LS-179
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The Effect of Small Ellipsoidal Material on the Resonant Frequency of a Cavity

We assume that the medium inside the cavity has no losses with $\varepsilon = \varepsilon_o = \frac{1}{36\pi}10^{-9}$ farads/meter and $\mu = \mu_o = 4\pi \times 10^{-7}$ henrys/meter. Excited at resonance, the fields inside the cavity can be written in the form

$$E(x, y, z, t) = E_o(x, y, z)e^{j\omega t}$$

$$H(x, y, z, t) = H_o(x, y, z)e^{j\omega t}.$$

The electric and magnetic fields are 90° out of phase. In other words, if E_o is real H_o is imaginary and vice versa. Insertion of a small piece of material with $\varepsilon \neq \varepsilon_o$ and $\mu \neq \mu_o$ will change the field values and the resonant frequency

$$E = (E_o + E_1)e^{j(\omega + \delta\omega)t} \tag{1}$$
$$H = (H_o + H_1)e^{j(\omega + \delta\omega)t}.$$

Note that $\delta\omega$ will be a complex quantity if the inserted material is lossy. Substitution of equations (1) in Maxwell's equations

$$\text{curl } E = -\frac{\partial B}{\partial t}$$
$$\text{curl } H = \frac{\partial D}{\partial t}$$

give

$$\text{curl } (E_o + E_1) = -j(\omega + \delta\omega)(B_o + B_1)$$
$$\text{curl } (H_o + H_1) = j(\omega + \delta\omega)(D_o + D_1).$$

Noting that the fields E_o, D_o, H_o and B_o satisfy the same Maxwell's equations we obtain:

$$\text{curl } E_1 = -j[\delta\omega B_o + (\omega + \delta\omega)B_1] \tag{2a}$$

$$\text{curl } H_1 = j[\delta\omega D_o + (\omega + \delta\omega)D_1]. \tag{2b}$$

Multiplication of equation (2a) by H_o and equation (2b) by E_o and addition give

$$\begin{aligned} H_o \cdot \text{curl } E_1 + E_o \cdot \text{curl } H_1 &= j(\omega + \delta\omega)(E_o \cdot D_1 - H_o \cdot B_1) \\ &+ j\delta\omega(E_o \cdot D_o - H_o \cdot B_o) \end{aligned} \tag{3}$$

Using the vector relation

$$\text{div } (A \times B) = B \cdot \text{curl } A - A \cdot \text{curl } B$$

we can rewrite the L.H.S. of equation (3) in the form

$$\begin{aligned} &H_o \cdot \text{curl } E_1 + H_o \cdot \text{curl } H_1 \\ &\quad = E_1 \cdot \text{curl } H_o + H_1 \cdot \text{curl } E_o \\ &\quad - \text{div}(H_o \times E_1 + E_o \times H_1), \end{aligned}$$

or

$$\begin{aligned} &H_o \cdot \text{curl } E_1 + E_o \cdot \text{curl } H_1 \\ &\quad = j\omega(E_1 \cdot D_o - H_1 \cdot B_o) \\ &\quad - \text{div}(H_o \times E_1 + E_o \times H_1). \end{aligned} \tag{4}$$

Comparison of equations (3) and (4) gives

$$\begin{aligned} &j(\omega+\delta\omega)\ (E_o \cdot D_1 - H_o \cdot B_1) + j\delta\omega(E_o \cdot D_o - H_o \cdot B_o) \\ &\quad = j\omega(E_1 \cdot D_o - H_1 \cdot B_o) - \text{div } (H_o \times E_1 + E_o \times H_1) \end{aligned}$$

In practice $\omega >> \delta\omega$ so that in the expression $(\omega + \delta\omega)\ (E_o \cdot D_1 - H_o \cdot B_1)$ we can neglect $\delta\omega$ with respect to ω and obtain

$$\begin{aligned} &j\omega(E_o \cdot D_1 - H_o \cdot B_1 - E_1 \cdot D_o + H_1 \cdot B_o) + j\delta\omega(E_o \cdot D_o - H_o \cdot B_o) \\ &\quad = - \text{div } (H_o \times E_1 + E_o \times H_1). \end{aligned} \tag{5}$$

Outside the ellipsoid we have

$$D_o = \varepsilon_o E_o, \quad D_1 = \varepsilon_o E_1, \quad B_o = \mu_o H_o, \quad B_1 = \mu_o H_1,$$

and equation (5) reduces to

$$j\delta\omega(E_o \cdot D_o - H_o \cdot B_o) = - \operatorname{div} (H_o \times E_1 + E_o \times H_1).$$

We integrate this equation over the volume bounded by the cavity wall and the surface of the ellipsoid

$$j\delta\omega \int_{V-\Delta V} (E_o \cdot D_o - H_o \cdot B_o) dV = - \int_{V-\Delta V} \operatorname{div}(H_o \times E_1 + E_o \times H_1) dV \qquad (6)$$

where V = volume of the cavity

and ΔV = volume of the ellipsoid.

Using the divergence theorem, the R.H.S. of equation (6) can be written as a surface integral

$$\int_{V-\Delta V} \operatorname{div}(H_o \times E_1 + E_o \times H_1) dV = \int_{s+\Delta s} (H_o \times E_1 + E_o \times H_1) \cdot ds$$

where $s + \Delta s$ is the surface bounding the volume V-ΔV. Since the cavity is assumed to be a good conductor, E_o and E_1 will be practically perpendicular to the cavity surface and the contribution of the cavity wall to the surface integral can be neglected. In this case we find

$$j\delta\omega \int_{V-\Delta V} (E_o \cdot D_o - H_o \cdot B_o) dV = \int_{\Delta s} (H_o \times E_1 + E_o \times H_1) \cdot ds \qquad (7)$$

where Δs = surface of the ellipsoid. Note that ds is in the direction of the now outward normal to the ellipsoid surface. Inside the ellipsoid we have

$$D_1 = \varepsilon_o E_1 + P \quad \text{and} \quad B_1 = \mu_o H_1 + M$$

where P = polarization or electric dipole moment per unit volume, and

M = magnetization or magnetic dipole moment per unit volume.

Substitution in equation (5) gives

$$j\omega(E_o \cdot P - H_o M) + j\delta\omega(E_o \cdot D_o - H_o \cdot B_o) = - \operatorname{div} (H_o \times E_1 + E_o \times H_1).$$

Integrating over the volume of the ellipsoid and using the divergence theorem we obtain

$$j\omega \int_{\Delta V} (E_o \cdot P - (H_o \cdot M))\Delta V + j\delta\omega \int_{\Delta V} (E_o \cdot D_o - H_o \cdot B_o)\Delta V \tag{8}$$
$$= -\int_{\Delta s} (H_o \times E_1 + E_o \times H_1) \cdot ds.$$

Comparison of equations (7) and (8) and some manipulation gives

$$\frac{\delta\omega}{\omega} = -\frac{\int_{\Delta V} (E_o \cdot P - H_o \cdot M)dV}{\int_V (E_o \cdot D_o - H_o \cdot B_o)dV}$$

or, since H_o and B_o are imaginary if E_o and D_o are real, we can write

$$\frac{\delta\omega}{\omega} = -\frac{1}{4U} \int_{\Delta V} (E_o \cdot P + H_o \cdot M)dV \tag{9}$$

where U = cavity stored energy and E_o, P, H_o and M are now all real quantities. For the small region in and around the ellipsoid, E_o and H_o are practically uniform. For an ellipsoid of semi-axis a, b and c with a field <u>parallel to the a-axis</u>, P and M are given by (see M. Mason and W. Weaver, Dover Publications, § 36).

$$P = \frac{\varepsilon_r - 1}{L(\varepsilon_r - 1) + 1}\varepsilon_o E_o$$

$$M = \frac{\mu_r - 1}{L(\mu_r - 1) + 1}\mu_o H_o$$

where ε_r = relative permittivity, μ_r = relative permeability, and

$$L = \frac{abc}{2}\int_o^\infty \frac{du}{(a^2 + u)\sqrt{(a^2 + u)(b^2 + u)(c^2 + u)}}$$

<u>Axial Symmetrical Ellipsoid</u>

For axial symmetry about the a-axis we have b = c, and this reduces the integral to an elementary one,

$$L = \frac{ab^2}{2}\int_o^\infty \frac{du}{(a^2+u)^{3/2}(b^2+u)}$$

Performing the integration we find for a oblate spheroid (a < b)

$$L = \frac{1+e^2}{e^3}(e - \arctan e), \quad e = \frac{1}{a}\sqrt{b^2 - a^2}$$

and for a prolate spheroid (a > b)

$$L = \frac{1-e^2}{e^3}\left(\frac{1}{2}\ln\frac{1+e}{1-e} - e\right), \quad e = \frac{1}{a}\sqrt{a^2 - b^2}$$

For both cases the spheroid reduces to a sphere for $e \longrightarrow 0$ and L = 1/3. For the limit as $a \longrightarrow 0$ ($e \longrightarrow \infty$), the oblate spheroid becomes a circular disk of radius b and L = 1. On the other hand, as $b \longrightarrow 0$ the prolate spheroid becomes a very thin rod of length 2a and L = 0.

If the field is perpendicular to the axis of revolution we have

$$L = \frac{ab^2}{2}\int_o^\infty \frac{du}{(b^2+u)^2\sqrt{a^2+u}}$$

For a prolate spheroid (a > b) we find

$$L = \frac{1-e^2}{4e^3}\left(\frac{2e}{1-e^2} - \ln\frac{1+e}{1-e}\right), \quad e = \frac{1}{a}\sqrt{a^2 - b^2}\ .$$

For an oblate spheroid (a < b)) we find

$$L = \frac{1+e^2}{8e^3}\left(\arctan e - \frac{e(1-e^2)}{(1+e^2)^2}\right), \quad e = \frac{1}{a}\sqrt{b^2 - a^2}.$$

In both cases for $e \longrightarrow 0$ (sphere), L = 1/3. For the prolate spheroid for e = 1 (rod), L = 1/2. For the oblate spheroid for $e = \infty$ (disk), L = 0.

<u>EXAMPLES</u>

(1) Dielectric Sphere:

Radius R, $\mu_r = 1, \quad P = 3\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\varepsilon_o E_o, \quad M = 0$

$$\frac{\delta f}{f} = -\frac{\pi R^3}{U}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)\varepsilon_o E_o.$$

(2) Metal Sphere:

Radius R, $\mu_r = O$, $\varepsilon_r = \infty$, $P = 3\varepsilon_o E_o$, $M = -3/2\ \mu_o H_o$

$$\frac{\delta f}{f} = -\frac{\pi R^3}{U}\left(\varepsilon_o E_o - \frac{1}{2}\mu_o H_o^2\right).$$

(3) Dielectric Needle:

Parallel to E_o, volume ΔV, $P = (\varepsilon_r - 1)\ \varepsilon_o E_o$, $M = O$

$$\frac{\delta f}{f} = -\frac{\Delta V}{4U}(\varepsilon_r - 1)\varepsilon_o E_o^2.$$

(4) Metal Needle:

Perpendicular to E_o and parallel to H_o, volume ΔV, $P = 2\ \varepsilon_o E_o$, $M = -\mu_o H_o$

$$\frac{\delta f}{f} = -\frac{\Delta V}{4U}\left(2\varepsilon_o E_o^{\ 2} - \mu_o H_o^{\ 2}\right).$$

(5) Dielectric Disk:

Perpendicular to E_o, volume ΔV, $P = \frac{\varepsilon_r - 1}{e_r}\varepsilon_o E_o, \ M = O$

$$\frac{\delta f}{f} = -\frac{\Delta V}{4U}\frac{\varepsilon_r - 1}{\varepsilon_r}\varepsilon_o E_o^{\ 2}.$$

(6) Metal Needle Parallel to E_o and Metal Disk Parallel or Perpendicular to E_o

For the metal needle parallel to E_o, the approximation of $e = \frac{1}{a}\sqrt{a^2 - b^2} = 1$ is not valid since for L = 0, $P \longrightarrow \infty$. For the metal disk, the approximation $e = \frac{1}{a}\sqrt{b^2 - a^2} = \infty$ is not valid since for L = 0, $P \longrightarrow \infty$ and for L = 1, $M \longrightarrow \infty$. In these cases, the actual value of L must be calculated and substituted in $P = \frac{1}{L}\varepsilon_o E_o$ and $M = \frac{1}{L-1}\mu_o H_o$.