\bar{x}}^r\bar{\Omega}^i + m\bar{\epsilon}_{kri}\bar{U}^l\bar{\Omega}^i + m\bar{\epsilon}_{kri}\bar{u}^l\Omega^i - m\bar{g}_{kl}\frac{\partial \bar{U}^l}{\partial t} - m\bar{g}_{kl}\frac{\partial \bar{u}^l}{\partial t} \tag{94}$$

To make a direct comparison with the usual expressions for Newton's law in a non-inertial frame we again specialize to Cartesian orthogonal coordinates in a Euclidean space. For these conditions we will shortly show that for those frames which are moving uniformly relative to an inertial that the last two terms of (94) are zero. If the motion is purely translational, the remaining terms vanish since these are proportional to Ω^i which is zero in this case. Thus the equations of motion in a uniformly translating coordinate system are the same as in an inertial coordinate system. If the motion is purely rotational $\bar{u}^r$ is zero and using (45) in conjunction with (55) with $C^k \triangleq 0$ we find that (94) takes the form

$$m\bar{a}^k = \bar{F}^k - m\left[2\bar{e}^{kir}\bar{\omega}_i\dot{\bar{y}}_r + \bar{e}^{kir}\bar{\omega}_i(\bar{e}_{rst}\bar{\omega}^s\bar{y}^t)\right] \tag{95}$$

In writing down (95) use was made of the fact that in Cartesian orthogonal coordinate systems indices may be raised or lowered at will. In the notation of vector analyses (95) can be written as [2]

$$m\underset{\sim}{a} = \underset{\sim}{F} - m[2\underset{\sim}{\omega} \times \dot{\underset{\sim}{r}} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r})] \tag{96}$$

In (96) the use of a bar over symbols to designate quantities in a non-inertial frame has been discontinued but it is to be understood that this equation is only valid in a non-inertial coordinate system. The second term on the right hand side of (95) or (96) is the Coriolis force while the last term is the centrifugal force.

In an earlier paragraph it was asserted that $\partial\overline{V}^k/\partial t$ was zero for a non-inertial frame moving uniformly relative to the inertial frame, that is, where ω^k and C^k are independent of time. But this is precisely the situation for which (55) gives the relationship between the inertial and non-inertial frames. Thus we are in a position to explicitly evaluate $\partial\overline{V}^k/\partial t$. By differentiation of (55) we have

$$V^l \equiv \frac{\partial y^l}{\partial t} = \dot{R}^l_k \overline{y}^k + \dot{B}^l_k C^k = \dot{\mathcal{R}}^l_k = \dot{R}^l_k \tag{97}$$

To get equation (97) in the same form as (48) it would be necessary to eliminate $\overline{y}^k$ in terms of y^l using (55) in the form

$$\overline{y}^k = \mathcal{R}^k_l y^l - \mathcal{R}^k_j B^j_l C^l$$

However we are interested here in $\overline{V}^k$ rather than V^l and thus we retain $\overline{y}^k$. Since

$$\overline{V}^k = \frac{\partial \overline{y}^k}{\partial y^l} V^l = \mathcal{R}^k_l V^l$$

We have as the analytical expression for $\overline{V}^k$

$$\overline{V}^k = \mathcal{R}^k_l \dot{R}^l_j \overline{y}^j + C^k$$

And since C^k is independent of t

$$\frac{\partial \overline{V}^k}{\partial t} = \frac{d}{dt}(\mathcal{R}^k_l \dot{R}^l_j)\overline{y}^j$$

By direct multiplication it can be readily shown that

$$\mathcal{R}^k_l \dot{R}^l_j = e^{kil}\omega_i \delta_{lj}$$

which is independent of t. Therefore we have immediately that $\partial\overline{V}^k/\partial t$ is zero for this case.

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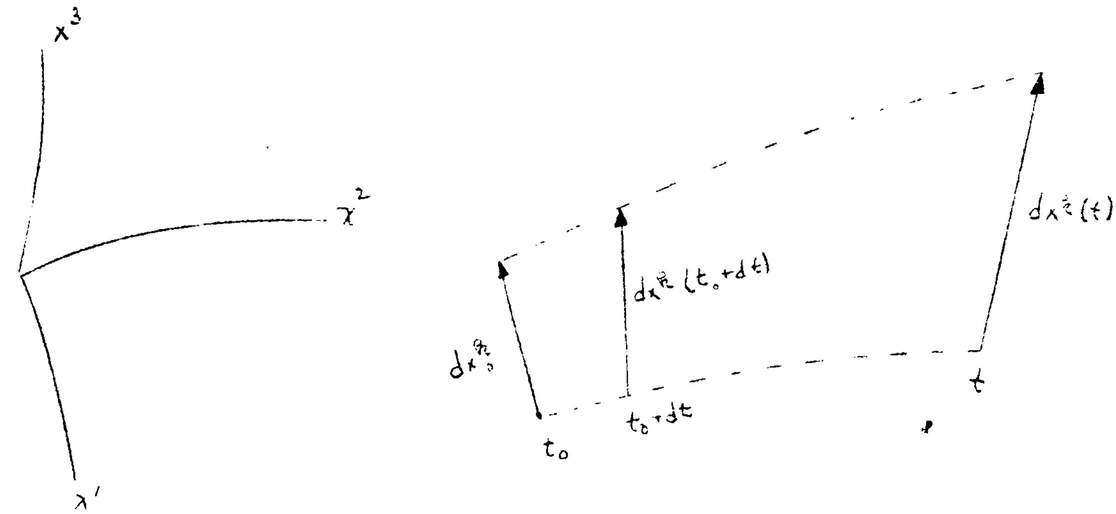

FIGURE 1.

FIGURE 2.