MULTIGRID METHODS FOR A SEMILINEAR PDE IN THE THEORY OF PSEUDOPLASTIC FLUIDS*

Van Emden Henson
Department of Mathematics
Naval Postgraduate School
Monterey, CA

A. W. Shaker
Department of Mathematics
Naval Postgraduate School
Monterey, CA

SUMMARY

We show that by certain transformations the boundary layer equations for the class of non-Newtonian fluids named *pseudoplastic* can be generalized in the form

$$\Delta u + p(x)u^{-\lambda} = 0, \quad x \in \Omega \subseteq R^n, \quad n \geq 1$$

under the classical conditions for steady flow over a semi-infinite flat plate. We provide a survey of the existence, uniqueness, and analyticity of the solutions for this problem. We also establish numerical solutions in one- and two-dimensional regions using multigrid methods.

INTRODUCTION

In the last two decades, solutions of the singular semilinear equation

$$\Delta u + p(x)u^{-\lambda} = 0, \quad x \in \Omega \subseteq R^n \tag{1}$$

have been extensively studied. Various existence and uniqueness results are given in [1], [2], and [3], to name a few. More recently, in [4], it is shown that by certain transformations the boundary layer equations for the class of non-Newtonian fluids named *pseudoplastic* can be generalized in the above form for the ODE case $n = 1$. Under this physical interpretation the above equation, considered in the context of partial differential equations ($n > 1$), has been the subject of much study. The equation has a unique classical solution with a bounded domain Ω, where $p(x)$ is a sufficiently regular function which is positive on $\bar{\Omega}$ [5]. There exist entire solutions with $\lambda \in (0, 1)$ for $p(x)$ sufficiently regular ([6], [7]). This is generalized to all $\lambda > 0$ via the upper and lower solution method ([8]) or other methods ([9]).

*This work was supported in part by the the Naval Postgraduate School Research Council under grant No. ZZ867-ZZ899/5986

The following sections provide a survey of both theoretical and numerical results in this area including a physical derivation [4], existence theorems for both the ODE and PDE cases with a proof of a main result [8], and our numerical results. We conclude with a discussion of a new technique and some open questions for further research.

PRELIMINARIES

A non-Newtonian fluid is called *pseudoplastic* if the shear stress τ and the strain rate $\dfrac{\partial u}{\partial y}$ are related as

$$|\tau| = k\left|\frac{\partial u}{\partial y}\right|^{\alpha}, \quad 0 < \alpha < 1$$

where k is a positive constant. That is, the absolute value of the shear stress increases with respect to the absolute value of the strain rate less than linearly.

In this paper, we study solutions of the singular semilinear equation (1) where $\lambda > 0$ and Ω is a domain in R^n, $n \geq 1$. In the following section we show that through a series of transformations the boundary layer equations for the class of pseudoplastic fluids under the classical conditions for a steady flow over a semi-infinite flat plate can be generalized into the well-known Blasius problem

$$f''' + ff' = 0,$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1$$

for the shear function, which arises from the standard Newtonian fluid case.

DERIVATION OF THE PROBLEM

For $n = 1$ equation (1) arises in the study of pseudoplastic fluids. We consider a two-dimensional incompressible flow of low viscosity along a plane wall. We denote by $\vec{v} = (u, v)$ as the fluid velocity in the boundary layer and $u_\infty(x)$ in the main stream. Since there is no velocity on the wall and the fluid takes the velocity of the main stream $u_\infty(x)$ outside the boundary layer, we see that $\dfrac{\partial u}{\partial y}$ is large near the wall which causes a significant transfer of momentum in the x direction.

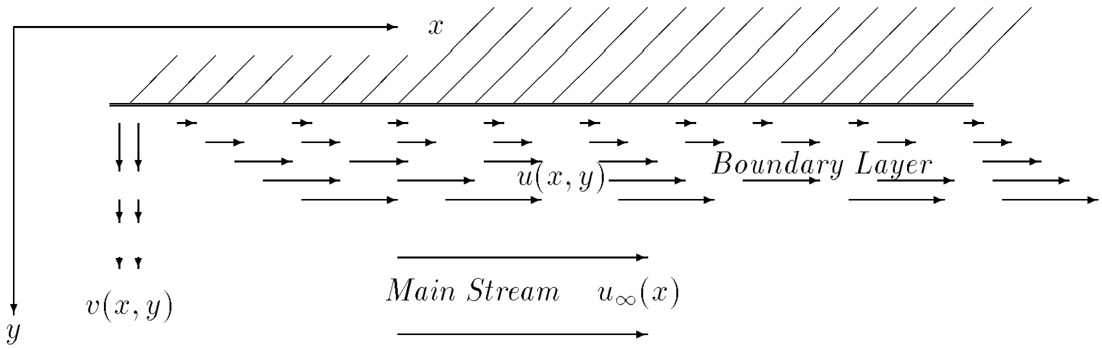
The boundary layer equations for this model include a continuity equations and a momentum equation in the x direction.

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} &= 0 \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y} \end{aligned} \tag{2}$$

with boundary condtions

$$u(x, 0) = v(x, 0) = 0,$$

$$u(x, \infty) = u_\infty(x)$$



$$\vec{v} = \begin{cases} (u, v) & \text{in boundary layer} \\ u_\infty(x) & \text{in main stream} \end{cases}$$

Figure 1: *2-D flow of low viscousity along a plane wall*

where $\tau_{xy} = K\left|\frac{\partial u}{\partial y}\right|^\alpha$ is the shear stress.

Note that (2) has 2 coupled equations in 3 dependent variables, u, v, and τ_{xy}. To reduce it to a single higher-order equation in only 1 dependent variable, we introduce the Lagrange stream function $\Phi(x, y)$ such that

$$u = \frac{\partial \Phi}{\partial y}, \quad v = -\frac{\partial \Phi}{\partial x}.$$

Then the momentum equation becomes

$$\frac{\partial \Phi}{\partial y} \cdot \frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial \Phi}{\partial x} \cdot \frac{\partial^2 \Phi}{\partial y^2} = \nu \frac{\partial}{\partial y}\left(\frac{\partial^2 \Phi}{\partial y^2}\right)^\alpha. \tag{3}$$

where $\nu = \frac{K}{\rho}$, while the continuity equation is clearly satisfied by Φ.

Let $f = \frac{a\Phi}{\sqrt{u_\infty \nu x}}$; $\eta = by\sqrt{\frac{u_\infty}{\nu x}}$, for some a, b. Then (3) becomes

$$f''' + f(f'')^{2-\alpha} = 0 \tag{4}$$

with

$$\begin{aligned} &f(0) = f'(0) = 0 \\ &f'(\infty) = 1 \end{aligned}$$

where $f = f(\eta)$. (Observe that if $\alpha = 1$ (4) is the well-known Blasius equation.) Employing the Crocco-like transformation

$$u = f'(\eta), \quad g(u) = u' = f''(\eta)$$

(4) becomes

$$g^\alpha g'' + (\alpha - 1)g^{\alpha-1}(g')^2 + u = 0$$

with $g'(0)=0$, $g(1)=0$, where $g=g(u)$. Finally the transformation $G=g^\alpha$ leads to the singular boundary value problem

$$\begin{aligned}&G''+\alpha uG^{-1/\alpha}=0, \quad 0<u,\alpha<1,\\ &G'(0)=G(1)=0\end{aligned}$$

of the form (e1) with $\lambda=\dfrac{1}{\alpha}$, $u=x$, $p=\dfrac{x}{\lambda}$.

EXISTENCE AND UNIQUENESS RESULTS

In the first part of this section we study the results in finite and infinite domains; in the second we discuss methods that are commonly used to approach the problem.

Theorems

Let Ω be a bounded domain in R^n, $n\geq 1$ with smooth boundary $\partial\Omega$ (of class $C^{2+\alpha}$, $0<\alpha<1$). Let $p(x)$ be of $C^\alpha(\bar{\Omega})$ and positive on $\bar{\Omega}$, $\lambda>0$.

Theorem 1 *(Lazer-McKenna [5]). The problem*

$$\begin{aligned}&\Delta u+p(x)u^{-\lambda}=0, \quad x\in\Omega\\ &u\mid_{\partial\Omega}=0\end{aligned}$$

has a unique positive solution $u(x)$ in Ω with $u\in C^{2+\alpha}(\Omega)\cap(\bar{\Omega})$. Furthermore let ϕ be an eigenfunction corresponding to the smallest eigenvalue λ_1 of the problem

$$\begin{aligned}&\Delta\phi+\lambda\phi=0, \quad x\in\Omega\\ &\phi\mid_{\partial\Omega}=0\end{aligned}$$

such that $\phi_1(x)>0$, $x\in\Omega$ and $\lambda>1$. Then there exists a unique $b_1,b_2>0$ such that

$$b_1\phi_1^{2/(1+\lambda)}\leq u\leq b_2\phi_1^{2/(1+\lambda)}$$

on $\bar{\Omega}$.

In the case $\Omega=R^n$, $n\geq 1$, wee study the results under conditions $n=1$, $n=2$, $n\geq 3$. Observe that if $n=1$, since $p,y>0$, $y''+py^{-\lambda}=0$ we have $y''>0$ and thus $y'\downarrow$. Hence $0<y'(\infty)<\infty$.

Theorem 2 *(Taliaferro [3]) The problem*

$$\begin{aligned}&y''+p(x)u^{-\lambda}=0\\ &y(c)=\alpha\\ &y'(\infty)=0\end{aligned}$$

has a unique positive solution $y(x)$ if

$$\int_1^\infty x^{-\lambda}p(x)\,dx<\infty$$

where α, $c\in R^1$, $\alpha>0$. Furthermore $y(\infty)<\infty$ if and only if $\int_0^\infty xp(x)\,dx<\infty$.

The following theorem describes the asymptotic behaviors of the solution.

Theorem 3 *(Taliaferro [3])*

- *If* $0 < y'(\infty) < \infty$ *and* $\int_1^\infty x^{-\lambda+1} p(x)dx < \infty$, $a, b > 0$ *then*

$$y(x) = ax + b - a{-}\lambda(1 + \circ(1)) \int_x^\infty (\xi - x)\xi^{-\lambda} p(\xi)d\xi.$$

- *If* $y'(\infty) = 0$ *and* $\int_0^\infty xp(x)dx < \infty$, $a > 0$ *then*

$$y(x) = a - a{-}\lambda(1 + \circ(1)) \int_x^\infty (\xi - x)p(\xi)d\xi.$$

- *if* $p, q > 0$ *are continuous on* $[0, \infty)$, $\lim_{x\to\infty} \frac{q(x)}{p(x)} = R > 0$ *and*

$$\begin{array}{ll} z'' + p(x)z^{-\lambda} = 0, & z'(\infty) = 0; \\ w'' + q(x)w^{-\lambda} = 0, & w'(\infty) = 0 \end{array}$$

  and $\int_0^\infty xp(x)dx = \infty$, *then* $\lim_{x\to\infty} w/z = R^{\frac{1}{1+\lambda}}$.

Theorem 4 *(Kusano-Swanson [7]). The problem*

$$\Delta u = f(|x|)u^{-\lambda} = 0, \quad x \in R^2,\ 0 < \lambda < 1$$

has an entire positive solution in R^2 *with logarithmic growth at* ∞ *if* $f(t) > 0$, $t > 0$, $f(t) \in C(0, \infty)$, *and*

$$\int_e^\infty t(\log t)^{-\lambda} f(t)dt < \infty.$$

A function $u(x)$ is said to be an *entire* solution of (1) if $u \in C^2_{loc}(R^n)$ and u satisfies the equation pointwise in R^n.

Theorem 5 *(Shaker [8]) The problem*

$$\Delta u + p(x)u^{-\lambda} = 0, \quad x \in R^n,\ \lambda > 0$$

has an entire positive solution $u(x)$ *such that*

$$c_1 \le u(x)|x|^{q|n-2|} \le c_2$$

for some c_1, c_2 *and* $0 < q < 1$ *as* $x \to \infty$ *if*

1. $p(x) \in C^\alpha_{loc}(R^n)$, $p(x) > 0$ *for* $x \in R^n \backslash \{0\}$;
2. *there exists* $0 < c < 1$ *such that* $c\phi(|x|) \le p(x) \le \phi(|x|)$ *where* $\phi(t) \equiv \max_{|x|=t} p(x)$, $t \in [0, \infty)$;
3. $\int_1^\infty t^{n-1+\lambda(n-2)}\phi(t)dt < \infty$.

Methods

In general there are two methods that are commonly used in proving existence and uniqueness of solutions for equations of type (1), namely *Schauder's fixed point thoerem* and *Barrier Methods.* Since the former is standard we elaborate here only on the latter.

Let Ω be a smoothly bounded domain in R^n. $\phi(x)$ is said to be an upper (lower) solution of the problem

$$\begin{aligned} \Delta u + f(x,u) = 0, \quad x \in \Omega \\ u \mid_{\partial\Omega} = 0 \end{aligned} \tag{5}$$

if $\Delta\phi + f(x,\phi) \leq 0,\ x \in \Omega,\ \phi(x) \geq 0\ x \in \partial\Omega$ $(\phi + f(x,\phi) \geq 0,\ x \in \Omega,\ \phi(x) \leq 0\ x \in \partial\Omega)$.

Theorem 6 *(Sattinger [10]). Let ϕ_1 be an upper solution and ϕ_2 be a lower solution of (5), and let f be locally Hölder continuous in Ω. If $\phi_1(x) \geq \phi_2(x)$ in Ω, then (5) has a solution u such that $\phi_2(x) \leq u(x) \leq \phi_1(x)$, $x \in \bar{\Omega}$.*

In the case when $\Omega = R^n$ we say ϕ is an upper (lower) solution of

$$\Delta u + f(x,u) = 0 \tag{6}$$

if $\Delta\phi + f(x,\phi) \leq 0\ x \in R^n$ (for lower solution, $\Delta\phi + f(x,\phi) \geq 0$).

Theorem 7 *(Ni [11]). Let ϕ_1 and ϕ_2 be an upper and a lower solution of equation (6), such that $\phi_1 \geq \phi_2\ x \in R^n$. If f is locally Hölder continuous in x and locally Lipschitz continuous in u, then (6) has a solution u with $\phi_2(x) \leq u(x) \leq \phi_1(x)$, $x \in R^n$.*

An Example.

Consider the problem

$$\begin{aligned} &u'' + \lambda u - u^3 = 0, && x \in (0,\pi) \\ &u = 0, && x = 0, \pi. \end{aligned}$$

It is easy to show that $\phi_1(x) = Rx^{1/2}$ for some R large is an upper solution, and $\phi_2(x) = \epsilon \sin x$ for some ϵ small is a lower solution of this problem. Clearly $\phi_1(x) \geq \phi_2(x)$ for $x \in [0,\pi]$. Thus by the above theorem there is a solution $u(x)$ such that $\epsilon \sin x \leq u(x) \leq Rx^{\frac{1}{2}}$, $x \in [0,\pi]$. Since the problem is homogeneous we conclude that the problem has at least three solutions, namely, u, $-u$ and the trivial solution.

MULTIGRID SOLUTION OF THE PROBLEM

In this section we present some numerical results for solving the problem

$$\begin{aligned} \Delta u \,+\, p(x)\, u^{-\lambda} \;=\; 0 \qquad & x \in \Omega \\ u(x) = 0 \qquad & x \in \partial\Omega. \end{aligned}$$

Specifically, we describe Newton's method for non-linear systems to solutions and multigrid $V-cycle$ and FMV methods. We have implemented all of these methods for both the one- and two-dimensional cases, using (respectively) the unit interval and the unit square for Ω. In each case we use a straightforward finite-difference discretization, employing the standard second-order difference approximation for the second derivative operator. For the one-dimensional problem we define the grid of $(N+1)$ points $x_k = jh$, for $k = 0, 1, \ldots N$, where h is the mesh parameter $1/N$. The second derivative operator is then approximated by

$$\left.\frac{d^2u}{dx^2}\right|_{x_k} = \frac{u_{k-1} - 2u_k + u_{k+1}}{h^2} + O(h^2), \tag{7}$$

where u_k approximates $u(x_k)$. For the non-linear term $p(x)u(x)^{-\lambda}$ we use the nodal values, $p_k\, u_k^{-\lambda}$. Since $u_0 = u_N = 0$, this results in the non-linear system of equations

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{bmatrix} + \begin{bmatrix} p_1\, u_1^{-\lambda} \\ p_2\, u_2^{-\lambda} \\ \vdots \\ p_{N-2}\, u_{N-2}^{-\lambda} \\ p_{N-1}\, u_{N-1}^{-\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{8}$$

Letting u represent the vector of unknowns, we may write the system as $Hu + g(u) = 0$, where H is the tridiagonal matrix and g is the non-linear vector function.

For the two-dimensional case we take the tensor product of the $(N+1)$-point grid in the x direction with an identical $(N+1)$-point grid in the y direction, yielding an $(N+1)^2$-point regular grid covering the unit square. The difference operator for the two-dimensional problem is

$$\left.\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right|_{x_{j,k}} = \frac{u_{j-1,k} - 2u_{j,k} + u_{j+1,k}}{h^2} + \frac{u_{j,k-1} - 2u_{j,k} + u_{j,k+1}}{h^2} + O(h^2). \tag{9}$$

Numbering the unknowns lexicographically by lines of constant y, we obtain the nonlinear system

$$\begin{bmatrix} A & B & & & \\ B & A & B & & \\ & \ddots & \ddots & \ddots & \\ & & B & A & B \\ & & & B & A \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-2} \\ w_{N-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{10}$$

where here u_j denotes the $(N-1)$-length vector of unknowns $u_{j,k}$ for $k = 1, 2, \ldots, N-1$ corresponding to the j^{th} grid-line in the y direction, and A and B are $(N-1)\times(N-1)$ matrices

$$A = \frac{1}{h^2}\begin{bmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 1 \\ & & & 1 & -4 \end{bmatrix} \qquad B = \frac{1}{h^2}\begin{bmatrix} -1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & -1 & \\ & & & & -1 \end{bmatrix}.$$

The $(N-1)$-length vectors w_j contain the non-linear entries $p_{j,k}\, u_{j,k}^{-\lambda}$, for $k = 1, 2, \ldots, N-1$. Once again, we may write the system as $Hu + g(u) = 0$, where H is the block tri-diagonal matrix and g is the non-linear vector function containing the w_j's.

Solution techniques

The classical solution technique for (8) or (10) is to apply Newton's method for non-linear systems. We write the system as $F(u) = 0$, where $F(u) = Hu + g(u)$. Each step of the iteration is then given by

$$u \leftarrow u - [J_F(u)]^{-1} F(u)$$

where the Jacobian of the system is given by

$$[J_F(u)] = H + D$$

with H the linear part of F and D a diagonal matrix whos diagonal entries are the derivatives of the entries of g, for example $-\lambda p(x_{j,k}) u_{j,k}^{-\lambda-1}$.

Naturally, the Jacobian is not inverted at each step, but rather, we solve the system $[J_F(u)]y = -F(u)$ and then make the correction $u \leftarrow u + y$. We examined two methods for solving the system at each step, namely LU decomposition and a multigrid FMV cycle. Newton's method converges quadratically. However, since each step involves inverting a system, it tends to be very slow. While the use of the FMV solver speeds the method up somewhat, it still is slower than the techniques we present next.

It has long been known ([12], [13]) that on certain problems non-linear analogs to the classical Jacobi or Gauss-Seidel iteration methods could be employed with some success. Technically, one sweep of such a method means that for $j = 1, 2, \ldots, N-1$ (or $(N-1)^2$ for the two-dimensional problem) one solves, via the scalar Newton's method, the j^{th} non-linear equation in the system $F(u) = 0$ for the j^{th} unknown. As in the linear case, if the old values u are used throughout the sweep this is the Newton-Jacobi method, while if the updated values are employed as they become available it is the Newton-Gauss-Seidel method. In practice the j^{th} equation is not actually solved, but rather, a few (one or two) steps of the scalar Newton's method is performed on each equation in turn.

The Newton-Jacobi and Newton-Gauss-Seidel iterations, however, typically behave in the same fashion that is observed in their linear counterparts. That is, the iteration generally progresses rapidly toward a solution with the first few sweeps, but then stalls out so that each additional sweep produces very little improvement. The reason behind this is the same as that seen in the linear case. The method stalls after the non-linear relaxation has successfully eliminated the oscillatory portion of the error, which it eliminates rapidly, but is unable to effectively treat the smooth portion of the error. This is precisely the difficulty that multigrid methods were devised to overcome.

At the heart of multigrid is the coarse-grid correction [14]. Many common relaxation iterative relaxation methods for solving a *linear* problem $Au = f$ have the property that the relaxation effectively eliminates the high-frequency (oscillatory) components of the error but leave the low frequency (smooth) components essentially unaffected. However, because the error is smooth after the relaxation, it may be represented accurately on a coarser grid, on which it also appears more oscillatory (relatively). Relaxation on this coarser grid then eliminates the oscillatory components of the coarse-grid error, which cannot be eliminated on the fine grid. The coarse-grid correction for a linear problem may be written as

$$u^h \leftarrow P^\nu u^h + I_{2h}^h (A^{2h})^{-1} I_h^{2h} (f^h - A^h P^\nu u^h) \tag{11}$$

where P is the relaxation matrix, ν is the number of relaxations, I_{2h}^h is a *prolongation* or *interpolation* matrix mapping coarse-grid vectors to the fine grid, I_h^{2h} is a *restriction* matrix mapping fine-grid vectors to the coarse grid, and A^{2h} is a coarse-grid version of the original matrix A. A crucial feature is that on the coarse grid Ω^{2h}, the problem to be solved is the *residual* equation $Ae = r$, where the residual is defined $r = f - Au$ and e is the *error*. That is, if u^* is the exact solution, then $Ae = A(u^* - u) = f - Au = r$.

For *nonlinear* problems the residual equation doesn't hold. Instead, we write the nonlinear equivalent of the residual equation,

$$F(u+e) - F(u) = r.$$

This equation is to be solved on the coarse grid, so we write

$$F^{2h}(I_h^{2h} u^h + e^{2h}) - F^{2h}(I_h^{2h} u^h) = I_h^{2h}(f^h - F^h(u^h)), \tag{12}$$

or

$$F^{2h}(u^{2h}) = I_h^{2h}(f^h - F^h(u^h)) + F^{2h}(I_h^{2h}).$$

The coarse-grid correction is then performed by solving (12) for $u^{2h} = I_h^{2h} u^h + e^{2h}$, and then making the correction $u^h \leftarrow u^h + I_{2h}^h(u^{2h} - I_h^{2h} u^h)$. This gives the full approximation scheme [15]

$$u^h \leftarrow P^\nu(u^h) + I_{2h}^h((F^{2h})^{-1}(I_h^{2h}(f^h - F^h(P^\nu(u^h))) + F^{2h}(I_h^{2h} P^\nu(u^h))) - I_h^{2h} P^\nu(u^h)),$$

where P is a nonlinear relaxation scheme.

For both the linear and nonlinear problems, the solution of the coarse-grid problem is computed using the same coarse-grid correction scheme as is being employed to solve the fine-grid problem. This leads to the multigrid *V-cycle* scheme, which (for the nonlinear problem using *FAS*) is described recursively as follows.

$$u^h \leftarrow FASV^h(u^h, f^h, \nu_1, \nu_2)$$

1. Perform ν_1 non-linear relaxation sweeps times on $F^h(u^h) = f^h$ with initial guess u^h.
2. If Ω^h is the coarsest grid, then go to 4. Else:
   $f^{2h} = I_h^{2h}(f^h - F^h(u^h)) + F^{2h}(I_h^{2h} u^h)$
   $u^{2h} \leftarrow 0$
   $u^{2h} \leftarrow FASV^{2h}(u^{2h}, f^{2h}, \nu_1, \nu_2)$.
3. Correct $u^h \leftarrow u^h + I_{2h}^h(u^{2h} - I_h^{2h} u^h)$.
4. Perform ν_2 non-linear relaxation sweeps times on $F^h(u^h) = f^h$ with initial guess u^h.

An important consideration for this (or any) iterative method is the choice of a good initial guess. Clearly a better initial guess will reduce the overall effort required to obtain an acceptable solution. A standard approach in multigrid is to obtain a good initial guess by first solving the problem on a coarse grid, and then interpolating that solution to the fine-grid for use as an initial guess. Solving this coarse-grid problem, in turn, will be easier if an initial guess is obtained by first solving the problem on a still coarser grid. Applying this idea recursively leads the the Full Multigrid *FMG* scheme, which (applied to the non-linear *FASV* scheme) may be described as follows:

$$u^h \leftarrow \ FASFMG^h(u^h, \nu_1, \nu_2)$$

1. If Ω^h is the coarsest grid, then go to 3. Else:
   $f^{2h} = I_h^{2h}(f^h - F^h(u^h)) + F^{2h}(I_h^{2h}u^h)$
   $u^{2h} \leftarrow 0$
   $u^{2h} \leftarrow FASFMG^{2h}(u^{2h}, f^{2h}, \nu_1, \nu_2)$.
2. Correct $u^h \leftarrow u^h \ + \ I_{2h}^h u^{2h}$.
3. $u^h \leftarrow FASV^h(u^h, f^h, \nu_1, \nu_2)$.

Numerical results for multigrid methods

We have implemented the *FASV* using Newton-Jacobi and Red-black Newton-Gauss-Seidel iteration schemes. (Our implementation was in *Matlab* using vector arithmetic. We elected not to analyse Newton-Gauss-Seidel since it is not vectorizable. We did encode it, however, and found that the slowness of the *for* loops overwhelmed the speed of convergence.) Several different choices for λ, $p(x)$ and $p(x,y)$ were used, as were several sets of relaxation parameters.

Table 1 gives some quantitative information regarding the performance of the method, comparing convergence rates for various choices of parameters. The results shown were obtained using the Red-black Newton-Gauss-Seidel relaxation. We find that for this problem we are able to obtain convergence rates that are similar to those obtained on the linear elliptic model problems for which multigrid is best known ([14], [16], [17]). Data for the one-dimensional problem are not shown, however, they are very similar to the two-dimensional case.

Dimension	$p(\vec{x})$	λ	Fine-grid size	Average V-cycle convergence factor
2	$2xy$	2	63×63	0.051
		5		0.050
		8		0.078
2	$2\sin(2\pi x)\sin(\pi y)$	2	63×63	0.060
		5		0.063
		8		0.104
2	x/y	2	63×63	0.059
		2		0.060
		8		0.086

Table 1

Additionally, we have implemented the *FASFMG* using Newton-Jacobi and Red-black Newton-Gauss-Seidel iteration schemes. Again, we find that the performance of the method is compatible with that found for FMG applied to the linear model problems ([15], [17]).

CONCLUSIONS

Our survey of existence and uniqueness results has shown the the problem

$$\Delta u \,+\, p(x)\, u^{-\lambda} \;=\; 0 \quad x \in \Omega$$

is guaranteed to have unique solutions under certain conditions, although these solutions will not be known in closed form. The problem arises in certain non-Newtonian fluids problems, so there is some interest in actually computing solutions. We have shown that for homogeneous Dirichlet boundary conditions on the unit interval and the unit square, multigrid methods appear to provide an efficient means of solution for reasonable choices of $p(x)$.

We note, however, that an actual convergence proof for the FAS method would be very difficult to obtain, in that such proofs normally require that we be able to decompose the space of grid functions into two operator-subspaces. Error components in one are anihilated by relaxation, while those in the other subspace are anihilated by coarse-grid correction. While such analysis is achieved for linear problems, non-linear problems generally can only be treated by linearization near a solution. In point of fact, the literature is remarkably sparse in the area of founding theory for the *FAS* method.

A new technique, called *multilevel projection methods (PML)* has recently been introduced, [18] in an effort to provide a unifying, thematic approach to the design of a multilevel solver for a given problem. The main feature of PML methods is that the only basic choices that must be made concern the subspaces that will be used in relaxation and coarsening. All other components of the method, such as interlevel transfers, scaling, coarse-level problems, etc., are determined by projection between appropriate subspaces. In [18], several prototypical problems are developed to illustrate the principals involved. It now appears that the best hope of obtaining a strong founding theory for multilevel treatment of nonlinear problems may well be through careful and judicious application of *PML*, and our future research into solution methods for the problems we have discussed here will be aimed in that direction.

ACKNOWLEDGEMENTS

The authors wish to thank Steve McCormick for his helpful suggestions throughout the project. This work was supported, in part, by the Research Council of the Naval Postgraduate School.

REFERENCES

[1] M. G. Crandall, P. H. Rabinowitz, and L. Tartar. On a dirichlet problem with a singular nonlinearity. *Comm. Part. Diff. Eq.*, 2(2):193–222, 1977.

[2] W. Fulks and L. S. Maybee. A singular nonlinear equation. *Osaka Math. J.*, 12:1–19, 1960.

[3] S. Taliaferro. On the positive solution of $y'' + \phi(t)y^{-\lambda} = 0$. *Nonlinear Analysis, Theory, Methods & Applications*, 2(4):437–446, 1978.

[4] A. J. Callegari and A. Nachman. A nonlinear singular boundary value problem in the theory of pseudoplastic fluids. *SIAM J. Appl. Math*, 30:275–281, 1980.

[5] A. C. Lazer and P. J. McKenna. On a singular nonlinear elliptic boundary value problem. *Proceedings of the AMS*, 111:721–730, 1991.

[6] A. Edelson. Entire solutions of singular elliptic equations. *J. Math. Anal. Appl.*, 139:523–532, 1989.

[7] T. Kusano and A. Swanson. Entire positive solutions of singular semilinear elliptic equations. *Japan J. Math.*, 11, 1985.

[8] A. W. Shaker. On singular semilinear elliptic equations. *J. Math. Anal. and Appl.*, 173(1):222–228, 1993.

[9] R. Dalmasso. Solutions d'equations elliptiques semilineaires singulieres. *Annali di matematica pura et applicata serie quarta*, 153:191–201, 1988.

[10] D. H. Sattinger. *Topics in stability snd bifurcation theory.* Springer-Verlag, Berlin, 1973.

[11] W. Ni. On the elliptic equation $\delta u + k(x)u^{(n+2)/(n-2)} = 0$, its generalization and application in geometry. *Indiana Univ. Math. J.*, 31(optional):493–525, 1982.

[12] Louis B. Rall. *Computational solution of nonlinear operator equations.* Robert E. Krieger Publishing Company, Huntington, New York, 1979.

[13] James M. Ortega. *Numerical analysis, a second course.* Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1979.

[14] William L. Briggs. *A Multigrid Tutorial.* Society for Industrial and Applied Mathematics, Philadelphia, 1987.

[15] Achi Brandt. *Mulitgrid techniques: 1984 guide with application to fluid dynamics.* GMD-Studien Nr. 85. Gesellschaft für Mathematik und Datenverarbeitung, St. Augustin, 1984.

[16] Jan Mandel, Stephen F. McCormick, and R. Bank. Variational multigrid theory. In Stephen F. McCormick, editor, *Multigrid methods*, volume 3 of *Frontiers in applied mathematics*, pages 131–177, Philadelphia, PA, 1987. Society for Industrial and Applied Mathematics.

[17] K. Stüben and Ullrich Trottenberg. Multigrid methods: Fundamental algorithms, model problem analysis and applications. In W. Hackbusch and U. Trottenberg, editors, *Multigrid methods, proceedings of a conference held at Köln-Porz, November 23-27, 1981*, volume 960 of *Lecture notes in mathematics*, pages 1–176, Berlin, 1982. Springer-Verlag.

[18] Stephen F. McCormick. *Multilevel projection methods for partial differential equations*, volume 62 of *CBMS-NSF regional conference series in applied mathematics.* Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.