Small Polyhedron Reconnection: A New Way to Eliminate Poorly-Shaped Tetrahedra

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Abstract. Local transformation, or topological reconnection, is one of effective procedures of mesh improvement method, especially for three-dimensional tetrahedral mesh. Although the existing local transformations such as 2-3/3-2 flip are effective in removing poorly-shaped tetrahedra, it is still possible to improve the quality of mesh further by expanding the space of transformation region. The authors recently proposed a new local transformation operation, *small polyhedron reconnection* (or *SPR* for abbreviating), which seeks the optimal tetrahedralization of a polyhedron with a certain number of vertexes and faces (typically composed of 20 to 40 tetrahedral elements). In this paper, the framework of SPR approach for mesh quality improvement based on the SPR operation is presented. The main idea is to take a poorly-shaped or "worst" element as the core and construct a small polyhedron by adding 20-40 elements surrounding it, then find the optimal tetrahedralization of this small polyhedron through SPR operation. By replacing the original tetrahedra with the optimal tetrahedralization, the quality of the mesh is improved. Experimental investigations with tetrahedral finite element meshes show that the SPR approach is quite effective in improvement of mesh quality with acceptable time cost, and works well in combining with a smoothing approach. Although further researches are required for a more definite conclusion, the presented approach can be utilized as a powerful and effective tool for tetrahedral mesh generation and mesh improvement. We believe that the superior performance of the SPR approach makes it worthy of further study.

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1. Introduction

Geometrical optimization (also called node repositioning or smoothing) and topological optimization (also called local transformation or reconnection) are two main categories of mesh improvement procedure. Geometrical optimization relocates mesh points to improve mesh quality without changing mesh topology [1-7], while topological optimization changes the topology of a mesh, i.e. node-element connectivity relationship [1-3, 8, 9]. This paper will focus on the latter, local transformation.

The most frequently used and most effective operations of reconnection for tetrahedral mesh are so-called basic or elementary flips [10], e.g. 2-3 flip, 3-2 flip, 2-2 flip, 4-4 flip. These topological transformations are usually called "local", since only a small number of tetrahedra (typically fewer than 5) are removed or introduced by a single transformation. Such flips are simple, easy to implement, but effective in removing poorly-shaped tetrahedra [2, 3, 8]. However, since these basic local transformations only simply make a selection from several possible configurations within a relatively small region composed of several tetrahedra, the effect for mesh quality improvement is limited.

In order to break such a limitation and improve the quality of mesh further, the authors recently proposed the strategy of *optimal tetrahedralization for small polyhedron* and corresponding *small polyhedron reconnection* (*SPR*) operation [11], which seeks the optimal tetrahedralization of a polyhedron with a certain number of vertexes and faces instead of choosing the best configuration from several possibilities within a small region that consists of a small number of tetrahedra. For a SPR operation, since the region – usually composed of 20 to 40 tetrahedral elements – is much larger than that in the local transformations mentioned, more quality improvement is expected. Up to now, to the best knowledge of the authors, no relevant studies have been reported in other literatures.

The previous work of the authors [11] mainly emphasized the concept and the idea of the SPR operation from the viewpoint of local topological transformation. The efficiency of the SPR operation is also discussed and tested. The time complexity of the searching algorithm in the SPR operation seems too high and the computational cost may not be afforded if the size of the small polyhedron is too large. However, by some deliberate speedup strategies, the efficiency of optimal searching algorithm can be significantly enhanced and the SPR operation can be applied to practical mesh improvement with acceptable payment of time cost.

In this paper, the framework of SPR approach for mesh quality improvement based on the SPR operation is presented. The main idea is as follows. First locate a poorly-shaped or “worst” element (in sense of some quality measurement). Then take this “worst” element as the core and construct a small polyhedron by adding 20-40 elements surrounding it. Next find the optimal tetrahedralization of this small polyhedron through the SPR operation. By replacing the original tetrahedra with the optimal tetrahedralization, the quality of the mesh is improved. The cycle continues until the improvement reaches its limit, that is no better tetrahedralization existed for the small polyhedron most recently constructed.

2. SPR Operation: Optimal Tetrahedralization for Small Polyhedron

To break the limitation of the existing elementary flips, the authors recently proposed a new local reconnection strategy [11], *optimal tetrahedralization for small polyhedron*, which is illustrated in form of two-dimensional case in Fig. 1. Rather than simply making a selection from several possible configurations within a small region that consists of a small number of tetrahedra as previous local transformation usually does, the new reconnection strategy seeks the optimal tetrahedralization of a polyhedron with a certain number of vertexes and faces. Since the region – usually composed of 20 to 40 tetrahedral elements – is much larger than that in the local transformations mentioned, more quality improvement is expected.

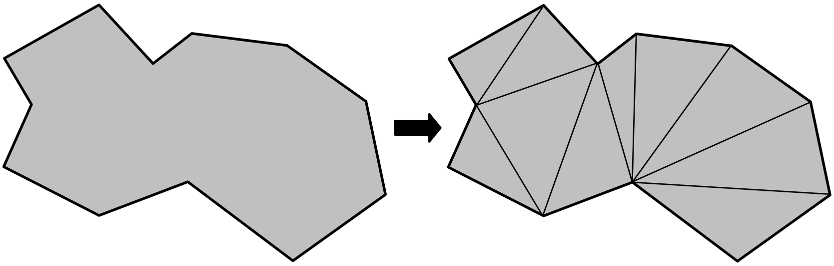

Fig. 1. Two-dimensional illustration for optimal tetrahedralization for small polyhedron

According to the strategy of optimal tetrahedralization for small polyhedron, two kinds of small polyhedron reconnection (SPR) operations are defined as follows.

SPR operation 1: For a given polyhedron with a certain number of triangles on boundary, seeks its optimal tetrahedralization without Steiner nodes added.

SPR operation 2: For a given polyhedron with a certain number of triangles on boundary, seeks its optimal tetrahedralization without Steiner nodes added under some extra geometric restrictions.

Note that the number of triangles on the boundary, *S*, is taken here to denote the size of the polyhedron instead of the number of tetrahedral elements. The SPR operation 2 is mainly applicable for boundary recovery and the details are not discussed here. This paper focuses on SPR operation 1 (for conciseness omits "1" in following text). In order to keep the completeness of the method proposed, the algorithm of the SPR operation [12] is first introduced. .

First choose a triangle *F* on the boundary of the polyhedron *P*, and construct an element (denoted by *ELE*) by *F* and one of the other vertexes of the polyhedron. Thus the original polyhedron is divided into the element *ELE* and a new smaller polyhedron (denoted by *Q*). Next solve the smaller problem for the new smaller polyhedron *Q* by the same algorithm recursively, and then merge its result with the element *ELE* to get a feasible solution for the original polyhedron *P*. Here, the so-called feasible solution is in some sense optimal, since it includes the optimal solution of the smaller polyhedron *Q*. This process is repeated for all the remain vertices. Finally choose the best tetrahedralization from all feasible solutions. Thus the final solution is exactly the optimal solution for the polyhedron *P*. The recursive procedure is illustrated in form of two-dimensional case in Fig. 2, and the pseudocode for the algorithm is listed in *Algorithm* 1.

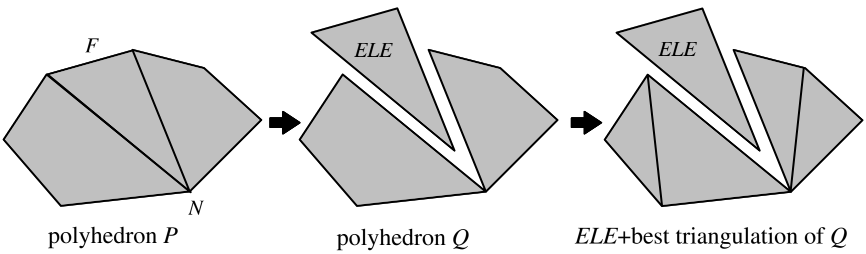

Fig. 2. Recursive procedure for the SPR operation illustrated in form of two-dimensional case (*ELE*+ the best triangulation of *Q* => a triangulation of *P*)

By the way, some more general and sophisticated local transformations, such as composite transformation operations [9], the general edge flip [10], may be considered as special cases of the SPR operation.

*Algorithm*1*: The recursive algorithm of the SPR operation*

int OptimalTetMeshForSmallPolyhedron (q_0, P, T)

input: q_0, quality of the initial mesh;
P, the small polyhedron.

output: T, the best triangulation. If there is no triangulation with quality better than q_0, T will be *NULL*.

return value: "*succeed*" or "*fail*".

temporary variables: T_c, the best triangulation among these already tested;
q_c, quality of T_c;
r_t, return value of the recursive call.

q_c=q_0, T_c=*NULL*
select a triangle F on the polyhedron
for each vertex N on the polyhedron, do
{
    if (F and N can construct a valid tetrahedron *ELE*
        and quality of *ELE* is better than q_c)
    {
        remove *ELE* from P, construct a new smaller
            polyhedron Q
        r_t= OptimalTetMeshForSmallPolyhedron (q_c, Q, T_Q)
        if (r_t is "succeed")
        {
            merge T_Q & *ELE* to create a new triangulation of P
            update T_c and q_c
        }
    }
}
if (a better mesh found) { T=T_c, return "*succeed*"}
else return "*fail*"

The algorithm given above is supposed to treat all of the triangulation cases and seems very time consuming. However, what's exhilarating is that many of the triangulation tries will be aborted and rejected in early stage since they produce a bad element or can not pass the valid test (such as overlapping or gap occurs). Similarly, though a large number of sub-problems appear in the searching process, many of them will be blocked by the valid test or quality test of the first element.

Moreover, if an initial triangulation for a polyhedron already exists, the optimal search process can be greatly accelerated, although the initial triangulation is not necessary for the SPR operation. This is the usual case for mesh improving.

The valid test and quality test discussed above has already made the SPR operation be able to improve quality for practical finite element meshes within acceptable time cost when the size of small polyhedron is limited to 15.

Additionally, a few further strategies are discussed in following to accelerate above searching algorithm. If a polyhedron can be subdivided into several smaller ones in digging process, it is unwise to still treat them as a whole one. We may readily achieve substantial speedup by solving several small sub-problems on smaller polyhedrons separately. Then the result is obtained by merging the result of all separate sub-problems. The testing results indicate this strategy can greatly enhance the algorithm efficiency. Furthermore a natural idea to speed the operation further is guiding the process to separate the small polyhedron as earlier as possible. So we select the digging face at the location where a tetrahedral element has just been removed.

In the thorough search algorithm, there is a drawback: the same sub-problems may be encountered several times. Here a simple strategy, storing and searching, is adopted to eliminate the repeated calculation of the same sub-problem. The sub-problems that have already been solved are stored in a bintree, in which each solved sub-problem has a record with information including its geometric description and the result of its optimal tetrahedralization. Before to be solved, any sub-problem will be searched first in the bintree. If its record has existed, just retrieve the result and return.

3. Using the SPR Operation to Remove Slivers and Other Poorly-Shaped Tetrahedra

The SPR operation can be applied to improve the mesh quality by removing slivers and other poorly-shaped element in a step by step manner. First find the worst element according to a specific quality measure. Then construct a small polyhedron that includes the worst element and its neighbors, and perform SPR operation to find out the best tetrahedralization of this polyhedron to improve the quality of local region adjacent to the worst element. Next, find another worst element and repeat above procedure. The procedure will stop when the tetrahedralization of the polyhedron that includes current worst element can not be improved. The SPR operations are usually performed in limited times in practice and the payment for time cost is reasonable.

There are two key steps in the above procedure: 1) Construction of the small polyhedron, 2) Optimal tetrahedralization of the small polyhedron by the SPR operation.

Suppose a bad element is a sliver, that its four nodes are nearly coplanar. This element together with some other elements which share one of 2 particular sliver edges with bigger dihedral can compose a small polyhedron (Fig. 3 and *Algorithm* 2).

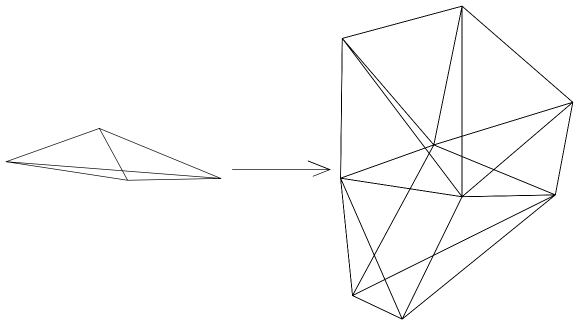

Fig. 3. A small polyhedron created surrounding a sliver

Algorithm 2: Constructing small polyhedron surrounding the worst element

Input: mesh *M*, the worst element *E* of *M*.
Output: a small polyhedron *P* with *E* as its core.

1 Calculate the six dihedral angles of *E*.
2 Get the two maximum angles, suppose their related edges are *ab* and *cd*.
3 Get all elements around *ab* and *cd*.
4 Merge all those elements to produce polyhedron *P*.

Usually, applying a SPR operation to such a polyhedron will eliminate thin element appropriately. In some cases, the polyhedron created is not big enough to produce a good local mesh. More elements should be included to mend the polyhedron, which is supposed to make the polyhedron something smoother. Our testing shows that using the small polyhedrons created in our algorithm can usually improve the mesh fairly well in a low time cost.

By the way, we also notice recent work of Moore and Saigal [13] to deal with sliver shaped elements in 3-dimensional finite element models, which first merges the slivers with neighboring elements to create a polyhedron, and then subdivides the polyhedron into a collection of local tetrahedra by connecting a temporary centroidal node added to all of the external triangular facets of the polyhedron, rather than searching for the best triangulation of the polyhedron without extra node added as we proposed.

4. Examples and Discussions

Several examples of finite element mesh are given to demonstrate the effectiveness of the presented SPR approach. The size of the small polyhedron, *S*, defined by the number of triangles, is set to 25. The finite element meshes in examples are generated by tetrahedral mesh generation package **AutoMesh3D** [14] through the ball-packing method [15, 16]. The presented SPR procedure is already embedded in **AutoMesh3D**. The γ coefficient [2, 17] is adopted as the quality measure for tetrahedral element. Most elements with γ less 0.1 are slivers in following examples. All tests are performed on the following platform: A Pentium IV PC (2.4 GHz CPU and 256 MB RAM) with compiler of Visual C++ 6.0.

The first finite element mesh shown in Fig. 4 consists of 22392 nodes and 113975 tetrahedral elements initially. Its quality is not good. There are 34 elements with the quality value below 0.03, and the lowest value is 0.00118. The statistics of initial quality and quality after the presented SPR approach are listed in Table 1, which shows remarkable improvement of mesh quality by the SPR approach. The minimum value of γ increases to 0.321. The substantial improvement in quality of large number of elements indicates that, as a new local transformation procedure, the SPR approach works effectively on optimizing mesh topology around the worst element, and hence improves the quality of whole mesh. In this example, the SPR operations with total number of 5754 are performed, and the running time (about 260 seconds) is acceptable considering substantial improvement in mesh quality.

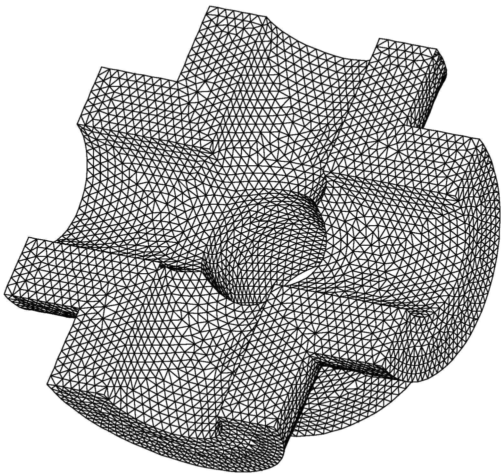

Fig. 4. The first finite element mesh

Table 1. Statistics of the quality distribution of elements in the first mesh

Range of γ	0.00~0.03	0.03~0.12	0.12~0.30	0.30~0.66	> 0.66
Initial mesh (min. γ 0.00118)	34	423	1763	13526	98229
After SPR only (min. γ 0.321)	0	0	0	10495	100975

The second finite element mesh includes 2726 nodes and 8359 tetrahedral elements initially (Fig. 5). Its quality is also not good enough. There are 13 elements with the quality value below 0.03, and the lowest value is 0.0036.

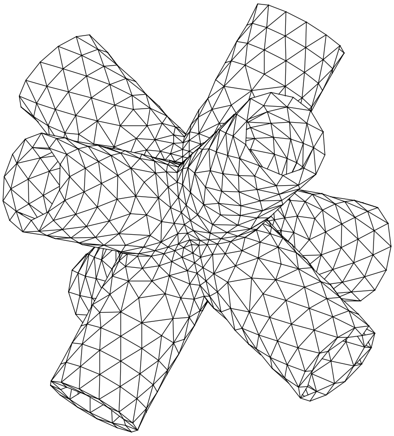

Fig. 5. The second finite element mesh

The elementary local transformations (or ELT for abbreviating) and the presented SPR approach are applied to the initial mesh, respectively. Due to the restriction of the geometry, the small polyhedrons created in the SPR operation are not big enough. Thus the benefit to mesh quality improvement is not evident compared with example 1. Table 2 shows the statistics of initial quality and quality after optimization. Both ELT and SPR procedures improve the mesh quality; however, as expected, the SPR

Table 2. Statistics of the quality distribution of elements in thesecond mesh

Range of γ	0.00~0.03	0.03~0.12	0.12~0.30	0.30~0.66	> 0.66
Initial mesh (min. γ 0.0036)	13	33	188	2500	5625
After ELT only (min. γ 0.181)	0	0	33	2304	5726
After SPR only (min. γ 0.275)	0	0	4	2358	5598

approach gives better result. The minimum value of γ increases from 0.0036 to 0.275 and there are only 4 elements with quality value lower than 0.30. The running time for SPR is about 7.5 seconds. We believe that the superiority in effectiveness makes the SPR approach more useful and become a potential replacement for previous local transformations in mesh topological optimization.

The results of above examples indicate that the proposed SPR procedure is able to significantly improve the quality of tetrahedral mesh. In practice, the topological modification and node reposition should be combined together to get more effective results. In next example, it can be seen that the combination of the proposed SPR procedure and smoothing will achieve substantial improvement in mesh quality.

The third finite element mesh illustrated in Fig. 6 consists of 11007 nodes and 53710 tetrahedral elements, and the minimum value of γ is 0.0110 initially. First, ELT and SPR procedures are applied to the initial mesh, respectively. The results listed in Table 3 indicate that the mesh quality has only limited improvement after ELT or SPR procedure. Almost same results are obtained for the two approaches. The minimum value of γ increases from 0.0110 to 0.0195. It is found that, by monitoring the optimization procedure, the processes for both approaches are quickly blocked by the same worst element, since no further improvement can be made by topological modification alone to the local small polyhedron that includes current worst element.

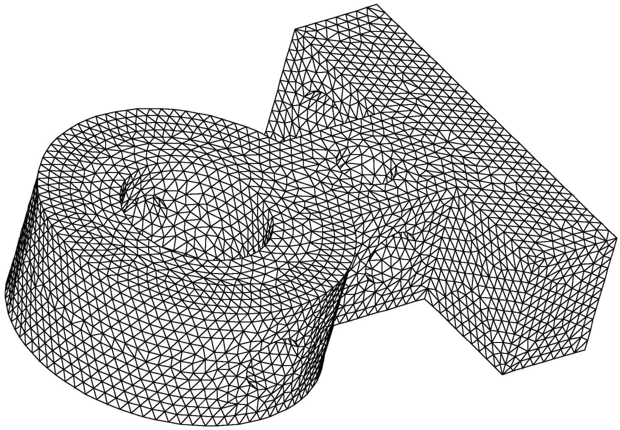

Fig. 6. The third finite element mesh

Table 3. Statistics of the quality distribution of elements in the third mesh

Range of γ	0.00~0.03	0.03~0.12	0.12~0.30	0.30~0.66	> 0.66
Initial mesh (min. γ 0.0110)	26	159	794	7605	45126
After ELT only (min. γ 0.0195)	11	159	794	7601	45130
After SPR only (min. γ 0.0195)	11	159	794	7600	45131
After ELT + smoothing + ELT (min. γ 0.0990)	0	1	494	11211	41988
After SPR + smoothing + SPR (min. γ 0.332)	0	0	0	9948	42598

In order to obtain further improvement in mesh quality, smoothing or node reposition is applied to combine with topological optimization. Here, an efficient smoothing approach based on chaos searching algorithm [5] is adopted. The running time for smoothing procedure is 76 seconds. After smoothing procedure, ELT and SPR procedures are performed respectively again. The direct effect on quality improvement by smoothing is not very distinct; however, the smoothing procedure has optimized node distribution or configuration around the worst element, and such an improvement provides favorable conditions for topological optimization and makes topological optimization work more effectively. It can be seen from Table 3 that both ELT and SPR procedures do actually take effect after the smoothing procedure. Similarly, the SPR procedure gives much better result while the running time of 120 seconds is acceptable. The minimum value of γ increases to 0.332.

Compared with ELT, the presented SPR approach is obviously more suitable for combining with smoothing approach, and combination of SPR and smoothing approach is a better choice for mesh improvement. The time cost of SPR approach is reasonable and worthy to be paid.

It can also be observed in above examples that the number of elements generally decreases by several percentages after topological optimization, since most of the bad elements which usually occupy small volumes are removed.

By the way, same quality measure should be adopted in smoothing and topological transformation procedures. Otherwise the optimization process may probably suffer "zigzag" problem since some quality measures are found to induce inconsistent evaluation for quality change of element in some circumstances [5, 18].

5. Conclusion and Future Work

The small polyhedron reconnection operation is a new and very effective way to improve tetrahedral meshes. Although further speedup is expected for the searching algorithm, examples show that the presented SPR approach can be applied to practical mesh improvement with acceptable payment of time cost and is able to give much better results than the most commonly used local transformations. In addition, the presented SPR approach works well in combining with smoothing approach. We believe that the superiority in effectiveness makes the SPR approach more useful with the further speedup of its efficiency and become a potential replacement for previous local transformations in mesh topological optimization.

The superior performance of the SPR approach makes it worthy of further study. Some works are in progress, including how to construct more appropriate polyhedron, developing more suitable data structure for supporting searching algorithm to eliminate the repeated calculation of the same sub-problem, choosing smartly digging face on the small polyhedron where the new elements are to be created, selecting the optimal digging directions and subdividing the polyhedron into several sub-polyhedrons as earlier as possible, etc. If a good tetrahedralization can be obtained in early stage, it will stop many unnecessary tries and block a lot of sub-problems to be created and treated.

The current work on the SPR approach mainly focuses on isotropic mesh. If changing the distance metric in quality measurement, the SPR approach might be extended to improvement of anisotropic mesh.

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