A.4 Eigensystems

1. The eigenvalue problem for a matrix A is defined as

$$A\vec{x} = \lambda\vec{x} \quad \text{or} \quad [A - \lambda I]\vec{x} = 0$$

and the generalized eigenvalue problem, including the matrix B, as

$$A\vec{x} = \lambda B\vec{x} \quad \text{or} \quad [A - \lambda B]\vec{x} = 0$$

2. If a square matrix with real elements is symmetric, its eigenvalues are all real. If it is skew-symmetric, they are all imaginary.
3. Gershgorin's theorem: The eigenvalues of a matrix lie in the complex plane in the union of circles having centers located by the diagonals with radii equal to the sum of the absolute values of the corresponding off-diagonal row elements.
4. In general, an $m \times m$ matrix A has $n_{\vec{x}}$ linearly independent eigenvectors with $n_{\vec{x}} \leq m$ and n_λ distinct eigenvalues (λ_i) with $n_\lambda \leq n_{\vec{x}} \leq m$.
5. A set of eigenvectors is said to be linearly independent if

$$a \cdot \vec{x}_m + b \cdot \vec{x}_n \neq \vec{x}_k \quad , \quad m \neq n \neq k$$

for any complex a and b and for all combinations of vectors in the set.
6. If A posseses m linearly independent eigenvectors then A is diagonalizable, i.e.,

$$X^{-1}AX = \Lambda$$

where X is a matrix whose columns are the eigenvectors,

$$X = \left[\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\right]$$

and Λ is the diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_m \end{bmatrix}$$

If A can be diagonalized, its eigenvectors completely span the space, and A is said to have a *complete* eigensystem.
7. If A has m distinct eigenvalues, then A is always diagonalizable. With each distinct eigenvalue there is one associated eigenvector, and this eigenvector cannot be formed from a linear combination of any of the other eigenvectors.

8. In general, the eigenvalues of a matrix may not be distinct, in which case the possibility exists that it cannot be diagonalized. If the eigenvalues of a matrix are not distinct, but all of the eigenvectors are linearly independent, the matrix is said to be *derogatory*, and it can still be diagonalized.

9. If a matrix does not have a complete set of linearly independent eigenvectors, it cannot be diagonalized. The eigenvectors of such a matrix cannot span the space, and the matrix is said to have a *defective* eigensystem.
10. Defective matrices cannot be diagonalized, but they can still be put into a compact form by a similarity transform, S, such that

$$J = S^{-1}AS = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_k \end{bmatrix}$$

where there are k linearly independent eigenvectors, and J_i is either a Jordan subblock or λ_i.
11. A Jordan submatrix has the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ 0 & 0 & \lambda_i & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & \lambda_i \end{bmatrix}$$

12. Use of the transform S is known as putting A into its *Jordan Canonical form*. A repeated root in a Jordan block is referred to as a *defective* eigenvalue. For each Jordan submatrix with an eigenvalue λ_i of multiplicity r, there exists one eigenvector. The other $r-1$ vectors associated with this eigenvalue are referred to as *principal vectors*. The complete set of principal vectors and eigenvectors are all linearly independent.
13. Note that if P is the permutation matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad , \quad P^T = P^{-1} = P$$

then

$$P^{-1}\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}P = \begin{bmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix}$$

14. Some of the Jordan blocks may have the same eigenvalue. For example, the matrix

$$\begin{bmatrix} \begin{bmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{bmatrix} & & & & \\ & \lambda_1 & & & \\ & & \begin{bmatrix} \lambda_1 & 1 \\ & \lambda_1 \end{bmatrix} & & \\ & & & \begin{bmatrix} \lambda_2 & 1 \\ & \lambda_2 \end{bmatrix} & \\ & & & & \lambda_3 \end{bmatrix}$$

is both defective and derogatory, having:

- 9 eigenvalues
- 3 distinct eigenvalues
- 3 Jordan blocks
- 5 linearly independent eigenvectors
- 3 principal vectors with λ_1
- 1 principal vector with λ_2

A.5 Vector and Matrix Norms

1. The *spectral radius* of a matrix A is symbolized by $\sigma(A)$ such that

$$\sigma(A) = |\sigma_m|_{max}$$

where σ_m are the eigenvalues of the matrix A.

2. A p-norm of the vector $\vec{v}$ is defined as

$$||v||_p = \left(\sum_{j=1}^{M} |v_j|^p \right)^{1/p}$$

3. A p-norm of a matrix A is defined as

$$||A||_p = \max_{x \neq 0} \frac{||Av||_p}{||v||_p}$$

4. Let A and B be square matrices of the same order. All matrix norms must have the properties

$$\begin{aligned} ||A|| &\geq 0, \quad ||A|| = 0 \text{ implies } A = 0 \\ ||c \cdot A|| &= |c| \cdot ||A|| \\ ||A + B|| &\leq ||A|| + ||B|| \\ ||A \cdot B|| &\leq ||A|| \cdot ||B|| \end{aligned}$$

5. Special p-norms are

$$||A||_1 = \max_{j=1,\cdots,M} \sum_{i=1}^{M} |a_{ij}| \quad \text{maximum column sum}$$

$$||A||_2 = \sqrt{\sigma(\overline{A}^T \cdot A)}$$

$$||A||_\infty = \max_{i=1,2,\cdots,M} \sum_{j=1}^{M} |a_{ij}| \quad \text{maximum row sum}$$

   where $||A||_p$ is referred to as the L_p norm of A.
6. In general $\sigma(A)$ does not satisfy the conditions in 4, so in general $\sigma(A)$ is *not* a true norm.
7. When A is normal, $\sigma(A)$ *is* a true norm; in fact, in this case it is the L_2 norm.
8. The spectral radius of A, $\sigma(A)$, is the lower bound of all the norms of A.

B. SOME PROPERTIES OF TRIDIAGONAL MATRICES

B.1 Standard Eigensystem for Simple Tridiagonal Matrices

In this work tridiagonal banded matrices are prevalent. It is useful to list some of their properties. Many of these can be derived by solving the simple linear difference equations that arise in deriving recursion relations.

Let us consider a *simple* tridiagonal matrix, i.e., a tridiagonal matrix with constant scalar elements a, b, and c; see Section 3.3.2. If we examine the conditions under which the determinant of this matrix is zero, we find (by a recursion exercise)

$$\det[B(M : a, b, c)] = 0$$

if

$$b + 2\sqrt{ac}\cos\left(\frac{m\pi}{M+1}\right) = 0 \quad , \quad m = 1, 2, \cdots, M$$

From this it follows at once that the eigenvalues of $B(a, b, c)$ are

$$\lambda_m = b + 2\sqrt{ac}\cos\left(\frac{m\pi}{M+1}\right) \quad , \quad m = 1, 2, \cdots, M \tag{B.1}$$

The right-hand eigenvector of $B(a, b, c)$ that is associated with the eigenvalue λ_m satisfies the equation

$$B(a, b, c)\vec{x}_m = \lambda_m \vec{x}_m \tag{B.2}$$

and is given by

$$\vec{x}_m = (x_j)_m = \left(\frac{a}{c}\right)^{\frac{j-1}{2}} \sin\left[j\left(\frac{m\pi}{M+1}\right)\right] \quad , \quad m = 1, 2, \cdots, M \tag{B.3}$$

These vectors are the columns of the right-hand eigenvector matrix, the elements of which are

$$X = (x_{jm}) = \left(\frac{a}{c}\right)^{\frac{j-1}{2}} \sin\left[\frac{jm\pi}{M+1}\right] \quad , \quad \begin{matrix} j = 1, 2, \cdots, M \\ m = 1, 2, \cdots, M \end{matrix} \tag{B.4}$$

Notice that if $a = -1$ and $c = 1$,

$$\left(\frac{a}{c}\right)^{\frac{j-1}{2}} = e^{i(j-1)\frac{\pi}{2}} \tag{B.5}$$

The left-hand eigenvector matrix of $B(a, b, c)$ can be written

$$X^{-1} = \frac{2}{M+1}\left(\frac{c}{a}\right)^{\frac{m-1}{2}} \sin\left[\frac{mj\pi}{M+1}\right] \quad , \quad \begin{matrix} m = 1, 2, \cdots, M \\ j = 1, 2, \cdots, M \end{matrix}$$

In this case notice that if $a = -1$ and $c = 1$

$$\left(\frac{c}{a}\right)^{\frac{m-1}{2}} = e^{-i(m-1)\frac{\pi}{2}} \tag{B.6}$$

B.2 Generalized Eigensystem for Simple Tridiagonal Matrices

This system is defined as follows

$$\begin{bmatrix} b & c & & & \\ a & b & c & & \\ & a & b & & \\ & & & \ddots & c \\ & & & a & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_M \end{bmatrix} = \lambda \begin{bmatrix} e & f & & & \\ d & e & f & & \\ & d & e & & \\ & & & \ddots & f \\ & & & d & e \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_M \end{bmatrix}$$

In this case one can show after some algebra that

$$\det[B(a - \lambda d, b - \lambda e, c - \lambda f] = 0 \tag{B.7}$$

if

$$b - \lambda_m e + 2\sqrt{(a - \lambda_m d)(c - \lambda_m f)} \cos\left(\frac{m\pi}{M+1}\right) = 0 \quad , \quad m = 1, 2, \cdots, M$$

If we define

$$\theta_m = \frac{m\pi}{M+1}, \quad p_m = \cos\theta_m$$

then

$$\lambda_m = \frac{eb - 2(cd + af)p_m^2 + 2p_m\sqrt{(ec - fb)(ea - bd) + [(cd - af)p_m]^2}}{e^2 - 4fdp_m^2}$$

The right-hand eigenvectors are

$$\vec{x}_m = \left[\frac{a - \lambda_m d}{c - \lambda_m f}\right]^{\frac{j-1}{2}} \sin[j\theta_m] \quad , \quad \begin{matrix} m = 1, 2, \cdots, M \\ j = 1, 2, \cdots, M \end{matrix}$$

These relations are useful in studying relaxation methods.

B.3 The Inverse of a Simple Tridiagonal Matrix

The inverse of $B(a, b, c)$ can also be written in analytic form. Let D_M represent the determinant of $B(M : a, b, c)$

$$D_M \equiv \det[B(M : a, b, c)]$$

Defining D_0 to be 1, it is simple to derive the first few determinants, thus

$$\begin{aligned} D_0 &= 1 \\ D_1 &= b \\ D_2 &= b^2 - ac \\ D_3 &= b^3 - 2abc \end{aligned} \tag{B.8}$$

One can also find the recursion relation

$$D_M = bD_{M-1} - acD_{M-2} \tag{B.9}$$

Eq. B.9 is a linear OΔE, the solution of which was discussed in Section 6.3. Its characteristic polynomial is $P(E) = E^2 - bE + ac$, and the two roots to $P(\sigma) = 0$ result in the solution

$$D_M = \frac{1}{\sqrt{b^2 - 4ac}} \left\{ \left[\frac{b + \sqrt{b^2 - 4ac}}{2} \right]^{M+1} - \left[\frac{b - \sqrt{b^2 - 4ac}}{2} \right]^{M+1} \right\}$$
$$M = 0, 1, 2, \cdots \tag{B.10}$$

where we have made use of the initial conditions $D_0 = 1$ and $D_1 = b$. In the limiting case when $b^2 - 4ac = 0$, one can show that

$$D_M = (M + 1)\left(\frac{b}{2}\right)^M$$

Then for $M = 4$

$$B^{-1} = \frac{1}{D_4} \begin{bmatrix} D_3 & -cD_2 & c^2D_1 & -c^3D_0 \\ -aD_2 & D_1D_2 & -cD_1D_1 & c^2D_1 \\ a^2D_1 & -aD_1D_1 & D_2D_1 & -cD_2 \\ -a^3D_0 & a^2D_1 & -aD_2 & D_3 \end{bmatrix}$$

and for $M = 5$

$$B^{-1} = \frac{1}{D_5} \begin{bmatrix} D_4 & -cD_3 & c^2D_2 & -c^3D_1 & c^4D_0 \\ -aD_3 & D_1D_3 & -cD_1D_2 & c^2D_1D_1 & -c^3D_1 \\ a^2D_2 & -aD_1D_2 & D_2D_2 & -cD_2D_1 & c^2D_2 \\ -a^3D_1 & a^2D_1D_1 & -aD_2D_1 & D_3D_1 & -cD_3 \\ a^4D_0 & -a^3D_1 & a^2D_2 & -aD_3 & D_4 \end{bmatrix}$$

The general element d_{mn} is

Upper triangle:

$$m = 1, 2, \cdots, M-1 \quad , \quad n = m+1, m+2, \cdots, M$$

$$d_{mn} = D_{m-1} D_{M-n} (-c)^{n-m} / D_M$$

Diagonal:

$$n = m = 1, 2, \cdots, M$$

$$d_{mm} = D_{M-1} D_{M-m} / D_M$$

Lower triangle:

$$m = n+1, n+2, \cdots, M \quad , \quad n = 1, 2, \cdots, M-1$$

$$d_{mn} = D_{M-m} D_{n-1} (-a)^{m-n} / D_M$$

B.4 Eigensystems of Circulant Matrices

B.4.1 Standard Tridiagonal Matrices

Consider the circulant (see Section 3.3.4) tridiagonal matrix

$$B_p(M : a, b, c) \tag{B.11}$$

The eigenvalues are

$$\lambda_m = b + (a+c) \cos\left(\frac{2\pi m}{M}\right) - i(a-c) \sin\left(\frac{2\pi m}{M}\right) \quad , \\ m = 0, 1, 2, \cdots, M-1 \tag{B.12}$$

The right-hand eigenvector that satisfies $B_p(a, b, c)\vec{x}_m = \lambda_m \vec{x}_m$ is

$$\vec{x}_m = (x_j)_m = e^{i\,j\,(2\pi m/M)} \quad , \quad j = 0, 1, \cdots, M-1 \tag{B.13}$$

where $i \equiv \sqrt{-1}$, and the right-hand eigenvector matrix has the form

$$X = (x_{jm}) = e^{ij\left(\frac{2\pi m}{M}\right)} \quad , \quad \begin{array}{l} j = 0, 1, \cdots, M-1 \\ m = 0, 1, \cdots, M-1 \end{array}$$

The left-hand eigenvector matrix with elements x' is

$$X^{-1} = (x'_{mj}) = \frac{1}{M} e^{-im\left(\frac{2\pi j}{M}\right)} \quad , \quad \begin{array}{l} m = 0, 1, \cdots, M-1 \\ j = 0, 1, \cdots, M-1 \end{array}$$

Note that both X and X^{-1} are symmetric and that $X^{-1} = \frac{1}{M} X^H$, where X^H is the conjugate transpose of X.

B.4.2 General Circulant Systems

Notice the remarkable fact that the elements of the eigenvector matrices X and X^{-1} for the tridiagonal circulant matrix given by Eq. B.11 do not depend on the elements a, b, c in the matrix. In fact, *all circulant matrices of order M have the same set of linearly independent eigenvectors*, even if they are completely dense. An example of a dense circulant matrix of order $M = 4$ is

$$\begin{bmatrix} b_0 & b_1 & b_2 & b_3 \\ b_3 & b_0 & b_1 & b_2 \\ b_2 & b_3 & b_0 & b_1 \\ b_1 & b_2 & b_3 & b_0 \end{bmatrix} \tag{B.14}$$

The eigenvectors are always given by Eq. B.13, and further examination shows that the elements in these eigenvectors correspond to the elements in a complex harmonic analysis or complex discrete Fourier series.

Although the eigenvectors of a circulant matrix are independent of its elements, the eigenvalues are not. For the element indexing shown in Eq. B.14 they have the general form

$$\lambda_m = \sum_{j=0}^{M-1} b_j e^{i(2\pi jm/M)}$$

of which Eq. B.12 is a special case.

B.5 Special Cases Found From Symmetries

Consider a mesh with an even number of interior points, such as that shown in Fig. B.1. One can seek from the tridiagonal matrix $B(2M : a, b, a,)$ the eigenvector subset that has even symmetry when spanning the interval $0 \leq x \leq \pi$. For example, we seek the set of eigenvectors $\vec{x}_m$ for which

$$\begin{bmatrix} b & a & & & & \\ a & b & a & & & \\ & a & \ddots & & & \\ & & & \ddots & a & \\ & & & a & b & a \\ & & & & a & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_2 \\ x_1 \end{bmatrix} = \lambda_m \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_2 \\ x_1 \end{bmatrix}$$

This leads to the subsystem of order M which has the form

$$B(M : a, \vec{b}, a)\vec{x}_m = \begin{bmatrix} b & a & & & & \\ a & b & a & & & \\ & a & \ddots & & & \\ & & & \ddots & a & \\ & & & a & b & a \\ & & & & a & b+a \end{bmatrix} \vec{x}_m = \lambda_m \vec{x}_m \qquad \text{(B.15)}$$

By folding the known eigenvectors of $B(2M : a, b, a)$ about the center, one can show from previous results that the eigenvalues of Eq. B.15 are

$$\lambda_m = b + 2a\cos\left(\frac{(2m-1)\pi}{2M+1}\right) \quad , \quad m = 1, 2, \cdots, M \qquad \text{(B.16)}$$

and the corresponding eigenvectors are

$$\vec{x}_m = \sin\left(\frac{j(2m-1)\pi}{2M+1}\right) \quad , \quad j = 1, 2, \cdots, M$$

Imposing symmetry about the same interval but for a mesh with an odd number of points (see Fig. B.1) leads to the matrix

$$B(M : \vec{a}, b, a) = \begin{bmatrix} b & a & & & & \\ a & b & a & & & \\ & a & \ddots & & & \\ & & & \ddots & a & \\ & & & a & b & a \\ & & & & 2a & b \end{bmatrix}$$

By folding the known eigenvalues of $B(2M-1 : a, b, a)$ about the center, one can show from previous results that the eigenvalues of Eq. B.16 are

$$\lambda_m = b + 2a\cos\left(\frac{(2m-1)\pi}{2M}\right) \quad , \quad m = 1, 2, \cdots, M$$

and the corresponding eigenvectors are

$$\vec{x}_m = \sin\left(\frac{j(2m-1)\pi}{2M}\right) \quad , \quad j = 1, 2, \cdots, M$$

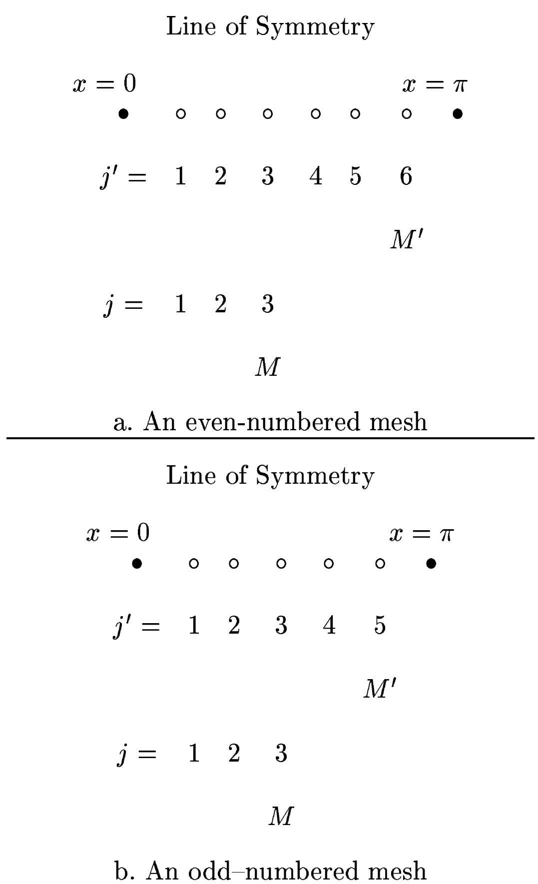

Figure B.1 – Symmetrical folds for special cases

B.6 Special Cases Involving Boundary Conditions

We consider two special cases for the matrix operator representing the 3-point central difference approximation for the second derivative $\partial^2/\partial x^2$ at all points away from the boundaries, combined with special conditions imposed at the boundaries.

Note: In both cases

$$
\begin{aligned}
m &= 1, 2, \cdots, M \\
j &= 1, 2, \cdots, M \\
-2 + 2\cos(\alpha) &= -4\sin^2(\alpha/2)
\end{aligned}
$$

When the boundary conditions are Dirichlet on both sides,

$$
\begin{bmatrix}
-2 & 1 & & & \\
1 & -2 & 1 & & \\
 & 1 & -2 & 1 & \\
 & & 1 & -2 & 1 \\
 & & & 1 & -2
\end{bmatrix}
\qquad
\begin{aligned}
\lambda_m &= -2 + 2\cos\left(\frac{m\pi}{M+1}\right) \\
\vec{x}_m &= \sin\left[j\left(\frac{m\pi}{M+1}\right)\right]
\end{aligned}
\tag{B.17}
$$

When one boundary condition is Dirichlet and the other is Neumann (and a diagonal preconditioner is applied to scale the last equation),

$$
\begin{bmatrix}
-2 & 1 & & & \\
1 & -2 & 1 & & \\
 & 1 & -2 & 1 & \\
 & & 1 & -2 & 1 \\
 & & & 1 & -1
\end{bmatrix}
\qquad
\begin{aligned}
\lambda_m &= -2 + 2\cos\left[\frac{(2m-1)\pi}{2M+1}\right] \\
\vec{x}_m &= \sin\left[j\left(\frac{(2m-1)\pi}{2M+1}\right)\right]
\end{aligned}
\tag{B.18}
$$