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ON THE REAL SINGULARITIES OF THE N-BODY PROBLEM

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Abstract

A collision singularity in the N-body problem has the property that all position vectors have finite limits and at least one mutual distance approaches zero as time approaches the instant of the singularity. General properties of a collision singularity are investigated and it is found that they are analogous to those of a collision of all bodies at the same point and time. A more detailed study of a simple type of such a singularity, namely that of several simultaneous binary collisions, is presented.

Introduction

Little is known about the singularities of the equations of motion of the N-body problem of celestial mechanics, particularly if complex values of the independent and dependent variables are admitted. Only the three simplest cases have received considerable attention, as manifested by the bibliography: the binary collision in the three-body problem (e.g., Sundman [4]) and in the N-body problem (e.g., Wintner [2]), the triple collision in the three-body problem (e.g., Sundman [1], Siegel [1]), and the collision of all N bodies at the same point and instant (e.g., Block [2], Wintner [2]); all of the above studies are restricted to real values of all variables. For a study of the binary collision in the N-body problem for complex values of the variables, see Sperling [3].

It is presently unknown which types of real singularities (in the sense of complex analysis as well as with regard to their dynamical interpretation) are possible in the N-body problem; the only general result is a remarkable statement by von Zeipel [2] (cf. 205), asserting that a singularity is a collision singularity (cf. 203) if the mutual distances remain bounded as time approaches the instant of the singularity. "Collision singularities" are such that the N bodies separate into distinct "clusters" with all bodies in each cluster colliding at the instant of the singularity at a well-defined point, while the clusters remain apart from each other. Since von Zeipel's statement seems to be virtually unknown and particularly in view of Wintner's [2, p. 431] remark: " A note of H. von Zeipel indicates a consideration to the effect that, if U becomes infinite when t tends to a finite value, then J must tend to infinity, unless all bodies tend to definite limiting positions. But it seems to be hard to fill in the gaps ", we will subsequently present a detailed proof, following essentially von Zeipel's ideas. Chazy's [6] attempt of a proof is only a brief sketch and must be considered at least vague. The introduction of " cluster coordinates " (cf. 103-104) enables us to prove in a simple manner that the existence of finite limits of the mutual distances implies the existence of the limits of the positions and relative positions as t approaches a finite value (cf. 216); Wintner [2, p. 327] mentions this as an undecided problem (cf. von Zeipel [2, p. 4]).

It will be shown that the known important properties of a collision of all bodies at the same instant hold, with obvious modifications, also for a collision singularity, the main result probably being that the bodies of each cluster are positioned close to a central configuration when the time is close to the instant of the singularity.

A detailed study of the case of several simultaneous binary collisions concludes the paper. This case has already been considered by Lahaye [3]; his treatment, however, is at least incomplete: his assumption, that the asymptotic behavior of a perturbed two-body problem equals that of the two-body problem as the instant of a binary collision is approached, is not justified without proof, and he does not show that his solutions are the only ones. The main result of the investigation is that this singularity gives rise to an algebraic branch point of the second order on the real time axis, the same as for <u>one</u> binary collision; in particular, the motion can be analytically continued in the real beyond this singularity.

While writing this paper, I was informed (an abstract in the Notices Amer. Math. Soc. 15 (1968) 631 and private communication) that H. Pollard has worked in the same field. Details of his work have not been available to me at the time of the completion of this paper.

On the notation

Let $t^* \neq \pm\infty$ (later $t^* = 0$) be the instant of the (collision) singularity; all limits are taken as $t \to t^*$ $(t < t^*)$.

c, $\underset{\sim}{c}$ denote positive constants

b, $\underset{\sim}{b}$ denote functions of t or $s = t^{\frac{1}{3}}$, bounded on an interval $[t_0, t^*[$ with suitable $t_0 < t^*$;

e, $\underset{\sim}{e}$ denote functions of t or s, with $e \to 0$, $\underset{\sim}{e} \to \underset{\sim}{0}$ as $t \to t^*$.

The above symbols have fixed meaning, at least within the given context, if they are indexed; without indexes they are unspecified, i.e., they may denote different constants or functions from one occurrence to the next. For example, in

$$\int_0^t b(\tau)\, d\tau = b(t) \quad \text{and in} \quad f(t) = b(t) + t\, b(t) \quad \text{the } b\text{'s}$$

occurring in the same equation may denote different functions.

The absolute value of a vector $\underset{\sim}{z}$ is z $(= |\underset{\sim}{z}| = +(\underset{\sim}{z}\underset{\sim}{z})^{\frac{1}{2}})$.

$\sim$ denotes asymptotic behavior as $t \to t^*$ $(= 0)$ or $s \to 0$: $f \sim g$ is equivalent to $f = g + o(g)$.

Differentiation with respect to time t is denoted by $\dot{}$: $df/dt = \dot{f}$.

Differentiation with respect to s is denoted by $'$: $df/ds = f'$.

The gradient of a scalar f with respect to a vector $\underset{\sim}{z}$ is denoted by $\text{grad}_{\underset{\sim}{z}} f = \dfrac{\partial f}{\partial \underset{\sim}{z}}$.

Subscript following semicolon: 0: some fixed value; 1, 2, 3: the three components of a vector.

100 Equations and integrals of motion

For reference we list here the equations and integrals of motion and some related concepts and equations, in the two coordinate systems used later. The geometric basis is the 3-dimensional euclidean space.

101 (Cartesian) inertial coordinates, the center of mass resting at the origin. N particles (=bodies) with masses $m_k > 0$ are moving according to Newton's law of gravitation. Let γ be the gravitational constant, $\underset{\sim}{z}_k$ the position vector from the origin to the body m_k, and

$$\underset{\sim}{z}_{jk} = \underset{\sim}{z}_k - \underset{\sim}{z}_j \tag{101.1}$$

the relative position vector from m_j to m_k.

Set

$$M = \sum_{k=1}^{N} m_k \quad . \tag{101.2}$$

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$$\ddot{\underset{\sim}{z}}_k = -\gamma \sum_{\substack{j=1\\ j\neq k}}^{N} m_j \frac{\underset{\sim}{z}_k - \underset{\sim}{z}_j}{|\underset{\sim}{z}_k - \underset{\sim}{z}_j|^3} = -\gamma \sum_{\substack{j=1\\ j\neq k}}^{N} m_j \frac{\underset{\sim}{z}_{jk}}{z_{jk}^3} = -\frac{1}{m_k}\frac{\partial U}{\partial \underset{\sim}{z}_k}$$

$$k = 1 \text{ -- } N \quad , \tag{102.1}$$

potential energy

$$U = -\gamma \sum\sum_{1\leq j<k\leq N} \frac{m_j m_k}{|\underset{\sim}{z}_k - \underset{\sim}{z}_j|} = -\gamma \sum\sum_{1\leq j<k\leq N} \frac{m_j m_k}{z_{jk}} \quad , \tag{102.2}$$

kinetic energy

$$V = \tfrac{1}{2}\sum_{j=1}^{N} m_j \dot{\underset{\sim}{z}}_j \dot{\underset{\sim}{z}}_j = \tfrac{1}{2}\sum_{j=1}^{N} m_j |\dot{\underset{\sim}{z}}_j|^2 \quad , \tag{102.3a}$$

$$V = \frac{1}{4M} \sum_{j=1}^{N} \sum_{k=1}^{N} m_j m_k \dot{\underset{\sim}{z}}_{jk} \dot{\underset{\sim}{z}}_{jk} \quad , \tag{102.3b}$$

polar inertia momentum

$$J = \sum_{j=1}^{N} m_j \underset{\sim}{z}_j \underset{\sim}{z}_j = \sum_{j=1}^{N} m_j z_j^2 \tag{102.4a}$$

$$= \frac{1}{2M} \sum_{j=1}^{N} \sum_{k=1}^{N} m_j m_k \underset{\sim}{z}_{jk} \underset{\sim}{z}_{jk} = \frac{1}{M} \sum\sum_{1<j<k\leq N} m_j m_k z_{jk}^2 \quad , \tag{102.4b}$$

angular momentum

$$\underset{\sim}{K} = \sum_{j=1}^{N} m_j \underset{\sim}{z}_j \times \dot{\underset{\sim}{z}}_j = \frac{1}{2M} \sum_{j=1}^{N} \sum_{k=1}^{N} m_j m_k \underset{\sim}{z}_{jk} \times \dot{\underset{\sim}{z}}_{jk} \quad , \tag{102.5}$$

conservation of energy (energy theorem)

$$V + U = h \quad (= \text{constant of energy}) \quad , \tag{102.6}$$

conservation of angular momentum

$$\underset{\sim}{K} = \underset{\sim}{\text{constant}} \quad , \tag{102.7}$$

center of mass rests at the origin

$$\sum_{j=1}^{N} m_j \underset{\sim}{z}_j = \underset{\sim}{0} \quad , \tag{102.8}$$

Lagrange-Jacobi equation

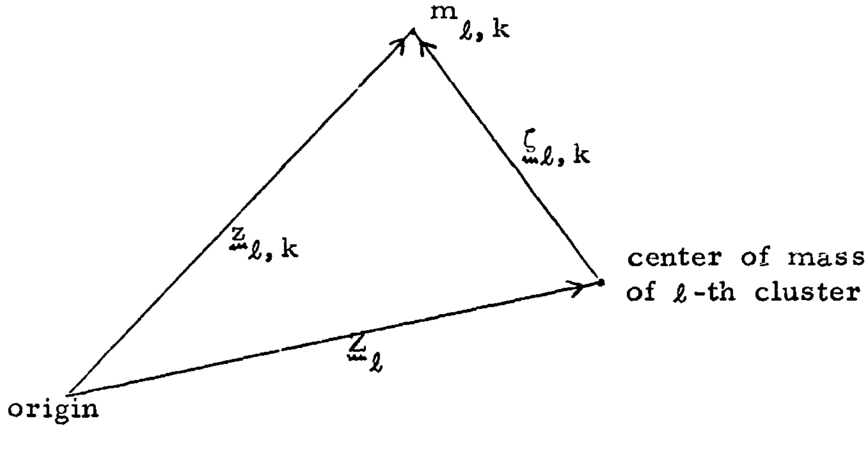
$$J'' = -2\,U + 4h = 2\,V + 2h \quad ; \tag{102.9}$$

the positions $\underset{\sim}{z}_k$ are expressed in terms of the relative positions by

$$M \underset{\sim}{z}_k = \sum_{j=1}^{N} m_j \underset{\sim}{z}_{jk} \quad . \tag{102.10}$$

103 "Cluster coordinates". Partition the set of all N bodies into L nonempty disjunct subsets, called "clusters" in the following. Let N_ℓ, $1 \le N_\ell \le N$, be the number of bodies in the ℓ-th cluster. Obviously, $N_1 + \ldots + N_L = N$. (Observe that subscripts are separated by commas in cluster coordinates and that $\ell = 1 -- L$, $\kappa = 1 -- N_\ell$). Notation:

$m_{\ell,k}$ = mass of the k-th body in the ℓ-th cluster,
$\underset{\sim}{z}_{\ell,k}$ = position vector from the origin to $m_{\ell,k}$,
$\underset{\sim}{Z}_\ell$ = position vector from the origin to the center of mass of the ℓ-th cluster,
$\underset{\sim}{\zeta}_{\ell,k}$ = position vector from the center of mass of the ℓ-th cluster to $m_{\ell,k}$.



Obviously,

$$\underset{\sim}{z}_{\ell,k} = \underset{\sim}{Z}_\ell + \underset{\sim}{\zeta}_{\ell,k} \quad ; \qquad (103.1)$$

define M_ℓ and $\underset{\sim}{Z}_\ell$ by

$$M_\ell = \sum_{k=1}^{N_\ell} m_{\ell,k} \quad \text{and} \quad M_\ell \underset{\sim}{Z}_\ell = \sum_{k=1}^{N_\ell} m_{\ell,k} \underset{\sim}{z}_{\ell,k} \quad ; \qquad (103.2)$$

then

$$\sum_{k=1}^{N_\ell} m_{\ell,k} \underset{\sim}{\zeta}_{\ell,k} = \underset{\sim}{0} \tag{103.3}$$

and

$$M_\ell \underset{\sim}{\zeta}_{\ell,k} = \sum_{j=1}^{N_\ell} m_{\ell,j} \left(\underset{\sim}{\zeta}_{\ell,k} - \underset{\sim}{\zeta}_{\ell,j} \right) \quad . \tag{103.4}$$

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$$\underset{\sim}{z}''_{\ell,k} = -\gamma \sum_{\substack{\lambda=1 \\ \ell,k \neq \lambda,\varkappa}}^{L} \sum_{\varkappa=1}^{N_\lambda} m_{\lambda,\varkappa} \frac{\underset{\sim}{z}_{\ell,k} - \underset{\sim}{z}_{\lambda,\varkappa}}{\left|\underset{\sim}{z}_{\ell,k} - \underset{\sim}{z}_{\lambda,\varkappa}\right|^3} \quad , \tag{104.1}$$

$$M_\ell \underset{\sim}{Z}''_\ell = -\gamma \sum_{\substack{\lambda=1 \\ \lambda \neq \ell}}^{L} \sum_{\varkappa=1}^{N_\lambda} \sum_{k=1}^{N_\ell} m_{\ell,k} m_{\lambda,\varkappa} \frac{\underset{\sim}{z}_{\ell,k} - \underset{\sim}{z}_{\lambda,\varkappa}}{\left|\underset{\sim}{z}_{\ell,k} - \underset{\sim}{z}_{\lambda,\varkappa}\right|^3}$$

(104.2)

$$\underset{\sim}{z}_{\ell,k} - \underset{\sim}{z}_{\lambda,\varkappa} = \underset{\sim}{Z}_\ell - \underset{\sim}{Z}_\lambda + \underset{\sim}{\zeta}_{\ell,k} - \underset{\sim}{\zeta}_{\lambda,\varkappa} \quad ,$$

$$\underset{\sim}{\zeta}''_{\ell,k} = -\gamma \sum_{\substack{j=1 \\ j \neq k}}^{N_\ell} m_{\ell,j} \frac{\underset{\sim}{\zeta}_{\ell,k} - \underset{\sim}{\zeta}_{\ell,j}}{\left|\underset{\sim}{\zeta}_{\ell,k} - \underset{\sim}{\zeta}_{\ell,j}\right|^3} \quad -$$

(104.3a)

$$\gamma \sum_{\substack{\lambda=1 \\ \lambda \neq \ell}}^{L} \sum_{\varkappa=1}^{N_\lambda} m_{\lambda,\varkappa} \left[\frac{\underset{\sim}{Z}_\ell - \underset{\sim}{Z}_\lambda + \underset{\sim}{\zeta}_{\ell,k} - \underset{\sim}{\zeta}_{\lambda,\varkappa}}{\left|\underset{\sim}{Z}_\ell - \underset{\sim}{Z}_\lambda + \underset{\sim}{\zeta}_{\ell,k} - \underset{\sim}{\zeta}_{\lambda,\varkappa}\right|^3} - \frac{1}{M_\ell} \sum_{\mu=1}^{N_\ell} m_{\ell,\mu} \frac{\underset{\sim}{Z}_\ell - \underset{\sim}{Z}_\lambda + \underset{\sim}{\zeta}_{\ell,\mu} - \underset{\sim}{\zeta}_{\lambda,\varkappa}}{\left|\underset{\sim}{Z}_\ell - \underset{\sim}{Z}_\lambda + \underset{\sim}{\zeta}_{\ell,\mu} - \underset{\sim}{\zeta}_{\lambda,\varkappa}\right|^3} \right]$$

or

$$\ddot{\underline{\zeta}}_{\ell, k} = - \frac{1}{m_{\ell, k}} \frac{\partial U_{\ell}}{\partial \underline{\zeta}_{\ell, k}} + \quad {}_{, k} \qquad . \tag{104.3b}$$

Define the potential energy of the ℓ-th cluster by

$$U_{\ell} = -\gamma \sum_{\substack{k=1 \\ k<j}}^{N_{\ell}} \sum_{j=1}^{N_{\ell}} \frac{m_{\ell, j} m_{\ell, k}}{|\underline{\zeta}_{\ell, j} - \underline{\zeta}_{\ell, k}|} \quad , \tag{104.4}$$

its kinetic energy by

$$V_{\ell} = \frac{1}{2} \sum_{k=1}^{N_{\ell}} m_{\ell, k} \dot{\underline{\zeta}}_{\ell, k} \dot{\underline{\zeta}}_{\ell, k} \quad , \tag{104.5}$$

its (total) energy by

$$h_{\ell} = V_{\ell} + U_{\ell} \tag{104.6}$$

(observe that h_{ℓ} is not a constant in general) ,

its polar inertia momentum by

$$J_{\ell} = \sum_{k=1}^{N_{\ell}} m_{\ell, k} \underline{\zeta}_{\ell, k} \underline{\zeta}_{\ell, k} \tag{104.7a}$$

$$= \frac{1}{2M_{\ell}} \sum_{j=1}^{N_{\ell}} \sum_{k=1}^{N_{\ell}} m_{\ell, j} m_{\ell, k} (\underline{\zeta}_{\ell, k} - \underline{\zeta}_{\ell, j})(\underline{\zeta}_{\ell, k} - \underline{\zeta}_{\ell, j}) \; , \tag{104.7b}$$

and its angular momentum by

$$\underline{K}_{\ell} = \sum_{k=1}^{N_{\ell}} m_{\ell,k}\, \underline{\zeta}_{\ell,k} \times \dot{\underline{\zeta}}_{\ell,k} \qquad ; \qquad (104.8)$$

define further the kinetic energy of the centers of mass of all clusters by

$$V^{\circ} = \tfrac{1}{2} \sum_{\ell=1}^{L} M_{\ell}\, \dot{\underline{Z}}_{\ell}\, \dot{\underline{Z}}_{\ell} \qquad , \qquad (104.9)$$

their polar inertia momentum by

$$J^{\circ} = \sum_{\ell=1}^{L} M_{\ell}\, \underline{Z}_{\ell}\, \underline{Z}_{\ell} \qquad , \qquad (104.10)$$

and their angular momentum by

$$\underline{K}^{\circ} = \sum_{\ell=1}^{L} M_{\ell}\, \underline{Z}_{\ell} \times \dot{\underline{Z}}_{\ell} \qquad ; \qquad (104.11)$$

write

$$U = \sum_{\ell=1}^{L} U_{\ell} + U^{+} = U^{*} + U^{+} \qquad , \qquad (104.12)$$

observing that U^{+} consists exactly of those terms of U which contain the difference of coordinates of masses in different clusters.

Using (103.3), it is easy to show that

$$V = \sum_{\ell=1}^{L} V_{\ell} + V^{\circ} = V^{*} + V^{\circ} \qquad , \qquad (104.13)$$

$$J = \sum_{\ell=1}^{L} J_{\ell} + J^{\circ} = J^{*} + J^{\circ} \qquad , \qquad (104.14)$$

$$\underset{\sim}{K} = \sum_{\ell=1}^{L} \underset{\sim}{K}_{\ell} + \underset{\sim}{K}^{\circ} = \underset{\sim}{K}^{*} + \underset{\sim}{K}^{\circ} \quad , \tag{104.15}$$

and

$$h = \sum_{\ell=1}^{L} h_{\ell} + V^{\circ} + U^{+} = h^{*} + V^{\circ} + U^{+} \quad . \tag{104.16}$$

200 Classification and properties of real singularities

As mentioned before, we restrict ourselves (necessarily) to real values of the independent variable time t and the dependent variables, the positions and velocities; consider only real analytic continuation of the solution of the equations of motion and exclude the two cases that $t = (\pm)\infty$, or that the finite t is an accumulation point of singularities from both sides: then, if there is a singularity at t^{*}, this point is the right or left endpoint of an open interval on which the solution is holomorphic. Without restriction of generality, we will assume that t^{*} is the right endpoint of such an interval.

201 For definiteness, we refer to the equations of motion (102.1), but analogous statements obviously hold for the equations in the form (104.2), (104.3). Let t_o, $\underset{\sim}{z}_{k;o}$, $\underset{\sim}{z}'_{k;o}$ be finite and real such that $|\underset{\sim}{z}_{k;o} - \underset{\sim}{z}_{j;o}| \neq 0$ for $j \neq k$; by the existence theorem for analytic differential equations there exists an interval $|t - t_o| < \delta$ and on it a unique holomorphic solution $\underset{\sim}{z}_k(t)$; $\underset{\sim}{z}'_k(t)$, $k = 1 \text{ -- } N$, of (102.1) satisfying $\underset{\sim}{z}_k(t_o) = \underset{\sim}{z}_{k;o}$, $\underset{\sim}{z}'_k(t_o) = \underset{\sim}{z}'_{k;o}$. It is noteworthy (and soon to be used) that an estimate of δ can be made involving only $\min|\underset{\sim}{z}_{k;o} - \underset{\sim}{z}_{j;o}|$ and $\max|\underset{\sim}{z}'_{k;o}|$, but not $\underset{\sim}{z}_{k;o}$ (cf. Siegel [3], pp. 23-25). Continue this solution holomorphically from t_o for increasing t; the finite point t^* is a singularity, if the solution is holomorphic on $[t_o, t^*[$, but not on $[t_o, t^*]$.

202 Theorem a): t^* is a singularity iff $\liminf(\min z_{jk}) = 0$ as $t \to t^*$.

b): t^* is a singularity iff $\lim(\min z_{jk}) = 0$ as $t \to t^*$.

Proof:

1. If t^* is a singularity then $\liminf(\min z_{jk}) = 0$: assume contrariwise that $\liminf(\min z_{jk}) \geq d_o > 0$.

Then there exist numbers t_1 and d_1, $t_o \leq t_1 < t^*$ and $0 < d_1 \leq d_o$, such that $z_{jk} \geq d_1$ on $[t_1, t^*[$; we infer that $|\underset{\sim}{z}_{jk} z_{jk}^{-3}| = z_{jk}^{-2} \leq d_1^{-2}$ and, using the equations of motion (102.1), $|\underset{\sim}{z}''_k| \leq \gamma M d_1^{-2}$. From

$$\underset{\sim}{z}_k = \underset{\sim}{z}_k(t_1) + (t - t_1)\underset{\sim}{z}'_k(t_1) + \int_{t_1}^{t} (t - \tau)\underset{\sim}{z}''_k(\tau)\, d\tau$$

it follows that $\underset{\sim}{z}_k$ and $\underset{\sim}{z}'_k$ have finite limits $\underset{\sim}{z}^*_k$ and $\underset{\sim}{z}'^*_k$ with $z^*_{jk} = |\underset{\sim}{z}^*_k - \underset{\sim}{z}^*_j| \geq d_1$ for $j \neq k$ as $t \to t^*$. The existence theorem (cf. 201), applied to initial values very close to t^*, $\underset{\sim}{z}_k^*$, $\underset{\sim}{z}_k^{*\prime}$, states that the solution remains holomorphic at t^*, contrary to the assumption.

2. If $\liminf(\min z_{jk}) = 0$, then t^* is a singularity: if at least one $\underset{\sim}{z}_k''$ is unbounded as $t \to t^*$, then there is a singularity at t^*. Assume now that all $\underset{\sim}{z}_k''$ remain bounded; then, by integration, all $\underset{\sim}{z}_k'$ are bounded. In

$$U' = \sum_{j=1}^{3}\sum_{k=1}^{N} \frac{\partial U}{\partial z_{k;j}} z'_{k;j} \quad \text{the} \quad \frac{\partial U}{\partial z_{k;j}}$$ are bounded by (102.1) and the

assumption; thus, U' and therefore U are bounded. This is inconsistent with the fact that $\liminf(\min z_{jk}) = 0$ implies $\limsup U = -\infty$.

3. If t^* is a singularity, then $\lim(\min z_{jk}) = 0$: assume contrariwise that $\limsup(\min z_{jk}) \geq d_0 > 0$. There exists a sequence $t_\nu \to t^*$ such that $z_{jk}(t_\nu) \geq d_1 > 0$, $-U \leq d_2$, and by the energy theorem $V \leq d_3$, implying $|\underset{\sim}{z}_k'(t_\nu)| \leq d_4$. Apply the existence theorem (cf. 201) to t_ν, $\underset{\sim}{z}_k(t_\nu)$, $\underset{\sim}{z}_k'(t_\nu)$ as initial values; if t_ν is sufficiently close to t^*, t^* is contained in the interval $|t - t_\nu| < \delta$ where the solution remains holomorphic, yielding a contradiction (observe that δ is a function of d_1 and d_4 only, but not of t_ν).

4. If $\lim(\min z_{jk}) = 0$, then t^* is a singularity: this is implied by 2.

Observe that a) does not require the energy theorem (102.6) for its proof.

203 Definition: A singularity t^* is called a collision singularity if all position vectors $\underset{\sim}{z}_k$ have finite limits as $t \to t^*$.

Hence, by 202 theorem, at least one of the mutual distances z_{jk} converges to zero as $t \to t^*$ for this type of singularity.

According to the original definition, the partition of the bodies into clusters (cf. 103) is quite arbitrary; in the following this partition is always used in conjunction with a collision singularity and assumed to be consistent with it, i.e.: two bodies m_j and m_k are in the same cluster iff z_{jk} is very small as $t \to t^*$ = instant of the collision singularity, or in terms of cluster coordinates:

$$\begin{array}{l} \text{all } |\underset{\sim}{\zeta}_{\ell,k} - \underset{\sim}{\zeta}_{\ell,j}| \text{ are very small compared with} \\ |\underset{\sim}{Z}_\ell - \underset{\sim}{Z}_\lambda| \, , \; \ell \neq \lambda \, , \text{ on the considered time interval.} \end{array} \tag{203.1}$$

204 Lemma: Let the $\underset{\sim}{z}_k = \underset{\sim}{z}_k(t)$ be holomorphic on $t_0 \le t < t^*$. Then $\lim J \le \infty$ exists as $t \to t^*$.

Proof (cf. Wintner [2] p. 327):

Let t^* be a singularity, otherwise the statement is trivial; then $\lim(\min z_{jk}) = 0$, hence $\lim(-U) = +\infty$. From $J'' = 4h - 2U$ it follows that J'' is ultimately positive; hence, J' is ultimately increasing and therefore ultimately does not change sign, and we conclude that J is ultimately monotonic (increasing or decreasing), implying the existence of the limit of J. Since $J \ge 0$, $\lim J$ is ∞ or a nonnegative number.

205 Theorem (von Zeipel): If there is a singularity at t^* and $\lim J < \infty$ as $t \to t^*$, then t^* is a collision singularity.

Proof: Assume that the solution of the N-body problem is holomorphic on $[t_0, t^*[$, $t_0 < t^*$.

206 Let first be $\lim J = 0$: then $z_k \to 0$ and $z_{jk} \to 0$ by (102.4), implying that $\mathbf{z}_k \to 0$ and $\mathbf{z}_{jk} \to 0$; all positions $\mathbf{z}_k$ and all relative positions $\mathbf{z}_{jk}$ have the finite limit zero and we have a collision singularity. All bodies collide at time t^* at the center of mass.

207 Now let $0 < \lim J = J(t^*) < \infty$; we partition the set of all mutual distances z_{jk} into four disjoint subsets as follows:

first type: $\lim z_{jk} > 0$ exists;

second type: $\lim z_{jk} = 0$ exists;

third type: $\lim z_{jk}$ does not exist and $\liminf z_{jk} > 0$;

fourth type: $\lim z_{jk}$ does not exist and $\liminf z_{jk} = 0$, $\limsup z_{jk} > 0$.

We show in the following that $0 < \lim J < \infty$ implies the existence and finiteness of all $\lim z_{jk}$. In a first case (cf. 208), we assume that there are no mutual distances of the fourth type and prove our assertion directly; in a second case (cf. 209-215), we assume that there is at least one distance of the fourth type and lead this assumption to a contradiction. 216 lemma then concludes the proof.

208 First case: there are no mutual distances of the fourth type. There exist a t_{oo} , $t_o \le t_{oo} < t^*$, and two constants a_1 and a_2 , $0 < a_1 < a_2$, such that on $[t_{oo}, t^*[$

for all mutual distances of the first and third type: $z_{jk} > a_2$,

and for all mutual distances of the second type: $z_{jk} < a_1$.

We consider now the solution on $[t_{00}, t^*[$. Introduce cluster coordinates such that two bodies are in the same cluster iff for their mutual distance $z_{jk} < a_1$ holds.

> Note: Because the mutual distances z_{jk} are continuous on $[t_{00}, t^*[$ and either $z_{jk} < a_1 < a_2$ or $z_{jk} > a_2 > a_1$, the separation of the bodies into clusters is unique on this interval; i.e., each body stays in one and the same cluster.

All mutual distances of bodies in the same cluster, say the ℓ-th, are by the definition of the cluster of the second type, i.e., $|\underset{\sim}{\zeta}_{\ell,k} - \underset{\sim}{\zeta}_{\ell,j}| \to 0$ as $t \to t^*$. By (103.4) this implies $\zeta_{\ell,k} \to 0$, hence $\underset{\sim}{\zeta}_{\ell,k} \to \underset{\sim}{0}$, i.e., all $\lim \underset{\sim}{\zeta}_{\ell,k}$ exist (and are zero).
Consider the equation of motion (104.2) for the center of mass $\underset{\sim}{Z}_\ell$ of the ℓ-th cluster; on the right of this equation only relative positions and mutual distances of bodies in different clusters occur, i.e., the mutual distances are of the first or third type; hence $|\underset{\sim}{z}_{\ell,k} - \underset{\sim}{z}_{\lambda,\varkappa}| > a_2$, $|(\underset{\sim}{z}_{\ell,k} - \underset{\sim}{z}_{\lambda,\varkappa})|\underset{\sim}{z}_{\ell,k} - \underset{\sim}{z}_{\lambda,\varkappa}|^{-3}| < a_2^{-2}$, and $|\underset{\sim}{Z}_\ell''| < \gamma M a_2^{-2}$. This together with

$$\underset{\sim}{Z}_\ell = \underset{\sim}{Z}_\ell(t_{00}) + (t - t_{00})\underset{\sim}{Z}_\ell'(t_{00}) + \int_{t_{00}}^{t} (t - \tau)\underset{\sim}{Z}_\ell''(\tau)\, d\tau$$

implies that $\lim \underset{\sim}{Z}_\ell$ exists and is finite as $t \to t^*$. We conclude that $\lim \underset{\sim}{z}_{\ell,k} = \lim \underset{\sim}{Z}_\ell + \lim \underset{\sim}{\zeta}_{\ell,k}$, $\lim \underset{\sim}{z}_{jk}$, $\lim z_k$, and $\lim z_{jk}$ all exist and are finite; i.e., in this first case all mutual distances have finite limits.

209 Second case: There is at least one mutual distance of the fourth type. The existence of $\lim J = J(t^*) > 0$ implies that for every $\varepsilon > 0$ there exists a t_{0*}, $t_0 \le t_{0*} < t^*$, such that

$$|J(t) - J(t^*)| < \varepsilon \qquad \text{on} \quad [t_{0*}, t^*[\quad . \tag{209.1}$$

Assume without restriction that $\varepsilon < \frac{1}{2} J(t^*)$; then $J_{min} > \frac{1}{2} J(t^*)$ on $[t_{0*}, t^*[$, and with (102.4) it follows that

$$\max z_{jk} > \left(\frac{J(t^*)}{M}\right)^{\frac{1}{2}} = a_3 \qquad \text{on} \quad [t_{0*}, t^*[\tag{209.2}$$

and

$$z_{jk} < \left(\frac{3\,M\,J(t^*)}{2\,m_{min}^2}\right)^{\frac{1}{2}} = a_4 \qquad \text{on} \quad [t_{0*}, t^*[\quad , \tag{209.3}$$

and by (102.10)

$$z_k < a_4 \quad , \quad k = 1 \text{ -- } N \, , \quad \text{on} \quad [t_{0*}, t^*[\quad . \tag{209.4}$$

210 Define $n(\tau, d)$ as the number of mutual distances z_{jk} for which $z_{jk}(\tau) \ge d$ holds; consider $n(\tau, d)$ on the set $\{ \tau, d : t_{0*} \le \tau < t^* \, , \; 0 < d \le a_3 \}$; $\frac{1}{2}N(N-1)$ being the total number of mutual distances, $n(\tau, d)$ satisfies

$$1 \le n(\tau, d) \le \tfrac{1}{2}N(N-1) \qquad \text{on this set} \; ; \tag{210.1}$$

the first inequality holding because of (209.2).

Define

$$\begin{aligned} &\mathfrak{N}(t, d) = \text{g.l.b.}\, \mathfrak{S}(t, d) \\ &\mathfrak{S}(t, d) = \{ \, n(\tau, d) : \tau \in [t, t^*[\; , \; t_{0*} \le t < t^* \, , \; 0 < d \le a_3 \, , \\ &\qquad\qquad t \text{ fixed} \, , \; d \text{ fixed} \, \} \; ; \end{aligned} \tag{210.2}$$

since each $n(\tau, d)$ is a natural number satisfying (210.1), their greatest lower bound exists and is the minimum of all competing $n(\tau, d)$. $\mathfrak{N}(t, d)$ satisfies

$$1 \leq \mathfrak{N}(t, d) \leq \tfrac{1}{2}N(N-1) - 1 \quad \text{on} \quad \{ t, d : t_{0*} \leq t < t^* , \ 0 < d \leq a_3 \} \ , \tag{210.3}$$

the second inequality holding because of $\lim(\min z_{jk}) = 0$. $\mathfrak{N}(t, d)$ is thus a number such that at every point of the interval $[t, t^*[$ at least $\mathfrak{N}(t, d)$ mutual distances are $\geq d$. Furthermore,

$$\mathfrak{N}(t_\sigma, d_\sigma) \geq \mathfrak{N}(t, d) \quad \text{for} \quad t \leq t_\sigma < t^* , \ 0 < d_\sigma \leq d : \tag{210.4}$$

for, $\mathfrak{S}(t_\sigma, d_\sigma)$ is a subset of $\mathfrak{S}(t, d)$ and hence $\text{g.l.b.}\mathfrak{S}(t_\sigma, d_\sigma) \geq \text{g.l.b.}\mathfrak{S}(t, d)$.

Define

$$\mathfrak{N}^* = \text{l.u.b.} \{ \mathfrak{N}(t, d) : t_{0*} \leq t < t^* , \ 0 < d \leq a_3 \} \quad ; \tag{210.5a}$$

since each $\mathfrak{N}(t, d)$ is a natural number satisfying (210.3), their least upper bound exists and is the maximum of all competing $\mathfrak{N}(t, d)$.

Let t_m and d_m, $t_{0*} \leq t_m < t^*$, $0 < d_m \leq a_3$, be such that

$$\mathfrak{N}^* = \mathfrak{N}(t_m, d_m) \quad ; \tag{210.5b}$$

by (210.4) we have

$$\mathfrak{N}^* = \mathfrak{N}(t_m, d_m) = \mathfrak{N}(t, d) \quad \text{for} \quad t_m \leq t < t^* , \ 0 < d \leq d_m \ . \tag{210.6}$$

Note: There are always at least $\mathfrak{N}(t, d)$ mutual distances with $z_{jk} \geq d$ on $[t, t^*[$, but the individual mutual distances satisfying this may change from one instant to another.

211 We define now the constants used in the subsequent analysis and further restrict in this context the interval on which we consider the solution. First, choose t_{o**}, $t_{o*} \le t_{o**} < t^*$, and $a_5 > 0$ such that for all mutual distances of the first and of the third type (i.e., those with $\liminf z_{jk} > 0$)

$$z_{jk} > a_5 \qquad \text{on} \qquad [t_{o**}, t^*[\qquad , \tag{211.1}$$

and that for all mutual distances of the fourth type

$$\limsup z_{jk} > a_5 \qquad \text{on} \qquad [t_{o**}, t^*[\qquad . \tag{211.2}$$

By (210.6) d_m can be replaced by a smaller positive number without changing $\mathfrak{N}^*$; replace d_m by d_* (subsequently defined). Then

$$\begin{array}{l}\text{at every point of } [t_{o**}, t^*[\text{ at least} \\ \mathfrak{N}^* \text{ mutual distances are } \ge d_* \, .\end{array} \tag{211.3}$$

Set

$$d_* = \left(10^{-Q} \, \frac{2\,M\,J(t^*)}{m^2_{\min}}\right)^{\frac{1}{2}} \qquad , \tag{211.4}$$

$$\epsilon_* = \left(10^{-Q-10} \, \frac{m^2_{\min}}{M^2}\right)^{\frac{1}{2}} d_* \qquad , \tag{211.5}$$

and determine the natural number Q such that

$$d_* < \min(d_m \,,\, a_5) \qquad \text{and} \qquad \epsilon_* < 10^{-1} d_* \qquad ; \tag{211.6}$$

set further

$$a_6 = \left(\gamma\, M^2 M^{-1}_{\ell \min}\, a_4\, d_*^{-2}\right)^{\frac{1}{2}} \qquad , \tag{211.7}$$

$$a_7 = a_6 + |t_{o**} - t^*|\, \gamma\, M\, d_*^{-2} \qquad , \tag{211.8}$$

$$a_8 = 2\,M\left(a_7^2 + \gamma\, M\, a_4\, d_*^{-2}\right) \quad , \tag{211.9}$$

$$\varepsilon = \frac{M}{2}\,\varepsilon_*^2 = 10^{-2Q-10}\, J(t^*) \quad . \tag{211.10}$$

Now a t_* , $t_* < t^*$, can be determined such that

(1) $t_* \geq \max(t_{0**}\ ,\ t_m\ ,\ t^* - \varepsilon a_8^{-1})$,

(2) all mutual distances of the second type satisfy $z_{jk} < \varepsilon_*$, and (211.11)

(3) $|J - J(t^*)| < \varepsilon$ with ε from (211.10) .

We show: there exist t_{1*} , t_2 , t_3 , $t_* < t_{1*} < t_2 < t_3 < t^*$, such that

(a) every mutual distance of the fourth type is $> a_5$ at least once on $[t_{1*}, t_2]$;

(b) exactly $\frac{1}{2}N(N-1) - \mathfrak{N}^*$ mutual distances are $< \varepsilon_*$ on $[t_2, t_3]$.

(a): Select a t_{1*} on $]t_*, t^*[$ and consider the mutual distances z_{pq} of the fourth type for t increasing from t_{1*} ; since they satisfy $\limsup z_{pq} > a_6$ (cf. (211.2)) , for each of them there is an instant t_{pq} on $[t_{1*}, t^*[$ such that $z_{pq}(t_{pq}) > a_6$. Let t_{2*} , $t_{1*} \leq t_{2*} < t^*$, be such that on $[t_{1*}, t_{2*}]$ for each of them there is such an instant t_{pq} where $z_{pq}(t_{pq}) > a_5$ ($t_{2*} = \max(t_{pq})$ will do) .

(b): Consider all mutual distances on $[t_{2*}, t^*[$; assume that contrary to the assertion at every point of this interval less than $\frac{1}{2}N(N-1) - \mathfrak{N}^*$ mutual distances are $< \varepsilon_*$; then always more than $\mathfrak{N}^*$ mutual distances are $\geq \varepsilon_*$, and we have $\mathfrak{N}(t_{2*}, \varepsilon_*) > \mathfrak{N}^*$, a contradiction to the maximum property of $\mathfrak{N}^*$ (cf. (210.6), (211.6), (211.1)) .

Correction to para. b
at bottom of p. 20

(b): Consider all mutual distances on $[t_{2*}, t^*[$. Assume that at every point of this interval more than $\mathfrak{N}^*$ mutual distances are $\geq \epsilon_*$, i.e. that $n(\tau, \epsilon_*) > \mathfrak{N}^*$, $\tau \in [t_{2*}, t^*[$; then $\mathfrak{N}(t_{2*}, \epsilon_*) > \mathfrak{N}^*$, a contradiction to the maximum property of $\mathfrak{N}^*$ (cf. (210.6), (211.6), (211.11)). Hence $n(\tau_\nu, \epsilon_*) \leq \mathfrak{N}^*$ for infinitely many τ_ν in $[t_{2*}, t^*[$, and since $\mathfrak{N}(t_{2*}, d_*) = \mathfrak{N}^*$ by the choice of t_* , ϵ_*, and d_*, we have also $n(\tau, \epsilon_*) \geq n(\tau, d_*) \geq \mathfrak{N}^*$ for $\tau \in [t_{2*}, t^*[$. It follows that $n(\tau_\nu, \epsilon_*) = \mathfrak{N}^*$, i.e. exactly $\frac{1}{2}N(N-1) - \mathfrak{N}^*$ mutual distances are $< \epsilon_*$ at τ_ν ; take one of these τ_ν for t_2 , then there is a nearby $t_3 > t_2$ such that by their continuity the number of mutual distances being $< \epsilon_*$ on $[t_2, t_3]$ still equals $\frac{1}{2}N(N-1) - \mathfrak{N}^*$.

212 We have shown that $\frac{1}{2}N(N-1) - \mathfrak{N}^*$ mutual distances are $< \epsilon_*$ on $[t_2, t_3]$, while all other $\mathfrak{N}^*$ mutual distances are $\geq d_*$ on this interval. Set

$$\mathfrak{S}_{\epsilon_*} = \{ z_{jk} : z_{jk} < \epsilon_* \text{ on } [t_2, t_3] \} \quad ,$$
$$\mathfrak{S}_{d_*} = \{ z_{jk} : z_{jk} \geq d_* \text{ on } [t_2, t_3] \} \quad ; \tag{212.1}$$

because the mutual distances are continuous functions of time, there is no exchange of elements between $\mathfrak{S}_{\epsilon_*}$ and $\mathfrak{S}_{d_*}$. We know that all mutual distances of the first and of the third type are $> d_*$ on $[t_2, t^*[$ and therefore in $\mathfrak{S}_{d_*}$ (cf. (211.1)), and all of the second type $< \epsilon_*$ on $[t_2, t^*[$ and therefore in $\mathfrak{S}_{\epsilon_*}$ (cf. (211.11)). We prove that at least one mutual distance of the fourth type must be in $\mathfrak{S}_{\epsilon_*}$, i.e., $< \epsilon_*$ on $[t_2, t_3]$: assume contrariwise that all (and at least one) mutual distances of the fourth type are in $\mathfrak{S}_{d_*}$. None of the mutual distances in $\mathfrak{S}_{\epsilon_*}$ (all of them are of the second type) can increase beyond ϵ_* on $[t_2, t^*[$, i.e., no mutual distance can change from $\mathfrak{S}_{\epsilon_*}$ to $\mathfrak{S}_{d_*}$, and therefore none of the $\mathfrak{N}^*$ mutual distances in $\mathfrak{S}_{d_*}$ can decrease below d_*, because then less than $\mathfrak{N}^*$ mutual distances would be $\geq d_*$: while by the definition of $\mathfrak{N}^*$ always at least $\mathfrak{N}^*$ mutual distances are $\geq d_*$ on $[t_*, t^*[$. This last result is contrary to the foregoing assumption that there exists a mutual distance of the fourth type in $\mathfrak{S}_{d_*}$, because this must necessarily decrease below d_* on $[t_3, t^*[$. Thus we have shown that there is a mutual distance z_{pq} of the fourth type in $\mathfrak{S}_{\epsilon_*}$.

It follows that there is a t_{3*} , $t_3 < t_{3*} < t^*$, such that $z_{pq}(t_{3*}) = d_*$.
Consider the mutual distances in $\mathfrak{S}_{\epsilon_*}$ for t increasing from t_3 ;
it follows now that

there is a t_4 , $t_3 < t_4 < t^*$, such that

(1) at least one mutual distance of $\mathfrak{S}_{\epsilon_*}$ equals d_* at t_4

and (212.2)

(2) all mutual distances of $\mathfrak{S}_{\epsilon_*}$ are $< d_*$ on $[t_2, t_4[$.

Considering the mutual distances in $\mathfrak{S}_{\epsilon_*}$ for t decreasing from t_2 ,
we find analogously (cf. 211 (a)) that

there is a t_1 , $t_* < t_1 < t_2$, such that

(1) at least one mutual distance of $\mathfrak{S}_{\epsilon_*}$ equals d_* at t_1

and (212.3)

(2) all mutual distances of $\mathfrak{S}_{\epsilon_*}$ are $< d_*$ on $]t_1, t_3]$.

Observe that

all mutual distances of $\mathfrak{S}_{d_*}$ remain $\geq d_*$ on $[t_1, t_4]$. (212.4)

Introduce cluster coordinates such that two bodies are in the same cluster iff their mutual distance is $< \epsilon_*$, i.e., it is in $\mathfrak{S}_{\epsilon_*}$ (, and in different clusters iff their mutual distance is $\geq d_*$, i.e., it is in $\mathfrak{S}_{d_*}$) on $[t_2, t_3]$.

By (104.7) and (10414) we have

$$J^* < \frac{M}{2}\epsilon_*^2 \qquad \text{on } [t_2, t_3] \tag{212.5}$$

and

$$J^* > \frac{m_{min}^2}{2M} d_*^2 \qquad \text{at } t_1 \text{ and at } t_4 \quad ; \tag{212.6}$$

it follows that J^* has a minimum on $]t_1, t_4[$, and (cf. (211.5))

$$J^*_{max} - J^*_{min} > \frac{m_{min}^2}{2M} d_*^2 (1 - 10^{-Q-10}) \qquad \text{on } [t_1, t_4] \ . \tag{212.7}$$

213 J^{o} has a maximum on $]t_1, t_4[$: with $J^{o} = J - J^{*}$ we find

$$J^{o} > J(t^{*}) - \varepsilon - \frac{M}{2}\,\varepsilon_{*}^{2} = J(t^{*})\,(1 - 2\cdot 10^{-2Q-10}) \quad \text{on } [t_2, t_3] \tag{213.1}$$

and

$$J^{o} < J(t^{*}) + \varepsilon - \frac{m_{\min}^{2}}{2M}\,d_{*}^{2} = J(t^{*})\,(1 - 10^{-Q} + 10^{-2Q-10}) \quad \text{at } t_1 \text{ and at } t_4 \;; \tag{213.2}$$

the right side of (213.1) is greater than the right side of (213.2) (recall that $Q \geq 1$, natural).

214 Let a maximum of J^{o} , existing by 213 , be at t_c , $t_1 < t_c < t_4$; then $J^{o\prime}(t_c) = 0 \quad , \quad J^{o\prime\prime}(t_c) \leq 0 \quad .$ (214.1)

We estimate $J^{o\prime\prime}$ on $[t_1, t_4]$: by (104.10)

$$J^{o\prime\prime} = 2\sum_{\ell=1}^{L} M_\ell\, \underline{Z}'_\ell \underline{Z}'_\ell + 2\sum_{\ell=1}^{L} M_\ell\, \underline{Z}_\ell \underline{Z}''_\ell \quad ; \tag{214.2}$$

all z_k are bounded (cf. (209.4)) , therefore, all Z_ℓ are bounded by (103.2) :

$$Z_\ell < a_4 \quad \text{on } [t_1, t_4] \quad . \tag{214.3}$$

Estimate of $\underline{Z}''_\ell$ (cf. (212.4)) :

$$|(\underline{z}_{\ell,k} - \underline{z}_{\lambda,\varkappa})|\underline{z}_{\ell,k} - \underline{z}_{\lambda,\varkappa}|^{-3}| \leq d_{*}^{-2} \quad \text{for } \ell \neq \lambda \tag{214.4}$$

and from the equation of motion (104.2) for $\underline{Z}_\ell$ we find

$$|\underline{Z}''_\ell| < \gamma M d_{*}^{-2} \quad . \tag{214.5}$$

Estimate of $\underline{Z}'_\ell$:

at t_c , (214.1) and (214.2) imply that

$$M_{\ell}\underset{\sim}{Z}'_{\ell}\underset{\sim}{Z}'_{\ell} \le \sum_{\ell=1}^{L} M_{\ell}\underset{\sim}{Z}'_{\ell}\underset{\sim}{Z}'_{\ell} \le \left|\sum_{\ell=1}^{L} M_{\ell}\underset{\sim}{Z}_{\ell}\underset{\sim}{Z}''_{\ell}\right| < M a_4 \gamma M d_*^{-2} \quad ,$$

hence (cf. (211.7))

$$|\underset{\sim}{Z}'_{\ell}(t_c)| < \left(\gamma M^2 M_{\ell\,\min}^{-1} a_4 d_*^{-2}\right)^{\frac{1}{2}} = a_6 \quad , \qquad (214.6)$$

and from

$$\underset{\sim}{Z}'_{\ell} = \underset{\sim}{Z}'_{\ell}(t_c) + \int_{t_c}^{t} \underset{\sim}{Z}''_{\ell}(\tau)\, d\tau$$

it follows that (cf. (211.8))

$$|\underset{\sim}{Z}'_{\ell}| \le |\underset{\sim}{Z}'_{\ell}(t_c)| + |t_1 - t_4|\gamma M d_*^{-2}$$

$$< a_6 + |t_{o**} - t^*|\gamma M d_*^{-2} = a_7 \quad \text{on} \quad [t_1, t_4] \; ; \qquad (214.7)$$

thus finally (cf. (211.9))

$$|J^{o''}| < 2M\left(a_7^2 + \gamma M a_4 d_*^{-2}\right) = a_8 \quad \text{on} \quad [t_1, t_4] \quad . \qquad (214.8)$$

215 Write (cf. (209.1))

$$J = J(t^*) + \mathcal{E}_1 \quad , \quad |\mathcal{E}_1| < \varepsilon \quad , \qquad (215.1)$$

and

$$J^o = J^o(t_c) + \mathcal{E}_2 \quad , \quad \mathcal{E}_2 = \int_{t_c}^{t} (t - \tau) J^{o''}(\tau)\, d\tau \quad ,$$

$$(215.2)$$

$$|\mathcal{E}_2| < |t_1 - t_4|\, a_8 < |t_* - t^*|\, a_8 \le \varepsilon \quad (\text{cf. } (211.11)) \; .$$

Hence,

$$J^* = J - J^o = J(t^*) - J^o(t_c) + \mathcal{E}_1 - \mathcal{E}_2 \quad , \qquad (215.3)$$

and

$$J^*_{max} - J^*_{min} < 4\varepsilon = 2M\varepsilon_*^2 = 2\cdot 10^{-Q-10}\,\frac{m^2_{min}}{M}\,d_*^2 \quad , \qquad (215.4)$$

this being inconsistent with the inequality (212.7).

Thus, the assumption that there is a mutual distance of the fourth type is false, and taking 208 into account, we deduce that all mutual distances have finite limits. The proof is concluded by the following

216 Lemma: Let the N-body motion be holomorphic on $[t_0, t^*[$, $t_0 < t^*$, and let all mutual distances z_{jk} have finite limits as $t \to t^*$. Then all relative positions $\underset{\sim}{z}_{jk}$ and all positions $\underset{\sim}{z}_k$ have finite limits as $t \to t^*$.

I.e., at t^* the motion remains holomorphic, or there occurs a collision singularity. The proof is analogous to that in 208, observing that the set of mutual distances of the third type is empty.

300 Properties of a collision singularity

301 This part generalizes results which have been established for a collision of all bodies ("simultaneous collision"), to each cluster of the configuration developing prior to a collision singularity. The principal result is that, for t close to t^*, the configuration of each cluster is close to a central configuration. Our proofs are similar to those of Wintner, with some extensions and modifications.

302 Observe that a collision singularity at t^* is characterized by: at least one mutual distance converges to zero, and $J \geq 0$ remains bounded and has a limit as $t \to t^*$. Without restriction of generality we may assume that $t^* = 0$ and that the approach is from $t < 0$. Also, without mentioning it always, we assume that $t < 0$ is very close to the instant of collision singularity $t = 0$, such that for the cluster coordinates holds:

$\underset{\sim}{\zeta}_{\ell,k} \to 0$ is very small on $[t, 0[$, and

$|\underset{\sim}{Z}_\ell - \underset{\sim}{Z}_\lambda| \geq$ a certain positive constant on $[t, 0[$, $\ell \neq \lambda$.

It follows also that $\underset{\sim}{Z}_\ell''$ is bounded (cf. (104.2)), consequently $\lim \underset{\sim}{Z}_\ell$ and $\lim \underset{\sim}{Z}_\ell'$, and finally $\lim \underset{\sim}{Z}_\ell''$ exist as $t \to 0$, as seen ~~by integrating, and using~~ from (104.2).

303 Differential equation and ultimate behavior of J^*:
Using (102.9): $J'' = 4h - 2U$, (104.12): $U = U^* + U^+$, and (104.14): $J = J^* + J^0$, we find

$$J^{*''} = -2U^* + 4h - 2U^+ - J^{0''} \quad ; \qquad (303.1)$$

both U^+ and $J^{0''}$ (cf. 214) are bounded on the considered interval, and we write

$$J^{*''} = -2U^* + b \quad . \qquad (303.2)$$

From $J^{*''} = -2U^* + b$ and $-U^* \to \infty$ as $t \to 0$ we conclude that:

$J^{*''} \to +\infty$,

$J^{*'}$ is monotonically increasing and ultimately does not change sign, and (303.3)

J^* is ultimately monotonically decreasing.

The last statement follows from the facts that J^* is ultimately monotonic (since $J^{*'}$ does not change sign) and $J^* > 0$, $J^* \to 0$.

304 Estimate of $\underset{\sim}{K}^*$:

The differential equation (104.3) for $\underset{\sim}{\zeta}_{\ell,k}$ and the definition (104.8) of $\underset{\sim}{K}_\ell$ yield

$$\underset{\sim}{K}_\ell' = \sum_{k=1}^{N_\ell} m_{\ell,k}\underset{\sim}{\zeta}_{\ell,k} \times \underset{\sim}{\zeta}_{\ell,k}'' = \sum_{k=1}^{N_\ell} m_{\ell,k}\underset{\sim}{\zeta}_{\ell,k} \times \underset{\sim}{P}_{\ell,k} \quad . \qquad (304.1)$$

Write, expanding in a binomial series,

$$|\underset{\sim}{Z}_\ell - \underset{\sim}{Z}_\lambda + \underset{\sim}{\zeta}_{\ell,r} - \underset{\sim}{\zeta}_{\lambda,s}|^{-3} = |\underset{\sim}{Z}_\ell - \underset{\sim}{Z}_\lambda|^{-3} \,(1 + W_{\ell\lambda rs}) , \qquad (304.2)$$

where $W_{\ell\lambda rs}$ is an absolutely convergent series, each term of which contains a $\zeta_{p,q}$ or a component of a $\underset{\sim}{\zeta}_{p,q}$ as a factor; it is easily seen that the terms in $\underset{\sim}{P}_{\ell,k}$ not containing such a factor cancel out, i.e., $\underset{\sim}{P}_{\ell,k} = \underset{\sim}{0}$ if all $\underset{\sim}{\zeta}_{p,q} = \underset{\sim}{0}$, and we conclude that each term of $\underset{\sim}{P}_{\ell,k}$ contains a factor of the types mentioned above.

From the definition (104.7) of J^* and (104.14)

$$\zeta_{p,q} < cJ^{*\frac{1}{2}} \quad , \quad \zeta_{p,q}\zeta_{r,s} < cJ^* \quad , \qquad (304.3)$$

and the same inequalities for the components of $\underset{\sim}{\zeta}_{p,q}$; hence,

$$|\underset{\sim}{K}_\ell'| < cJ^* \quad \text{and} \quad \underset{\sim}{K}_\ell' = J^*\underset{\sim}{b} \quad . \qquad (304.4)$$

Integrating from t to $t = 0$ and observing that (cf. (303.3))

$$J^* = J^*(t) = \max J^*(\tau) \quad , \quad \tau \in [t, 0[\quad , \qquad (304.5)$$

we find

$$\mathbf{K}_\ell = \mathbf{K}_{\ell;o} + tJ^*\mathbf{b} \tag{304.6}$$

and

$$\mathbf{K}^* = \mathbf{K}^*_o + tJ^*\mathbf{b} \quad , \qquad K^{*2} = K^{*2}_o + tJ^*b \quad . \tag{304.7}$$

305 $\frac{1}{4}J^{*\prime 2} + K^{*2}_o \leq J^*(J^{*\prime\prime} + d)$:

Using on

$$\tfrac{1}{2}J^{*\prime} = \sum_{\ell=1}^{L} \sum_{k=1}^{N_\ell} m_{\ell,k}\boldsymbol{\zeta}_{\ell,k}\boldsymbol{\zeta}'_{\ell,k}$$

the inequality

$$\left(\sum a_\nu b_\nu\right)^2 \leq \left(\sum a_\nu^2\right)\left(\sum b_\nu^2\right)$$

with

$$a_\nu = m^{\frac{1}{2}}_{\ell,k} \ \zeta_{\ell,k} \quad , \qquad b_\nu = m^{\frac{1}{2}}_{\ell,k}\zeta'_{\ell,k} \quad ,$$

we find

$$\frac{1}{4} J^{*\prime 2} \leq J^* \sum\sum m_{\ell,k} (\boldsymbol{\zeta}_{\ell,k}\boldsymbol{\zeta}'_{\ell,k})^2 \ \zeta^{-2}_{\ell,k}$$

and similarly

$$K^{*2} \leq J^* \sum\sum m_{\ell,k}(\boldsymbol{\zeta}_{\ell,k} \times \boldsymbol{\zeta}'_{\ell,k})^2 \zeta^{-2}_{\ell,k} \quad .$$

Using $(\mathbf{a}\mathbf{b})^2 + (\mathbf{a}\times\mathbf{b})^2 = a^2b^2$, addition yields

$$\frac{1}{4}J^{*\prime 2} + K^{*2} \leq J^* \sum\sum m_{\ell,k}\boldsymbol{\zeta}'_{\ell,k}\boldsymbol{\zeta}'_{\ell,k} \ = \ 2J^*V^* \tag{305.1}$$

(cf. (104.5) , (104.13)) .

Substitute for K^{*2} from (304.7) and from (104.13): $V = V^* + V^o$, from (104.14): $J = J^* + J^o$, and (102.9): $J'' = 2h + 2V$; then

$$\frac{1}{4} J^{*\prime 2} + K^{*2}_o \leq J^*(J^{*\prime\prime} - 2h + J^{o\prime\prime} - 2V^o - tb) \quad ,$$

and since $-2h + J^{o\backslash\backslash} - 2V^{o} - tb$ is bounded on $[t, 0[$, we finally get

$$\frac{1}{4} J^{*\backslash 2} + K_{o}^{*2} \leq J^{*}(J^{*\backslash\backslash} + d) \quad . \tag{305.2}$$

306 The inequality (305.2) implies that $\underset{\sim}{K}_{o}^{*} = \underset{\sim}{0}$ and that $\lim J^{*^{-\frac{1}{2}}} J^{*\backslash 2} \geq 0$ exists:

define $$Q = dJ^{*\frac{1}{2}} + (\frac{1}{4} J^{*\backslash 2} + K_{o}^{*2})/J^{*\frac{1}{2}} \quad ; \tag{306.1}$$

then

$$Q^{\backslash} = \tfrac{1}{2} J^{*^{-\frac{1}{2}}} J^{*\backslash} \left\{ d + J^{*\backslash\backslash} - (\frac{1}{4} J^{*\backslash 2} + K_{o}^{*2})/J^{*} \right\} \quad ,$$

and since $\{..\} \geq 0$, we see that $J^{*\backslash}$ and $Q^{\backslash}$ are of the same sign, implying that Q ultimately cannot increase, since J^{*} ultimately decreases; hence, Q has a limit Q_{o} as $t \to 0$, $0 \leq Q_{o} < \infty$, and this implies because of $J^{*} \to 0$ that

$$K_{o}^{*2} = 0 \quad \text{and} \quad (0 \leq) \lim J^{*^{-\frac{1}{2}}} J^{*\backslash 2} \; (< \infty) \quad \text{exists} \quad . \tag{306.2}$$

307 $J^{*\frac{1}{2}} J^{*\backslash\backslash} \geq c > 0$:

$J^{*\backslash\backslash} = -2U^{*} + b$ implies

$$J^{*\frac{1}{2}} J^{*\backslash\backslash} = -2J^{*\frac{1}{2}} U^{*} + J^{*\frac{1}{2}} b \quad ; \tag{307.1}$$

now

$$J^{*} > c \min \zeta_{\ell,k}^{2} \quad , \quad J^{*\frac{1}{2}} > c \min \zeta_{\ell,k} \quad , \quad -U^{*} > c(\min \zeta_{\ell,k})^{-1} \quad ,$$

therefore $J^{*\frac{1}{2}}(-U^{*}) > c > 0$. The second term in (307.1) goes to zero, and the desired inequality

$$J^{*\frac{1}{2}} J^{*\backslash\backslash} \geq c > 0 \tag{307.2}$$

follows.

308 $J^{*\frac{1}{2}} J^{*''} \ge c > 0$ and $\frac{1}{4} J^{*'2} \le J^* (J^{*''} + d)$ imply $\lim J^{*-\frac{1}{2}} J^{*'2} > 0$:

define (cf. (306.1))

$$Q = d J^{*\frac{1}{2}} + \frac{1}{4} J^{*-\frac{1}{2}} J^{*'2} \quad ;$$

then

$$\begin{aligned} Q J^{*\frac{1}{2}} &= dJ^* + \frac{1}{4} J^{*'2} \\ &= dJ^* + \tfrac{1}{2} \int_0^t J^{*'} J^{*''} dt \quad , \quad \text{since} \quad J^* \to 0 \quad \text{by (306.2)} . \end{aligned}$$

Substitute from (307.2), then

$$Q J^{*\frac{1}{2}} \ge dJ^* + \frac{c}{2} \int_0^t J^{*-\frac{1}{2}} J^{*'} dt = dJ^* + c J^{*\frac{1}{2}} \quad ,$$

and this implies

$$Q \ge d J^{*\frac{1}{2}} + c \quad ,$$

i.e.,

$$\lim Q = \frac{1}{4} \lim J^{*-\frac{1}{2}} J^{*'2} = \mu^* > 0 \quad . \tag{308.1}$$

309 From (308.1) asymptotic expressions for J^* and $J^{*'}$ as $t \to 0$ are easily derived:

(308.1) is equivalent to

$$J^{*\frac{1}{2}} J^{*'2} = 4\mu^* + e \quad , \quad e = e(t) \to 0 \quad ; \tag{309.1}$$

separating and integrating we find

$$J^* = \left(\frac{3}{2} \mu^{*\frac{1}{2}}\right)^{\frac{4}{3}} t^{\frac{4}{3}} + t^{\frac{4}{3}} e$$

or (309.2)

$$J^* \sim \left(\frac{9}{4} \mu^*\right)^{\frac{2}{3}} t^{\frac{4}{3}} \quad .$$

Substitution into (309.1) yields

$$J^{*\prime} = (12\mu^{*2})^{\frac{1}{3}}\, t^{\frac{1}{3}} + t^{\frac{1}{3}}\, e$$

or (309.3)

$$J^{*\prime} \sim (12\mu^{*2})^{\frac{1}{3}}\, t^{\frac{1}{3}} \quad .$$

310 $|J^{*\prime\prime\prime}| < cJ^{*\prime\prime\frac{5}{2}}$ ultimately:

differentiation of (303.1) yields

$$J^{*\prime\prime\prime} = -2U^{*\prime} - 2U^{+\prime} - J^{o\prime\prime\prime} \quad ; \tag{310.1}$$

by (104.10) $J^{o\prime\prime\prime}$ is a linear combination of terms $\underset{\sim}{Z}'_{\ell}\underset{\sim}{Z}''_{\ell}$ and $\underset{\sim}{Z}_{\ell}\underset{\sim}{Z}'''_{\ell}$, $\underset{\sim}{Z}_{\ell}$, $\underset{\sim}{Z}'_{\ell}$, and $\underset{\sim}{Z}''_{\ell}$ are bounded, and by the equation of motion (104.2) for $\underset{\sim}{Z}_{\ell}$, $\underset{\sim}{Z}'''_{\ell}$ is a linear combination of the components of the $\underset{\sim}{\zeta}'_{p,q}$ with bounded coefficients, plus a bounded function. Since $|\underset{\sim}{\zeta}'_{p,q}| < c\, V^{*\frac{1}{2}}$ and this inequality holds also for the components of $\underset{\sim}{\zeta}'_{p,q}$, we get

$$|J^{o\prime\prime\prime}| < c\, V^{*\frac{1}{2}} \quad . \tag{310.2}$$

Similarly, U^{+} is a linear combination of the components of the $\underset{\sim}{\zeta}'_{p,q}$ with bounded coefficients, plus a bounded function, hence

$$|U^{+\prime}| < c\, V^{*\frac{1}{2}} \quad . \tag{310.3}$$

$U^{*\prime}$ is a linear combination of terms $(\underset{\sim}{\zeta}_{p,q} - \underset{\sim}{\zeta}_{p,s})(\underset{\sim}{\zeta}'_{p,q} - \underset{\sim}{\zeta}'_{p,s})|\underset{\sim}{\zeta}_{p,q} - \underset{\sim}{\zeta}_{p,s}|^{-3}$,

this term's absolute value being $|\underset{\sim}{\zeta}'_{p,q} - \underset{\sim}{\zeta}'_{p,s}|\,|\underset{\sim}{\zeta}_{p,q} - \underset{\sim}{\zeta}_{p,s}|^{-2}$; since

$|\underset{\sim}{\zeta}_{p,q} - \underset{\sim}{\zeta}_{p,s}|^{-2} < c\, U^{*2}$, we have

$$|U^{*\prime}| < c\, V^{*\frac{1}{2}}\, U^{*2} \quad . \tag{310.4}$$

By means of the energy equation $U^{*} + V^{*} = h - U^{+} - V^{0}$ and the Lagrange-Jacobi equation (303.2): $J^{*\prime\prime} = -2U^{*} + b$, both V^{*} and U^{*}

can be replaced by $J^{*''}$, and we finally get the desired inequality

$$|J^{*'''}| < c\, J^{*''\frac{5}{2}} \quad . \tag{310.5}$$

311 Asymptotic expression for $J^{*''}$. We prove $J^{*\frac{1}{2}} J^{*''} \sim \mu^*$:

$\frac{1}{4} J^{*'2} \leq J^* (J^{*''} + d)$ (cf. (305.2), (306.2)) implies

$$\liminf J^{*\frac{1}{2}} J^{*''} \geq \frac{1}{4} \lim J^{*-\frac{1}{2}} J^{*'2} = \mu^* \quad ; \tag{311.1}$$

it remains to be shown that

$$\limsup J^{*\frac{1}{2}} J^{*''} \leq \mu^* \quad . \tag{311.2}$$

Introduce

$$F = J^{*'3} \quad ;$$

then (311.3)

$$F' = 3J^{*'2} J^{*''} \quad , \quad F'' = 6J^{*'} J^{*''2} + 3J^{*'2} J^{*'''} \quad ,$$

and as consequence of (309.3) we have

$$F \sim 12\mu^{*2} t \quad . \tag{311.4}$$

Consider now F' (≥ 0 obviously) instead of $J^{*\frac{1}{2}} J^{*''}$: we get

$$\begin{aligned} \liminf F' &= 3 \liminf (J^{*-\frac{1}{2}} J^{*'2})(J^{*\frac{1}{2}} J^{*''}) \\ &= 12\mu^* \liminf J^{*\frac{1}{2}} J^{*''} \geq 12\mu^{*2} \quad , \end{aligned} \tag{311.5}$$

and it remains to be shown that

$$\limsup F' \leq 12\mu^{*2} \quad . \tag{311.6}$$

Case 1: Assume that $\liminf F' = \limsup F' = \lim F' = d < \infty$; integration of this relation $F' \sim d$ to $F \sim dt$ yields by comparison with (311.4) $d = 12\mu^{*2}$, and the proof is complete.

Case 2: Assume that $\lim F' = +\infty$ and let $d > 0$ be arbitrarily large; then ultimately $F' > d$ and by integration $F > dt$, contradicting (311.4).

Case 3: Assume finally that $\liminf F' < \limsup F'$, implying that $(\liminf F' < +\infty)$.

There exist two positive numbers d_1 and d_2 and an infinite sequence of intervals $t_\alpha < t < t_\beta$, both t_α and t_β going to zero, such that

$$0 < 12\mu^{*2} < d_1 = F'(t_\alpha) < F'(t) < F'(t_\beta) = d_2 \quad , \quad t \in \,]t_\alpha, t_\beta[\ . \qquad (311.7)$$

Write

$$F'' = \frac{1}{F}\left(\frac{2}{3} F'^2 + \quad 3^{-\frac{3}{2}} F'^{\frac{5}{2}} \frac{J^{*'''}}{J^{*''\frac{5}{2}}}\right)$$

and use (310.5) and (311.4) ; then

$$|F''| < \frac{c}{t}\left(F'^2 + F'^{\frac{5}{2}}\right)$$

and on $]t_\alpha, t_\beta[$, since $F' \leq d_2$:

$$|F''| < \frac{c}{t} \qquad .$$

Integration over $[t_\alpha, t_\beta]$ yields

$$0 < d_2 - d_1 = F'(t_\beta) - F'(t_\alpha) = \int_{t_\alpha}^{t_\beta} F''(t)\, dt \leq \int_{t_\alpha}^{t_\beta} |F''(t)|\, dt$$

$$< c\left|\log \frac{t_\beta}{t_\alpha}\right| = c \log \frac{-t_\alpha}{-t_\beta}$$

(observe that $t_\alpha < t_\beta < 0$) ; this implies

$$\log \frac{-t_\alpha}{-t_\beta} > \frac{d_2 - d_1}{c} \quad , \text{ i.e., ultimately } \frac{-t_\alpha}{-t_\beta} \geq T > 1 \quad . \qquad (311.8)$$

From (311.7) it follows that

$$F(t_\beta) - F(t_\alpha) = \int_{t_\alpha}^{t_\beta} F'(t)\, dt > d_1 (t_\beta - t_\alpha) \qquad ,$$

implying

$$\frac{F(t_\beta)}{-t_\beta} + d_1 > \left(\frac{F(t_\alpha)}{-t_\alpha} + d_1 \right) \frac{-t_\alpha}{-t_\beta} \qquad ;$$

this last inequality is ultimately inconsistent, since

by (311.4) $\frac{F(t_\beta)}{-t_\beta}$ and $\frac{F(t_\alpha)}{-t_\alpha}$ both $\to -12\mu^{*2}$,

by (311.7) $12\mu^{*2} \neq d_1$, and

by (311.8) ultimately $\frac{-t_\alpha}{-t_\beta} > 1$.

Thus, $J^{*\frac{1}{2}} J^{*''} \sim \mu^*$ is true, and it follows that

$$J^{*''} \sim \left(\frac{2}{3}\mu^*\right)^{\frac{2}{3}} t^{-\frac{2}{3}} \qquad . \tag{311.9}$$

312 h_ℓ remains bounded as $t \to 0$: the energy h_ℓ of the ℓ-th cluster is by (104.5) and (104.6)

$$h_\ell = \tfrac{1}{2} \sum_{k=1}^{N_\ell} m_{\ell,k} \underset{\sim}{\zeta}'_{\ell,k} \underset{\sim}{\zeta}'_{\ell,k} + U_\ell \qquad , \tag{312.1}$$

and by differentiation we get

$$h'_\ell = \sum_{k=1}^{N_\ell} m_{\ell,k} \underset{\sim}{\zeta}'_{\ell,k} \underset{\sim}{\zeta}''_{\ell,k} + U'_\ell \qquad ;$$

substitute from the equation of motion (104.3): then

$$h'_\ell = - \sum_{k=1}^{N_\ell} (\partial U_\ell / \partial \underset{\sim}{\zeta}_{\ell,k}) \underset{\sim}{\zeta}'_{\ell,k} + \sum_{k=1}^{N_\ell} m_{\ell,k} \underset{\sim}{\zeta}'_{\ell,k} \underset{\sim}{P}_{\ell,k} + U'_\ell \qquad ,$$

the first and last term on the right cancel each other, thus

$$h'_{\ell} = \sum_{k=1}^{N_\ell} m_{\ell,k}\, \boldsymbol{\zeta}'_{\ell,k}\, \boldsymbol{P}_{\ell,k}$$

and by integration

$$h_{\ell} = \text{const.} + \sum_{k=1}^{N_\ell} m_{\ell,k} \int \boldsymbol{\zeta}'_{\ell,k}\, \boldsymbol{P}_{\ell,k}\, dt \quad . \tag{312.2}$$

Now $J^{*''} \sim \left(\frac{2}{3} \quad \right)^{\frac{2}{3}} t^{-\frac{2}{3}}$ implies $V^* < c\, t^{-\frac{2}{3}}$ and

$$|\boldsymbol{\zeta}'_{\ell,k}| < c\,(-t)^{-\frac{1}{3}} \quad , \tag{312.3}$$

and since $\boldsymbol{P}_{\ell,k}$ is bounded on $[t, 0[$, we get

$$\left| \int_t^0 \boldsymbol{\zeta}'_{\ell,k}\, \boldsymbol{P}_{\ell,k}\, dt \right| < c \left| \int_t^0 (-t)^{-\frac{1}{3}}\, dt \right| = c\, t^{\frac{2}{3}} \quad , \tag{312.4}$$

together with (312.2) implying the boundedness of h_ℓ .

313 The Lagrange-Jacobi equation for J_ℓ:
from the definition (104.7) we get

$$J''_{\ell} = 2 \sum_{k=1}^{N_\ell} m_{\ell,k}\, \boldsymbol{\zeta}'_{\ell,k}\, \boldsymbol{\zeta}'_{\ell,k} + 2 \sum_{k=1}^{N_\ell} m_{\ell,k}\, \boldsymbol{\zeta}_{\ell,k}\, \boldsymbol{\zeta}''_{\ell,k} \quad ,$$

and substituting V_ℓ from (104.5) and for $\boldsymbol{\zeta}''_{\ell,k}$ from the equation of motion (104.3),

$$J''_{\ell} = 4 V_\ell - 2 \sum_{k=1}^{N_\ell} \boldsymbol{\zeta}_{\ell,k}\, (\partial U_\ell / \partial \boldsymbol{\zeta}_{\ell,k}) + 2 \sum_{k=1}^{N_\ell} m_{\ell,k}\, \boldsymbol{\zeta}_{\ell,k}\, \boldsymbol{P}_{\ell,k} \quad ;$$

on the right, the third term is bounded, and the second term equals $+2U_\ell$ (by Euler's theorem about homogeneous functions), and using $h_\ell = V_\ell + U_\ell$, where h_ℓ is bounded (cf. 312), we find

$$J_\ell'' = -2U_\ell + b \quad . \tag{313.1}$$

As in (303.3) we conclude (observing that $J_\ell \to 0$ by the assumption of a collision singularity):

$$\left.\begin{array}{l} J_\ell'' = -2U_\ell + b \quad \text{and} \quad U_\ell \to -\infty \quad \text{as} \quad t \to 0 \quad \text{imply:} \\ J_\ell'' \to +\infty \\ J_\ell' \ \text{ultimately does not change sign (and is negative)} \\ J_\ell \ \text{is ultimately monotonically decreasing to zero.} \end{array}\right. \tag{313.2}$$

314 Estimate of $\underset{\sim}{K}_\ell$ (cf. 304) :
(304.1) , together with

$$\zeta_{\ell,k} < c J_\ell^{\frac{1}{2}} \quad , \tag{314.1}$$

yields

$$|K_\ell'| < c J_\ell^{\frac{1}{2}} \quad \text{or equivalently} \quad \underset{\sim}{K}_\ell' = J_\ell^{\frac{1}{2}} \underset{\sim}{b} \quad , \tag{314.2}$$

and integrating from t to $t = 0$, observing that by (313.2) $J_\ell = J_\ell(t) = \max J_\ell(\tau)$, $\tau \in [t, 0[$, we have

$$\underset{\sim}{K}_\ell = \underset{\sim}{K}_{\ell;o} + J_\ell^{\frac{1}{2}} t \underset{\sim}{b} \quad \text{and} \quad K_\ell^2 = K_{\ell;o}^2 + J_\ell^{\frac{1}{2}} t b \quad . \tag{314.3}$$

315 $\frac{1}{4} J_\ell'^2 + K_{\ell;o}^2 \le J_\ell(J_\ell'' + d)$:

analogous to the process used in 305 leading to (305.1), we arrive at

$$\frac{1}{4} J_\ell'^2 + K_\ell^2 \le 2 J_\ell V_\ell \quad ; \tag{315.1}$$

by (313.1): $2V_\ell = 2h_\ell - 2U_\ell = J_\ell'' + 2h_\ell - b$; using (314.3), and observing that h_ℓ is bounded, we get

$$\frac{1}{4} J_\ell'^2 + K_{\ell;o}^2 \le J_\ell(J_\ell'' + d_1) + J_\ell^{\frac{1}{2}}(-t) d_2 \quad . \tag{315.2}$$

As in 307 for J^*, it can be shown that (313.1): $J_\ell'' = -2U_\ell + b$ implies $J_\ell^{\frac{1}{2}} J_\ell'' \geq c > 0$, hence $J_\ell^{\frac{1}{2}} \leq c J_\ell J_\ell''$; substitute this into the last term of (315.2):

$$\frac{1}{4} J_\ell'^2 + K_{\ell;o}^2 \leq J_\ell (J_\ell'' + d_1 + c J_\ell'' \cdot (-t)) \quad . \tag{315.3}$$

Finally, with $J_\ell'' \leq J^{*''} + c$, $J^{*''} \leq c t^{-\frac{2}{3}}$ (cf. (311.9)):

$(-t) J_\ell'' \leq c(-t)^{\frac{1}{3}}$, and we get

$$\frac{1}{4} J_\ell'^2 + K_{\ell;o}^2 \leq J_\ell (J_\ell'' + d) \quad . \tag{315.4}$$

316 In full analogy to the development in 306 - 311, the following results can be derived:

$$\underset{\sim}{K}_{\ell;o} = \underset{\sim}{0} \quad \text{or} \quad K_{\ell;o} = 0 \quad ; \tag{316.1}$$

$$\lim J_\ell^{-\frac{1}{2}} J_\ell'^2 = 4\mu_\ell > 0 \quad \text{exists} \quad ; \tag{316.2}$$

$$J_\ell \sim \left(\frac{9}{4}\mu_\ell\right)^{\frac{2}{3}} t^{\frac{4}{3}} \quad , \tag{316.3}$$

$$J_\ell' \sim (12\mu_\ell^2)^{\frac{1}{3}} t^{\frac{1}{3}} \quad , \tag{316.4}$$

$$J_\ell'' \sim \left(\frac{2}{3}\mu_\ell\right)^{\frac{2}{3}} t^{-\frac{2}{3}} \quad . \tag{316.5}$$

317 Direct proof of $\underset{\sim}{K}_{\ell;o} = \underset{\sim}{0}$:

we had seen that (cf. (312.3))

$$|\underset{\sim}{\zeta}_{\ell,k}'| < c(-t)^{-\frac{1}{3}} \quad \text{or equivalently} \quad \underset{\sim}{\zeta}_{\ell,k}' = t^{-\frac{1}{3}} \underset{\sim}{b} \; , \tag{317.1}$$

and by integration from t to $t = 0$

$$\underset{\sim}{r}'_{\ell,k} = t^{-\frac{1}{3}} \underset{\sim}{b} \tag{317.2}$$

results. Therefore

$$\underset{\sim}{K}_{\ell} = \sum_{k=1}^{N_{\ell}} m_{\ell,k} \underset{\sim}{r}_{\ell,k} \times \underset{\sim}{r}'_{\ell,k} = t^{\frac{1}{3}} \underset{\sim}{b} \quad , \tag{317.3}$$

i.e., $\underset{\sim}{K}_{\ell} \to \underset{\sim}{0}$ as $t \to 0$. (317.4)

(317.4) together with (314.3) implies

$$\underset{\sim}{K}_{\ell} = J_{\ell}^{\frac{1}{2}}\, t\, \underset{\sim}{b} \qquad \text{and} \qquad K_{\ell}^{2} = J_{\ell}\, t^{2}\, b \quad ,$$

leading to a simple derivation of the inequality (315.4).

318 Corollary. The estimate of $\underset{\sim}{K}_{\ell}$ (cf. 314) can be developed in full analogy with that of $\underset{\sim}{K}^{*}$ (cf. 304) : indeed,

$$\underset{\sim}{P}_{\ell,k} = \underset{\sim}{0} \qquad \text{if} \qquad \underset{\sim}{\zeta}_{\ell,\varkappa} = \underset{\sim}{0} \ , \quad \varkappa = 1 \text{ -- } N_{\ell} \quad , \tag{318.1}$$

and writing

$$\underset{\sim}{Z}_{\ell} - \underset{\sim}{Z}_{\lambda} - \underset{\sim}{\zeta}_{\lambda,\varkappa} + \underset{\sim}{\zeta}_{\ell,\nu} = \underset{\sim}{w}_{\lambda\varkappa} + \underset{\sim}{\zeta}_{\ell,\nu}$$

and

$$|\underset{\sim}{w}_{\lambda,\varkappa} + \underset{\sim}{\zeta}_{\ell,\nu}|^{-3} = w_{\lambda\varkappa}^{-3} \left(1 + \zeta_{\ell,\nu} R_{\lambda\varkappa\nu} \right) \quad ,$$

where $R_{\lambda\varkappa\nu}$ is bounded on $[t_{o}, t^{*}[$, t_{o} sufficiently close to t^{*} , (318.1) shows that all terms in $\underset{\sim}{P}_{\ell,k}$ not containing a factor $\zeta_{\ell,\nu}$ or $\underset{\sim}{\zeta}_{\ell,\nu}$ cancel out. Since

$$\zeta_{\ell,\nu} < c\, J_{\ell}^{\frac{1}{2}} \quad ,$$

it follows that

$$|\underset{\sim}{K}'_{\ell}| < c\, J_{\ell} \qquad \text{and} \qquad \underset{\sim}{K}'_{\ell} = J_{\ell}\, \underset{\sim}{b} \quad ;$$

for the further steps see (304.4) ff. .

319 In 319 - 323 we will show that each cluster's configuration is close to a central configuration if $t < 0$ is close to the instant $t = 0$ of a collision singularity; more precisely: the N_ℓ bodies of the ℓ-th cluster are close to a central configuration in the N_ℓ-body problem.

We consider throughout the following an arbitrary, but fixed cluster, say the ℓ-th.

Let ψ be one of the components of $\underset{\sim}{\zeta}_{\ell, k}$; the conditions that the ℓ-th cluster form a central configuration are

$$J_\ell \frac{\partial U_\ell}{\partial \psi} = - \tfrac{1}{2} U_\ell \frac{\partial J_\ell}{\partial \psi} \qquad \text{for all } \psi \quad . \tag{319.1}$$

320 We introduce new dependent variables $\underset{\sim}{\zeta}_k^{\sim}$ by

$$\underset{\sim}{\zeta}_k^{\sim} = t^{-\frac{2}{3}} \underset{\sim}{\zeta}_{\ell, k} \quad , \qquad \psi^{\sim} = t^{-\frac{2}{3}} \psi \quad ; \tag{320.1}$$

then

$$\zeta^{\sim}_{k} = t^{-\frac{2}{3}} \zeta_{\ell, k} \quad , \tag{320.2}$$

$$J^{\sim} = t^{-\frac{4}{3}} J_{\ell} \quad , \tag{320.3}$$

and

$$U^{\sim} = t^{\frac{2}{3}} U_{\ell} \quad . \tag{320.4}$$

The conditions (319.1) for a central configuration transform into

$$J^{\sim} \frac{\partial U^{\sim}}{\partial \psi^{\sim}} = -\tfrac{1}{2} U^{\sim} \frac{\partial J^{\sim}}{\partial \psi^{\sim}} \qquad \text{for all } \psi^{\sim} \quad , \tag{320.5}$$

the Lagrange-Jacobi equation $J''_{\ell} = -2U_{\ell} + b_{\ell}$ into

$$t^2 J^{\sim''} + \frac{8}{3} t J^{\sim'} + \frac{4}{9} J^{\sim} = -2U^{\sim} + t^{\frac{2}{3}} b \tag{320.6}$$

and the equation of motion (104.3b) into

$$m_{\ell, k}\left(t^2 \zeta^{\sim''}_{k} + \frac{4}{3} t \zeta^{\sim'}_{k} - \frac{2}{9} \zeta^{\sim}_{k}\right) = - \frac{\partial U^{\sim}}{\partial \zeta^{\sim}_{k}} + t^{\frac{4}{3}} m_{\ell, k} P_{\ell, k} \quad ; \tag{320.7}$$

differentiating this last equation we get

$$\begin{aligned} m_{\ell, k}&\left(t^3 \zeta^{\sim'''}_{k} + \frac{10}{3} t^2 \zeta^{\sim''}_{k} + \frac{10}{9} t \zeta^{\sim'}_{k}\right) \\ &= -t \frac{d}{dt} \frac{\partial U^{\sim}}{\partial \zeta^{\sim}_{k}} + t^{\frac{7}{3}} m_{\ell, k} \frac{d}{dt} P_{\ell, k} + \frac{4}{3} t^{\frac{4}{3}} m_{\ell, k} P_{\ell, k} \quad . \end{aligned} \tag{320.8}$$

321 From (316.3) - (316.5) it follows that

$$t J^{\sim'} = - \frac{4}{3} t^{-\frac{4}{3}} J_{\ell} + t^{-\frac{1}{3}} J'_{\ell} \sim 0 \quad , \tag{321.1}$$

and similarly

$$t^2 J^{\sim''} \sim 0 \quad ; \tag{321.2}$$

these relations together with (320.6) imply that

$$\frac{2}{9} J^{\sim} \sim -U^{\sim} \quad . \tag{321.3}$$

Substitute $\boldsymbol{\zeta}'_{\ell,k} = t^{\frac{2}{3}} \tilde{\boldsymbol{\zeta}}'_k + \frac{2}{3} t^{-\frac{1}{3}} \tilde{\boldsymbol{\zeta}}_k$ into the energy equation (312.1); we get

$$\tfrac{1}{2} \sum{}' \, m_{\ell,k} (t\tilde{\boldsymbol{\zeta}}'_k)(t\tilde{\boldsymbol{\zeta}}'_k) + \frac{1}{3} t J^{\sim\prime} + \frac{2}{9} J^{\sim} = t^{\frac{2}{3}} h_{\ell} - U^{\sim} \quad , \tag{321.4}$$

implying by (321.2) and (321.3) that

$$t \tilde{\zeta}'_k \sim 0 \quad . \tag{321.5}$$

By (316.5) or (316.3) and (321.3) $U^{\sim}$ is bounded, therefore $\liminf \tilde{\zeta}_k > 0$; hence also $\dfrac{\partial U^{\sim}}{\partial \tilde{\boldsymbol{\zeta}}_k}$ is bounded, and from (320.7) it follows that

$$t^2 \tilde{\boldsymbol{\zeta}}''_k \text{ is bounded as } t \to 0 \quad . \tag{321.6}$$

In (320.8) the first and second term on the right are bounded since $t\tilde{\boldsymbol{\zeta}}'_k \to 0$ by (321.5), and the third term is obviously bounded; therefore

$$t^3 \tilde{\boldsymbol{\zeta}}'''_k \text{ is bounded as } t \to 0 \quad . \tag{321.7}$$

322 (Tauberian) Lemma: If $t f'(t)$ has a finite limit Ω and $t^3 f'''(t)$ remains bounded as $t \to 0$, then $\lim t^2 f''(t) = -\Omega$ as $t \to 0$. (cf. Wintner [2] pp. 279 - 280)

323 Applying 322 to (321.5), (321.6), and (321.7), we conclude that

$$t^2 \tilde{\boldsymbol{\zeta}}''_k \sim 0 \quad ; \tag{323.1}$$

using this, together with (321.5) in the equation of motion (320.7), the asymptotic relation

$$- \frac{2}{9} m_{\ell, k} \underset{\sim}{r}_{k}^{\sim} \sim - \frac{\partial U^{\sim}}{\partial \underset{\sim}{r}_{k}^{\sim}} \quad \text{or} \quad - \frac{2}{9} m_{\ell, k} \psi^{\sim} \sim - \frac{\partial U^{\sim}}{\partial \psi^{\sim}} \tag{323.2}$$

results. The left side is $-\frac{1}{9} \frac{\partial J^{\sim}}{\partial \psi^{\sim}}$ and, multiplying both sides by $-J^{\sim} \neq 0$ and using (321.3), the desired result (cf. (320.5))

$$J^{\sim} \frac{\partial U^{\sim}}{\partial \psi^{\sim}} \sim -\tfrac{1}{2} U^{\sim} \frac{\partial J^{\sim}}{\partial \psi^{\sim}} \tag{323.3}$$

follows.

400 Simultaneous binary collisions

We shall consider the situation that there are L_1 pairs of bodies such that the two bodies in each pair collide at $t = 0$, while the centers of mass of the pairs and the remaining L_2 bodies approach distinct positions; thus

$$L_1 + L_2 = L = \text{number of clusters,}$$
$$2L_1 + L_2 = N = \text{number of bodies.}$$

401 Change to a new coordinate system such that the motion of the second body in each cluster is referred to its first body. Introduce the new coordinate

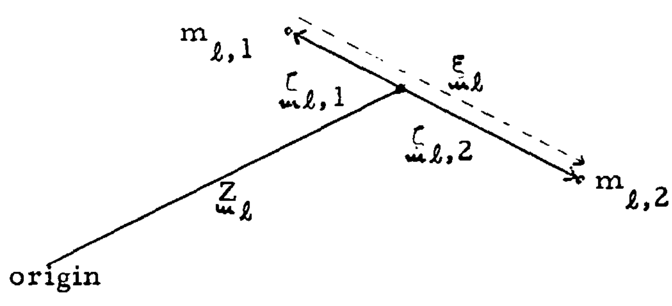
$$\boldsymbol{\xi}_\ell = \boldsymbol{\zeta}_{\ell,2} - \boldsymbol{\zeta}_{\ell,1} \quad ; \tag{401.1}$$

using $m_{\ell,1}\boldsymbol{\zeta}_{\ell,1} + m_{\ell,2}\boldsymbol{\zeta}_{\ell,2} = \mathbf{0}$ (cf. (103.3)), we find

$$m_{\ell,2}\boldsymbol{\xi}_\ell = -(m_{\ell,1} + m_{\ell,2})\boldsymbol{\zeta}_{\ell,1}$$

and (401.2)

$$m_{\ell,1}\boldsymbol{\xi}_\ell = +(m_{\ell,1} + m_{\ell,2})\boldsymbol{\zeta}_{\ell,2} \quad .$$



402 The transformed equations of motion (104.3) read

$$\boldsymbol{\xi}_\ell'' = -\Gamma_\ell \frac{\boldsymbol{\xi}_\ell}{\xi_\ell^3} + \mathbf{Q}_\ell \quad , \quad \Gamma_\ell = \gamma(m_{\ell,1} + m_{\ell,2}) \quad , \quad \mathbf{Q}_\ell = -\frac{m_{\ell,1}+m_{\ell,2}}{m_{\ell,2}}\mathbf{P}_\ell , \tag{402.1}$$

the energy equation (104.6) changes into

$$\Gamma_{\ell} = V_{\ell} + \frac{1}{2}\frac{m_{\ell,1}m_{\ell,2}}{m_{\ell,1}+m_{\ell,2}}\,\underset{\sim}{\xi}^{\backprime}_{\ell}\underset{\sim}{\xi}^{\backprime}_{\ell} - \gamma\frac{m_{\ell,1}m_{\ell,2}}{\xi_{\ell}} \quad ; \qquad (402.2)$$

observing that

$$J_{\ell} = \frac{m_{\ell,1}m_{\ell,2}}{m_{\ell,1}+m_{\ell,2}}\,\xi^2_{\ell} \quad , \qquad (402.3)$$

the Lagrange-Jacobi equation reads

$$\left(\xi^2_{\ell}\right)^{\backprime\backprime} = \frac{2\Gamma_{\ell}}{\xi_{\ell}} + b \qquad (402.4)$$

403 From the Lagrange-Jacobi equation (402.4) we derive now the asymptotic expression

$$\xi_{\ell} = \left(\frac{9}{2}\Gamma_{\ell}\right)^{\frac{1}{3}} t^{\frac{2}{3}} (1 \quad b) \qquad \text{as} \qquad t \to 0 \; : \qquad (403.1)$$

Abbreviate temporarily

$$\xi^2_{\ell} = \Xi_{\ell} \qquad ;$$

(313.2) implies that Ξ_{ℓ} ultimately decreases monotonically to zero, and (316.3) that

$$\Xi_{\ell} = b t^{\frac{4}{3}} \qquad . \qquad (403.2)$$

From the energy expression (402.2) follows $\xi^2_{\ell}\underset{\sim}{\xi}^{\backprime}_{\ell}\underset{\sim}{\xi}^{\backprime}_{\ell} \to 0$, and since $\underset{\sim}{K}_{\ell} \to \underset{\sim}{0}$, we find $\Xi^{\backprime}_{\ell} = \frac{1}{2}\xi_{\ell}\xi^{\backprime}_{\ell} \to 0$ by means of the identity

$$\underset{\sim}{K}_{\ell}\underset{\sim}{K}_{\ell} + (\underset{\sim}{\xi}_{\ell}\underset{\sim}{\xi}^{\backprime}_{\ell})(\underset{\sim}{\xi}_{\ell}\underset{\sim}{\xi}^{\backprime}_{\ell}) = \xi^2_{\ell}\underset{\sim}{\xi}^{\backprime}_{\ell}\underset{\sim}{\xi}^{\backprime}_{\ell} \qquad ;$$

Ξ_ℓ' is not identically zero since Ξ_ℓ is not constant.

Multiply the Lagrange-Jacobi equation

$$\Xi_\ell'' = \frac{2\Gamma_\ell}{\Xi_\ell^{\frac{1}{2}}} + b$$

by Ξ_ℓ' and integrate from $t = 0$ to t, $t < 0$; we get

$$\Xi_\ell'^2 = 8\Gamma_\ell \Xi_\ell^{\frac{1}{2}} + \int_0^t \Xi_\ell'(\tau)\, b(\tau)\, d\tau \qquad . \qquad (403.3)$$

Taking into account that by (402.4) Ξ_ℓ'' is ultimately positive, implying that Ξ_ℓ' ultimately increases, i.e., does not change sign,

~~[illegible]~~

by the first mean value theorem

the last integral can be written as $b\Xi_\ell$; hence,

$$\Xi_\ell'^2 = 8\Gamma_\ell \Xi_\ell^{\frac{1}{2}} (1 + b\Xi_\ell^{\frac{1}{2}}) \qquad ,$$

and substituting into the last term from (403.2), we get

$$\Xi_\ell'^2 = 8\Gamma_\ell \Xi_\ell^{\frac{1}{2}} (1 + bt^{\frac{2}{3}}) \qquad ,$$

yielding (403.1) by integration.

404 In order to get an asymptotic expression for $\underset{\sim}{\xi}_\ell$ as $t \to 0$, we first introduce a new independent variable s by

$$s = t^{\frac{1}{3}} \tag{404.1}$$

into the equations of motion (402.1); denoting differentiation with respect to s by a prime $'$ and substituting for ξ_ℓ from (403.1), we get

$$s^2 \underset{\sim}{\xi}''_\ell - 2s\underset{\sim}{\xi}'_\ell + 2\underset{\sim}{\xi}_\ell = s^2 b \underset{\sim}{\xi}_\ell + s^6 \underset{\sim}{Q}_\ell \quad . \tag{404.2}$$

Introduce further new dependent variables $\underset{\sim}{u}_\ell$ by

$$\underset{\sim}{\xi}_\ell = s\underset{\sim}{u}_\ell \quad ; \tag{404.3}$$

it follows that $\xi_\ell = s u_\ell$, and since $\xi_\ell = s^2 b$ by (403.1), both u_ℓ and $\underset{\sim}{u}_\ell$ remain bounded and $\underset{\sim}{u}_\ell$ can be written as $\underset{\sim}{u}_\ell = s\underset{\sim}{b}$. The application of (404.3) transforms (404.2) into

$$\underset{\sim}{u}_\ell'' = b\underset{\sim}{u}_\ell + s^3 \underset{\sim}{Q}_\ell \quad ; \tag{404.4}$$

substitute $\underset{\sim}{u}_\ell = s\underset{\sim}{b}$ and integrate $\underset{\sim}{u}_\ell'' = s\underset{\sim}{b} + s^3\underset{\sim}{Q}_\ell$ twice; then

$$\underset{\sim}{u}_\ell = s\underset{\sim}{a}_\ell + s^3\underset{\sim}{b} \quad , \tag{404.5}$$

$$\underset{\sim}{\xi}_\ell = s^2\underset{\sim}{a}_\ell + s^4\underset{\sim}{b} \quad , \tag{404.6}$$

and by comparison with (403.1)

$$a_\ell^2 = \left(\frac{9}{2}\Gamma_\ell\right)^{\frac{2}{3}} \neq 0 \quad . \tag{404.7}$$

405 Without restriction of generality, we can assume that all (finitely many) components of the $\mathbf{a}_\ell$ are different from zero; this can always be achieved by a constant rotation of the coordinates. We now transform the equations of motion (104.2) and (402.1) into a standard form of a system of differential equations of the first order, using again $s = t^{\frac{1}{3}}$ as the independent variable and introducing new dependent variables $\mathbf{w}_\ell$, $\boldsymbol{\omega}_\ell$, $\mathbf{W}_\ell$, and $\boldsymbol{\Omega}_\ell$ by

$$\mathbf{Z}_\ell = \mathbf{Z}_{\ell;o} + s^2\,\mathbf{W}_\ell \quad , \qquad \boldsymbol{\Omega}_\ell = s\,\mathbf{W}'_\ell \quad , \tag{405.1}$$

$$\xi_{\ell,j} = s^2\,a_{\ell;j}\,(\,1 + s\,w_{\ell;j}) \quad , \qquad \boldsymbol{\omega}_\ell = s\,\mathbf{w}'_\ell \qquad j=1,2,3\,. \tag{405.2}$$

Since $\mathbf{Z}_\ell = \mathbf{Z}_{\ell;o} + s^3\,\dot{\mathbf{Z}}_{\ell;o} + s^6\,\mathbf{b}$, we have $\mathbf{W}_\ell \to \mathbf{0}$ as $s \to 0$, and (404.6) implies that $\mathbf{w}_\ell \to \mathbf{0}$.

Write for the further discussion (104.2) as

$$\underset{\sim}{Z}_{\ell}'' = \underset{\sim}{B}_{\ell} \tag{405.3}$$

and (402.1) as

$$\underset{\sim}{\xi}_{\ell}'' = -\Gamma_{\ell} \frac{\underset{\sim}{\xi}_{\ell}}{\xi_{\ell}^{3}} + \underset{\sim}{b}_{\ell} \quad , \tag{405.4}$$

where $\underset{\sim}{B}_{\ell}$ and $\underset{\sim}{b}_{\ell}$ are bounded as $s, t \to 0$ and can be expanded into power series of s, $W_{\ell;j}$, and $w_{\ell;j}$, which converge for sufficiently small absolute values of those quantities.

406 Transformation of (405.3) by (405.1) gives

$$\begin{aligned} s\underset{\sim}{\Omega}_{\ell}' &= -\underset{\sim}{\Omega}_{\ell} + 2\underset{\sim}{W}_{\ell} + 9s^{4}\underset{\sim}{B}_{\ell} \\ s\underset{\sim}{W}_{\ell}' &= \underset{\sim}{\Omega}_{\ell} \quad . \end{aligned} \tag{406.1}$$

407 From the first equation of (405.2) we find

$$\xi_{\ell}^{2} = s^{4}\left(a_{\ell}^{2} + 2s\chi_{\ell} + s^{2}b\right) \quad ,$$

where (407.1)

$$\chi_{\ell} = a_{\ell;1}^{2} w_{\ell;1} + a_{\ell;2}^{2} w_{\ell;2} + a_{\ell;3}^{2} w_{\ell;3} \quad ,$$

hence

$$\xi_{\ell}^{-3} = s^{-6}\frac{2}{9\Gamma_{\ell}}\left(1 - 3a_{\ell}^{-2}\chi_{\ell} + s^{2}b\right) \quad . \tag{407.2}$$

Introduction of s and $\underset{\sim}{w}_{\ell}$ into (405.4) and substitution from (407.2) lead to

$$s^{2}w_{\ell;j}'' + 4s\,w_{\ell;j}' = -2w_{\ell;j} + 6a_{\ell}^{-2}\chi_{\ell} + s\,b_{\ell;j}^{*} \quad , \tag{407.3}$$

where $\underset{\sim}{b}^*_\ell$ has the same properties as $\underset{\sim}{b}_\ell$ in 405. With $\underset{\sim}{\omega}_\ell$ (cf. (405.2)) we finally get the following system:

$$\left.\begin{aligned} s\,\omega'_{\ell;j} &= -3\,\omega_{\ell;j} - 2\,w_{\ell;j} + 6\,a_\ell^{-2}\,\chi_\ell + s\,b^*_{\ell;j} \\ s\,w'_{\ell;j} &= \omega_{\ell;j} \quad . \end{aligned}\right\} \quad j=1,2,3 \qquad (407.4)$$

408 The full system of the transformed equations of motion (406.1) and (407.4) reads

$$\left.\begin{aligned} s\,\Omega'_{\ell;j} &= -\Omega_{\ell;j} + 2\,W_{\ell;j} + s\,B^*_{\ell;j} \\ s\,W'_{\ell;j} &= \Omega_{\ell;j} \end{aligned}\right\} \quad \begin{aligned} &j=1,2,3 \\ &\ell=1 \text{ -- } L_1+L_2 \end{aligned} \qquad (408.1a)$$

$$\left.\begin{aligned} s\,\omega'_{\ell;j} &= -3\,\omega_{\ell;j} - 2\,w_{\ell;j} + 6\,a_\ell^{-2}\,\chi_\ell + s\,b^*_{\ell;j} \\ s\,w'_{\ell;j} &= \omega_{\ell;j} \quad ; \end{aligned}\right\} \quad \begin{aligned} &j=1,2,3 \\ &\ell=1 \text{ -- } L_1 \end{aligned} \qquad (408.1b)$$

the nonlinear parts $s\,B^*_{\ell;j}$ and $s\,b^*_{\ell;j}$ are power series in all variables, converging if their absolute values are sufficiently small .

It is apparent that the matrix of the coefficients of the linear part on the right of (408.1) is in "block form"; consequently, the set of its elementary divisors is the union of the sets of the elementary divisors of the individual blocks (a block is the 6x6 matrix for a particular ℓ in (408.1a) or (408.1b)).
The eigenvalues of a block in (408.1a) are +1, +1, +1, -2, -2, -2 , and all elementary divisors are linear; similarly, the eigenvalues of

a block in (408. 1b) are $+1$, -1, -1, -2, -2, -4, and again all elementary divisors are linear.

By a well-knowntheorem about systems of differential equations of the type (408. 1) it follows that the solution with the property that all $\underset{\sim}{W}_\ell$, $\underset{\sim}{\Omega}_\ell$, $\underset{\sim}{w}_\ell$, and $\underset{\sim}{\omega}_\ell$ converge to zero as $s \to 0$ can be expanded into power series of s, which converge for sufficiently small $|s|$; the solution depends on $4L_1 + 3L_2$ parameters.

409 The theorem used in 408 has been proved by J. Horn: Ueber die Reihenentwicklung der Integrale eines Systems von Differentialgleichungen in der Umgebung gewisser singulärer Stellen. J. Reine Angew. Math. 116 (1896) 265 - 306 ; 117 (1897) 104 - 128 ; 254 - 266. For details and a survey of this and related problems see H. Dulac: Points singuliers des équations différentielles. Gauthier-Villars, Paris 1934. A modern simple proof (of a special case) can be found in Siegel [1] pp. 156 - 165.

410 Alternate proof of the boundedness of h_ℓ. The principal tool for the study of the multiple binary collisions is the Lagrange-Jacobi equation (cf. (402.4)), enabling us to derive the asymptotic expression (403.1) for ξ_ℓ; its applicability, in turn, is based on the boundedness of the energy h_ℓ as t approaches the instant of collision. We deduced this boundedness from general results abou ollisions (cf. 312). In the following we shall give a direct proof of the boundedness of h_ℓ, independent of the previous general results; for the method see Arenstorf [3].

According to 318, $\underset{\sim}{Q}_\ell$ in equation (402.1) can be written as $\xi_\ell \underset{\sim}{b}$, provided that ξ_ℓ is sufficiently small, i.e., that t is sufficiently close to the instant of collision $t=0$. From

$$h'_\ell = \frac{m_{\ell,1} m_{\ell,2}}{m_{\ell,1} + m_{\ell,2}} \underset{\sim}{\xi}'_\ell \underset{\sim}{Q}_\ell \qquad \text{(cf. 312)}$$

$$= \xi_\ell \underset{\sim}{\xi}'_\ell \underset{\sim}{b} \tag{410.1}$$

we get

$$h'^2_\ell \le \xi^2_\ell b^2 \underset{\sim}{\xi}'_\ell \underset{\sim}{\xi}'_\ell \qquad (b^2,\text{ to indicate } \underline{\text{positive}} \text{ bounded function})$$

and substituting for $\underset{\sim}{\xi}'_\ell \underset{\sim}{\xi}'_\ell$ from (402.2),

$$h'^2_\ell \le \xi_\ell b^2 (\xi_\ell h_\ell + \gamma m_{\ell,1} m_{\ell,2})$$

and because of $\xi_\ell \to 0$:

$$h'^2_\ell \le 1 + |h_\ell| \qquad (\text{for sufficiently small } \xi_\ell).$$

Denote by $H_\ell(t_1)$ the maximum of $|h_\ell|$ on $[t_0, t_1]$; then by integration from t_0 to t and then to t_1 , $t_0 < t \le t_1$:

$$|h_\ell(t) - h_\ell(t_0)| \le \left(1 + H_\ell(t_1)\right)^{\frac{1}{2}}(t - t_0) \le \left(1 + H_\ell(t_1)\right)^{\frac{1}{2}}(t_1 - t_0) \quad ,$$

hence

$$|h_\ell(t)| \le |h_\ell(t_0)| + \left(1 + H_\ell(t_1)\right)^{\frac{1}{2}}(t_1 - t_0) \qquad \text{on} \quad [t_0, t_1]$$

and

$$H_\ell(t_1) = \max_{t\in[t_0,t_1]} |h_\ell(t)| \le |h_\ell(t_0)| + \left(1 + H_\ell(t_1)\right)^{\frac{1}{2}}(t_1 - t_0) \quad .$$

For nonnegative a, b, x the inequality $x \le a + b(1+x)^{\frac{1}{2}}$ implies $x \le a + \frac{1}{2}b^2 + b(1 + a + \frac{1}{4}b^2)^{\frac{1}{2}} \le a + b + ba^{\frac{1}{2}} + b^2$; applying this, we find

$$H_\ell(t_1) \le |h_\ell(t_0)| + (t_1 - t_0)\left(1 + |h_\ell(t_0)|^{\frac{1}{2}} + (t_1 - t_0)\right) \quad ,$$

and it is obvious that the right side remains bounded as $t_1 \to 0$.

411 We finally formulate the following

Theorem: Let all variables be real and let the solution of the N-body problem be holomorphic on $[t_0, 0[$, $t_0 < 0$; let there be at $t = 0$ a collision singularity such that several binary collisions (and only these) take place at this instant. Then the position vectors, distances, and the products $t^{\frac{1}{3}} \cdot$(velocity vector) can be expanded into power series of $t^{\frac{1}{3}}$. Therefore, the solution can be analytically continued in the real beyond $t = 0$ into $t > 0$.

Bibliography on the singularities
of the equations of motion of celestial mechanics

This bibliography is intended to be a complete listing of papers on the title subject; the major part of the papers deals with the classical N-body problem (with newtonian gravitational forces acting on particles), but publications on the following topics have been included: regularization of the equations of motion, non-newtonian forces, and central configurations and homographic solutions, these last ones in view of their importance for the collision problem.

The only publications known to me as containing more than just a few references are (Sokolov [30],) Szebehely [13] , Vernić [3] , and Wintner [2] . Being a first attempt at collecting the pertinent literature, this bibliography will have its share of errors and omissions; particularly, no systematic search could be made for the Russian literature. Also, since to many of the included papers I do not have access, there will be listings which are only marginally or not at all connected with the topics intended to be covered. No attempt has been made to include all publications on collision and regularization in (1) the two-body problem and (2) in Euler's problem of two fixed centers; likewise, papers only applying regularizing transformations to other problems have not been listed systematically.

An effort has been made to provide references to review journals; the publications and the abbreviations used are:

AJ Astronomischer Jahresbericht [1899 -]
JFM Jahrbuch über die Fortschritte der Mathematik [1868 - 1944]
MR Mathematical Reviews [1940 -]
Z' Zentralblatt für Mathematik und ihre Grenzgebiete [1931 -];

other review publications (and most of JFM) are not available to me at present. The abbreviations of names of journals have been chosen as much as possible as in MR 32 (1966).

An asterisk * preceding the page number of a reference to a review publication indicates that the paper is listed only, but not reviewed. A tilde ~ indicates that the name of the journal is the same as in the foregoing reference.

I shall be grateful for any comments as to errors, omissions, etc., that will lead to an improvement of this bibliography. In particular, I would appreciate receiving reprints of new papers dealing with the singularity problem and/or related topics.

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