Sympathetic cooling of trapped ions for quantum logic

D. Kielpinski, B. E. King, C. J. Myatt, C. A. Sackett, Q. A. Turchette, W. M. Itano, C. Monroe, and D. J. Wineland
Time and Frequency Division, National Institute of Standards and Technology, Boulder, Colorado 80303

W. H. Zurek
T-6 (Theoretical Astrophysics), MS B288, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
(Received 27 August 1999; published 16 February 2000)

One limit to the fidelity of quantum logic operations on trapped ions arises from heating of the ions' collective modes of motion. Sympathetic cooling of the ions during the logic operations may eliminate this source of errors. We discuss the benefits and drawbacks of this proposal, and describe possible experimental implementations. We also present an overview of trapped-ion dynamics in this scheme.

PACS number(s): 03.67.−a, 32.80.Pj

I. INTRODUCTION

One of the most attractive physical systems for generating large entangled states and realizing a quantum computer [1] is a collection of cold trapped atomic ions [2]. The ion trap quantum computer stores one or more quantum bits (qubits) in the internal states of each trapped ion, and quantum logic gates (implemented by interactions with externally applied laser beams) can couple qubits through a collective quantized mode of motion of the ion Coulomb crystal. Loss of coherence of the internal states of trapped ions is negligible under proper conditions but heating of the motion of the ion crystal may ultimately limit the fidelity of logic gates of this type. In fact, such heating is currently a limiting factor in the National Institute of Standards and Technology (NIST) ion-trap quantum logic experiments [3,4].

Electric fields from the environment readily couple to the motion of the ions, heating the ion crystal [3–7]. If the ion trap is much larger than the ion crystal size, we expect these electric fields to be nearly uniform across the crystal. Uniform fields will heat only modes that involve center-of-mass motion (COM motion), in which the crystal moves as a rigid body. Motional modes orthogonal to the COM motion, for instance, the collective breathing mode, require field gradients to excite their motion. The heating of these modes is therefore suppressed [4]. However, even if quantum logic operations use such a ''cold'' mode, the heating of the COM motion can still indirectly limit the fidelity of logic operations. Since the laser coupling of an internal qubit and a motional mode depends on the total wave-packet spread of the ion containing the qubit, the thermal COM motion can reduce the logic fidelity [3,4].

In this paper, we examine sympathetic cooling [8] in a particular scheme for which we can continuously laser cool the COM motion while leaving undisturbed the coherences of both the internal qubits and the mode used for quantum logic. In this method, one applies continuous laser cooling to only the center ion of a Coulomb-coupled string of an odd number of ions. One can address the center ion alone if the center ion is of a different ion species than that composing the rest of the string [9]. Alternatively, one can simply focus the cooling beams so that they affect only the center ion. In either case, the cooling affects only the internal states of the center ion, leaving all other internal coherences intact. If the logic operations use a mode in which the center ion remains at rest, the motional coherences in that mode are also unaffected by the cooling. On the other hand, the sympathetic cooling keeps the COM motion cold, reducing the thermal wave packet spread of the ions. In the following, we will discuss the dynamics of an ion string in which all ions are identical except the center ion, assuming heating by a uniform electric field. Our results give guidelines for implementing the sympathetic cooling scheme. Similar results would apply to two- and three-dimensional ion crystals [10–13].

II. AXIAL MODES OF MOTION

We consider a crystal of N ions, all of charge q, in a linear radiofrequency (RF) trap [10,11]. The linear RF trap is essentially an RF quadrupole mass filter with a static confining potential along the filter axis $\hat{z}$. If the radial confinement is sufficiently strong compared to the axial confinement, the ions will line up along the z axis in a string configuration [10,11]. There is no RF electric field along $\hat{z}$, so we can write the axial confining potential as $\phi(z)=\mathsf{q}a_0 z^2/2$ for a_0 a constant. The potential energy of the string is then given by

$$V(z_1,\ldots,z_n)=\frac{1}{2}\mathsf{q}a_0\sum_{i=1}^{N} z_i^2+\frac{\mathsf{q}^2}{8\pi\epsilon_0}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{1}{|z_i-z_j|} \quad (1)$$

for z_i the position of the ith ion in the string (counting from the end of the string). The first term in the potential energy expresses the influence of the static confining potential along the z axis, while the second arises from the mutual Coulomb repulsion of the ions. For a single ion of mass m, the trap frequency along z is just $\omega_z=\sqrt{\mathsf{q}a_0/m}$.

We can compute the equilibrium positions of the ions in the string by minimizing the potential energy of Eq. (1). Defining a length scale ℓ by $\ell^3=\mathsf{q}/(4\pi\epsilon_0 a_0)$ and normalizing the ion positions by $u_i=z_i/\ell$ gives a set of equations for the u_i as

$$u_i-\sum_{j=1}^{i-1}\frac{1}{(u_i-u_j)^2}+\sum_{j=i+1}^{N}\frac{1}{(u_i-u_j)^2}=0,\quad i=1,\ldots,N, \tag{2}$$

which has analytic solutions only up to $N=3$. Steane [1] and James [14] have computed the equilibrium positions of ions in strings with N up to 10. The potential energy is independent of the mass, so the equilibrium positions of ions in a string are independent of the elemental composition of the string if all the ions have the same charge.

In a real ion trap the ions will have some nonzero temperature and will move about their equilibrium positions. If the ions are sufficiently cold, we can write their positions as a function of time as $z_i(t)=\ell u_i+q_i(t)$, where $q_i(t)$ is small enough to allow linearizing all forces. We focus on the case of an odd number of ions N, where all ions have mass m, except for the one at the center of the string which has mass M. The ions are numbered $1,\ldots,N$, with the center ion labeled by $n_c=(N+1)/2$. Following James [14], the Lagrangian for the resulting small oscillations is

$$L=\frac{m}{2}\sum_{\substack{i=1\\ i\neq n_c}}^{N}\dot{q}_i^2+\frac{M}{2}\dot{q}_{n_c}^2-\frac{1}{2}\sum_{i,j=1}^{N}\left.\frac{\partial^2 V}{\partial z_i\partial z_j}\right|_{\{q_i\}=0}q_iq_j \tag{3}$$

$$=\frac{m}{2}\sum_{\substack{i=1\\ i\neq n_c}}^{N}\dot{q}_i^2+\frac{M}{2}\dot{q}_{n_c}^2-\frac{1}{2}\mathrm{q}a_0\sum_{i,j=1}^{N}A_{ij}q_iq_j, \tag{4}$$

where

$$A_{ij}=\begin{cases}1+2\displaystyle\sum_{\substack{k=1\\ k\neq i}}^{N}\frac{1}{|u_i-u_k|^3} & i=j\\ -2\dfrac{1}{|u_i-u_j|^3} & i\neq j.\end{cases} \tag{5}$$

We define a normalized time as $T=\omega_z t$. In treating the case of two ion species, we write $\mu=M/m$ for the mass ratio of the two species and normalize the amplitude of the ion vibrations $q_i(t)$ as $Q_i=q_i\sqrt{\mathrm{q}a_0}$, $i\neq n_c$, $Q_{n_c}=q_{n_c}\sqrt{\mathrm{q}a_0\mu}$. The Lagrangian becomes

$$L=\frac{1}{2}\sum_{i=1}^{N}\left(\frac{dQ_i}{dT}\right)^2-\frac{1}{2}\sum_{i,j=1}^{N}A'_{ij}Q_iQ_j, \tag{6}$$

where

$$A'_{ij}=\begin{cases}A_{ij} & i,j\neq n_c\\ A_{ij}/\sqrt{\mu} & i \text{ or } j=n_c, i\neq j\\ A_{ij}/\mu & i=j=n_c\end{cases} \tag{7}$$

generalizing the result of James [14].

The Lagrangian is now cast in the canonical form for small oscillations in the coordinates $Q_i(t)$. To find the normal modes, we solve the eigenvalue equation

$$\mathbf{A}'\cdot\vec{v}^{(k)}=\zeta_k^2\vec{v}^{(k)}\quad k=1,\ldots,N \tag{8}$$

for the frequencies ζ_k and (orthonormal) eigenvectors $\vec{v}^{(k)}$ of the N normal modes. Because of our normalization of the Lagrangian (6), the ζ_k are normalized to ω_z and the $\vec{v}^{(k)}$ are expressed in terms of the normalized coordinates $Q_i(t)$. In terms of the physical time t, the frequency of the kth mode is $\zeta_k\omega_z$. If the kth mode is excited with an amplitude C, we have

$$q_i(t)=\mathrm{Re}[Cv_i^{(k)}e^{i(\zeta_k\omega_z t+\phi_k)}]\quad i\neq n_c, \tag{9}$$

$$q_{n_c}(t)=\mathrm{Re}\left[C\frac{1}{\sqrt{\mu}}v_{n_c}^{(k)}e^{i(\zeta_k\omega_z t+\phi_k)}\right] \tag{10}$$

in terms of the physical coordinates $q_i(t)$.

We can solve for the normal modes analytically for $N=3$. Exact expressions for the normal-mode frequencies are

$$\zeta_1=\left[\frac{13}{10}+\frac{1}{10\mu}(21-\sqrt{441-34\mu+169\mu^2})\right]^{1/2}, \tag{11}$$

$$\zeta_2=\sqrt{3}, \tag{12}$$

$$\zeta_3=\left[\frac{13}{10}+\frac{1}{10\mu}(21+\sqrt{441-34\mu+169\mu^2})\right]^{1/2}, \tag{13}$$

normalized to ω_z. The mode eigenvectors are

$$\vec{v}^{(1)}=N_1\left(1,\frac{\sqrt{\mu}}{8}(13-5\zeta_1^2),\,1\right), \tag{14}$$

$$\vec{v}^{(2)}=N_2(1,0,-1), \tag{15}$$

$$\vec{v}^{(3)}=N_3\left(1,\frac{\sqrt{\mu}}{8}(13-5\zeta_3^2),\,1\right), \tag{16}$$

in terms of $Q_i(t)$. Here N_1, N_2, and N_3 are normalization factors. In the case of three identical ions ($\mu=1$), we can express the mode eigenvectors in terms of the $Q_i(t)$ as $\vec{v}^{(1)}=(1,1,1)/\sqrt{3}$, $\vec{v}^{(2)}=(1,0,-1)/\sqrt{2}$, and $\vec{v}^{(3)}=(1,-2,1)/\sqrt{6}$. The mode eigenvectors, in this special case, also give the ion oscillation amplitudes in terms of the physical coordinates $q_i(t)$. For three identical ions, then, pure axial COM motion constitutes a normal mode. (This result holds for an arbitrary number of identical ions.) We also note that the center ion does not move in mode #2; hence the frequency and eigenvector of mode #2 are independent of μ. For any odd number N of ions there are $(N-1)/2$ modes for which the center ion does not move. These modes will likewise have frequencies and eigenvectors independent of μ. Moreover, they have $v_{n_c-m}^{(k)}=-v_{n_c+m}^{(k)}$ and so they are orthogonal to the COM motion and do not couple to uniform electric fields. The center ion moves in the other $(N+1)/2$ modes, and unless $\mu=1$, each of these $(N+1)/2$ modes has a component of axial COM motion and therefore couples to uniform electric fields.

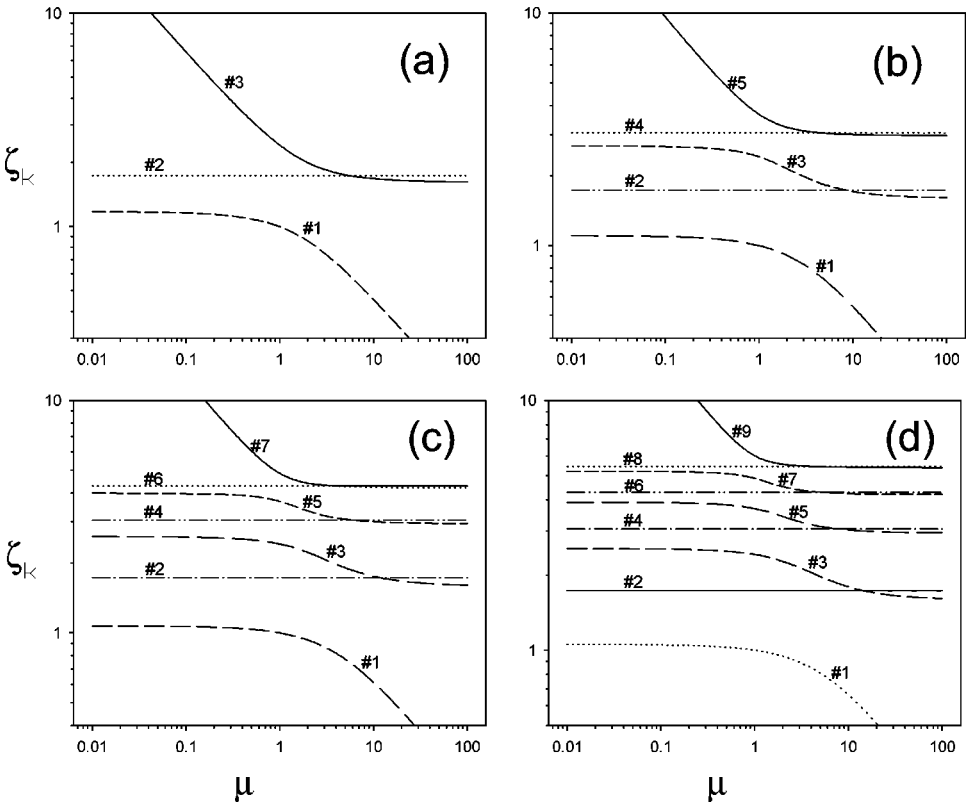

FIG. 1. Normalized axial mode frequencies as a function of μ for (a) 3, (b) 5, (c) 7, and (d) 9 ions.

For $N=5$ and higher, the normal mode frequencies depend on μ in a complicated way. However, it is easy to find the frequencies numerically. Figure 1 shows the mode frequencies for $N=3$, 5, 7, and 9 as a function of μ for $0.01<\mu<100$. The modes are numbered in order of increasing frequency (at $\mu=1$), and are normalized to ω_z. In each case, the lowest-lying mode has all ions moving in the same direction and consists of pure COM motion for $\mu=1$. The even-numbered modes correspond to the $(N-1)/2$ modes for which the center ion does not move. Their frequencies are therefore independent of μ. For both very large and very small μ the modes pair up, as shown in Fig. 1. For each pair there is some value $\mu>1$ for which the modes become degenerate. The relative spacing between modes in a pair is also smaller in the large-μ limit than in the small-μ limit.

If the static confining potential of the trap is not perfectly harmonic, the normal modes of motion will exchange energy with each other. This problem is addressed in Ref. [3] (Sec. 4.1.8). These effects are small if all modes are reasonably cold. Moreover, mode cross-coupling is a resonant process which requires the mode frequencies to be related as sums or differences. Avoiding such resonances by tuning the trap potentials reduces the rate of energy transfer by a large factor.

The Rabi frequency of the laser-ion interaction sets the speed of quantum logic gates and the linewidths of transitions between vibrational states of the ion crystal. If we perform quantum logic on a normal mode whose frequency is too close to that of another ''spectator'' mode, we must reduce our Rabi frequency, and thus our gate speed, to avoid driving transitions on the spectator mode; otherwise we suffer a loss of fidelity. Only the lowest-frequency mode is well separated from all other modes for μ very large. We will show below (Sec. IV) that this mode is undesirable for use in quantum logic. Hence, in order to maximize gate speed, it is best to use a cooling ion that is of the same mass as, or lighter than, the logic ions. In this case mode 2 is well separated from all other modes, as shown in Fig. 1.

III. TRANSVERSE MODES OF MOTION

We now consider the motion of the ions transverse to the z axis. The ions experience an RF potential $\chi\cos(\Omega t)(x^2-y^2)/2$ for a suitable choice of axes x and y perpendicular to z, where Ω is the frequency of the RF field and χ is a constant. The static confining potential can be written $(\mathrm{q}a_0/2)(z^2-\alpha x^2-(1-\alpha)y^2)$ at the position of the ions (with α a constant), so there is also a transverse static electric field. To analyze the ion motion, we work in the pseudopotential approximation [15], in which one time averages the motion over a period of the RF drive to find the ponderomotive force on the ion. If the static potential is negligible, the RF drive gives rise to an effective transverse confining potential of $\frac{1}{2}m\omega_{r0}^2(x^2+y^2)$, where $\omega_{r0}=\mathrm{q}\chi/(\sqrt{2}\Omega m)$ for an ion of mass m. If we include the effects of the static field, the transverse potential becomes $\frac{1}{2}m(\omega_x^2x^2+\omega_y^2y^2)$, where $\omega_x=\omega_{r0}\sqrt{1-\alpha\omega_z^2/\omega_{r0}^2}$, $\omega_y=\omega_{r0}\sqrt{1-(1-\alpha)\omega_z^2/\omega_{r0}^2}$. Below we will assume $\alpha=1/2$, so that $\omega_y=\omega_x$. In any case, the transverse potential is that of a simple harmonic oscillator, as we saw also for the axial potential. However, the transverse potential depends directly on the ion's mass, so the center ion of a string feels a different trap potential than the others for $\mu\neq1$.

We define $\epsilon=\omega_{r0}/\omega_z$, so that $\omega_x=\omega_z\sqrt{\epsilon^2-1/2}$. Then the

normalized Lagrangian for the motion along x is

$$L=\frac{1}{2}\sum_{i=1}^{N}\left(\frac{dX_i}{dT}\right)^2-\frac{1}{2}\sum_{i,j=1}^{N}B'_{ij}X_iX_j, \tag{17}$$

where $X_i=x_i\sqrt{\mathrm{q}a_0}$ for $i\neq n_c$ and $X_{n_c}=x_i\sqrt{\mathrm{q}a_0\mu}$ are normalized ion vibration amplitudes along x. Here

$$B'_{ij}=\begin{cases}B_{ij} & i,j\neq n_c\\ B_{ij}/\sqrt{\mu} & i \text{ or } j=n_c, i\neq j\\ B_{ij}/\mu & i=j=n_c\end{cases} \tag{18}$$

and

$$B_{ij}=\begin{cases}\epsilon^2-\frac{1}{2}-\sum\limits_{\substack{k=1\\k\neq i}}^{N}\frac{1}{|u_i-u_k|^3} & i=j, j\neq n_c\\ \frac{\epsilon^2}{\mu}-\frac{1}{2}-\sum\limits_{\substack{k=1\\k\neq i}}^{N}\frac{1}{|u_i-u_k|^3} & i=j=n_c\\ \frac{1}{|u_i-u_j|^3} & i\neq j.\end{cases} \tag{19}$$

We can describe the normal mode frequencies and oscillation amplitudes in terms of the eigenvectors and eigenvalues of B'_{ij}, just as for the axial case above. The normalizations of the time and position coordinates remain the same as in the axial case.

In the previous section, we assumed that the radial confinement of the ions was strong enough that the configuration of ions in a string along the z axis was always stable. However, for sufficiently small ϵ, the string configuration becomes unstable. The stable configurations for different values of ϵ can be calculated [16,17], and several of these configurations have been observed for small numbers of ions [10,11]. Rather than review the theory of these configurations, we will simply find the range of validity of our small-oscillation Lagrangian for the string configuration. The string will remain stable for all ϵ greater than some $\epsilon_s=\epsilon_s(\mu)$; ϵ_s also varies with N. On the boundary between stable and unstable regions, the frequency of some mode goes to zero. Recalling that the determinant of a matrix is equal to the product of its eigenvalues, we see that $\epsilon_s(\mu)$ is the maximum value of ϵ satisfying $\det B'(\epsilon,\mu)=0$ for μ fixed. Figure 2 shows $\epsilon_s(\mu)$ as a function of μ for 3, 5, 7, and 9 ions. In each case, there is a cusp in $\epsilon_s(\mu)$ corresponding to the crossing of the two largest solutions to $\det B'(\epsilon,\mu)=0$. The position of the cusp varies with the number of ions, but lies between $\mu=0.1$ and $\mu=1$ for $N\leqslant 9$. The positions of the cusps are labeled with arrows in Fig. 2. For μ greater than the value at the cusp, $\epsilon<\epsilon_s(\mu)$ corresponds to instability of the zigzag mode, so that the string breaks into a configuration in which each ion is displaced in the opposite direction to its neighbors [16,17]. For μ smaller than the value at the cusp, ϵ_s is independent of μ. In this regime, $\epsilon<\epsilon_s$ creates an instability in a mode similar to the zigzag mode, except that the center ion remains fixed.

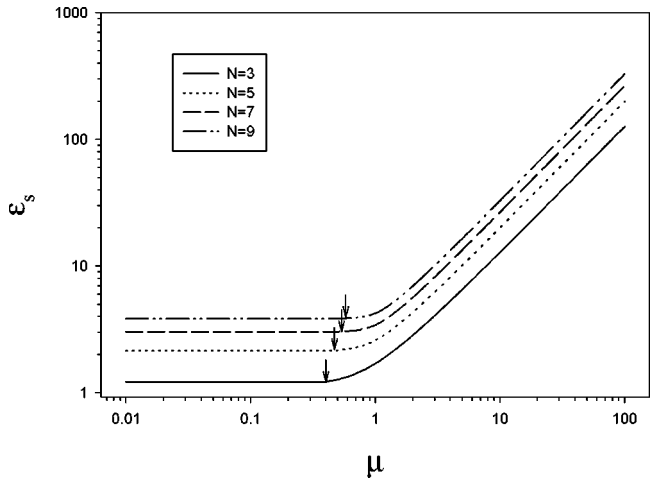

FIG. 2. Trap anisotropy at instability of the string configuration as a function of μ for 3, 5, 7, and 9 ions. Arrows indicate the cusps discussed in the text.

We can proceed to calculate the frequencies of the transverse modes for values $\epsilon>\epsilon_s(\mu)$. Again, these frequencies are normalized to the axial frequency of a single ion of mass m. Figure 3 shows the transverse mode frequencies for 3, 5, 7, and 9 ions as a function of μ, where ϵ is taken equal to $1.1\epsilon_s(\mu)$. The modes are numbered in order of increasing frequency at $\mu=1$ (all ions identical). In this numbering scheme, the central ion moves in odd-numbered modes but not in even-numbered modes. The frequencies of the even-numbered modes appear to depend on μ because they are calculated at a multiple of $\epsilon_s(\mu)$; for constant ϵ these frequencies are independent of μ. The cusps in the mode frequencies in Fig. 3 arise from the cusps of $\epsilon_s(\mu)$ at the crossover points between the two relevant solutions of $\det B'=0$. Mode frequencies plotted for a constant value of ϵ do not exhibit these cusps. As in the case of axial motion, the mode frequencies form pairs of one even- and one odd-numbered mode for small μ. However, for large μ all but one of the transverse modes become degenerate. The only nondegenerate transverse mode in this case is the zigzag mode. In general, the modes are most easily resolved from their neighbors for $\mu=1$, as in the case of axial motion. Increasing ϵ reduces the frequency spacing between nearly degenerate modes. At $\epsilon=1.1\epsilon_s(\mu)$ and $\mu=1$, for instance, the fractional spacing between the cold transverse mode of 3 ions and its nearest neighbor is 0.20, but for $\epsilon=1.5\epsilon_s(\mu)$ the same spacing is 0.09.

The near degeneracy of the modes for large or small μ and for ϵ/ϵ_s significantly greater than 1 limits the usefulness of these modes because of possible mode cross-coupling, just as for the axial modes. Resolving a particular transverse mode requires operating the trap near the point at which the string configuration becomes unstable, i.e., ϵ near $\epsilon_0(\mu)$. In this regime, the collective motion of the ions is quite sensi-

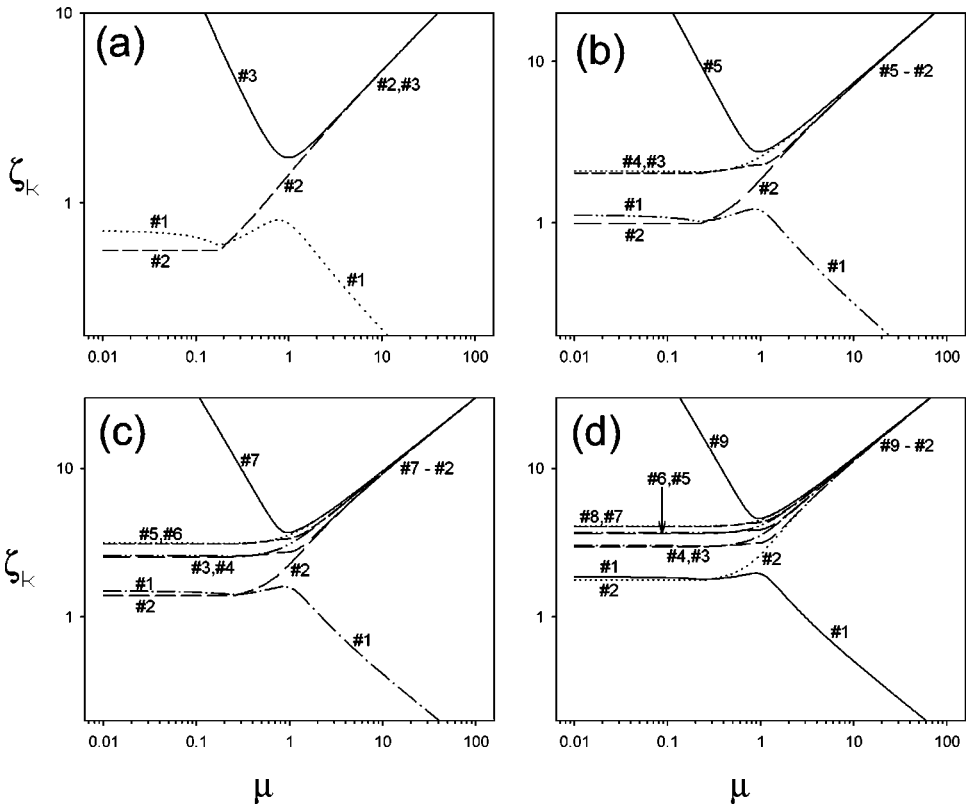

FIG. 3. Normalized frequencies of the transverse modes as a function of μ with $\epsilon = 1.1\epsilon_0(\mu)$ for (a) 3, (b) 5, (c) 7, and (d) 9 ions.

tive to uncontrolled perturbations, which may pose significant technical problems for using a transverse mode in quantum logic operations.

IV. MODE HEATING

Stochastic electric fields present on the ion trap electrodes, for instance, from fluctuating surface potentials, can heat the various normal modes of motion incoherently. For ion trap characteristic dimension d_{trap} much larger than the size of the ion crystal d_{ions}, these fields are approximately uniform across the ion crystal, so they couple only to the COM motion. The $(N-1)/2$ even-numbered modes are orthogonal to the COM motion, so they are only heated by fluctuating electric-field gradients. The heating rates of these modes are reduced by a factor of at least $(d_{ions}/d_{trap})^2 \ll 1$ as compared to the heating of the other modes [4]. In the following, therefore, we will neglect the effects of fluctuating field gradients, so that the even-numbered modes do not heat at all.

The analysis of Secs. II and III shows that the motion of a crystal of N ions is separable into the $3N$ normal modes, each of which is equivalent to a simple harmonic oscillator. Hence we can quantize the crystal motion by quantizing the normal modes. The kth normal mode gives rise to a ladder of energy levels spaced by $\hbar\zeta_k\omega_z$, with $3N$ such ladders in all. If we now write the uniform electric-field power spectral density as $S_E(\omega)$, we can generalize the result of Ref. [18] to give

$$\dot{\bar{n}}_k = \frac{q^2 S_E(\zeta_k\omega_z)}{4m\hbar\zeta_k\omega_z}\left(\frac{v_{n_c}^{(k)}}{\sqrt{\mu}} + \sum_{\substack{j=1\\ j\neq n_c}}^{N} v_j^{(k)}\right)^2 \qquad (20)$$

for the heating rate of the kth mode, expressed in terms of the average number of quanta gained per second. Recall that $v_i^{(k)}$ is the oscillation amplitude of the ith ion in the kth normal mode, expressed in the normalized coordinates. It is useful to normalize the heating rate in Eq. (20) to the heating rate of the lowest-lying axial mode of a string of identical ions. This normal mode consists entirely of COM motion and we write $v_j^{COM} = 1/\sqrt{N}$ for all ions. The normalized heating rate of the kth mode is then

$$\frac{\dot{\bar{n}}_k}{\dot{\bar{n}}_{COM}} = \frac{1}{N\zeta_k}\left(\frac{v_{n_c}^{(k)}}{\sqrt{\mu}} + \sum_{\substack{j=1\\ j\neq n_c}}^{N} v_j^{(k)}\right)^2, \qquad (21)$$

where we have assumed that the spectral density $S_E(\omega)$ is constant over the frequency range of the normal modes, i.e., $S_E(\omega_z) = S_E(\zeta_k\omega_z)$.

Figure 4 shows plots of the normalized heating rates of the axial modes for $N=3$, 5, 7, and 9 as a function of μ. Figure 5 is the same, but for the transverse modes, with $\epsilon = 1.1\epsilon_s$. The numbering of modes on the plots of heating rate matches the numbering on the corresponding plots of mode frequency (Figs 1 and 2).

In both axial-mode and transverse-mode plots, the even-numbered modes have the center ion at rest, while the center ion moves for all odd-numbered modes. We see from Figs. 4 and 5 that the modes for which the center ion is fixed can never heat, while all the other modes always heat to some extent for $\mu \neq 1$. We will refer to these modes as ''cold'' and ''hot'' modes, respectively. If the ions are identical, only the modes with all ions moving with the same amplitude (COM

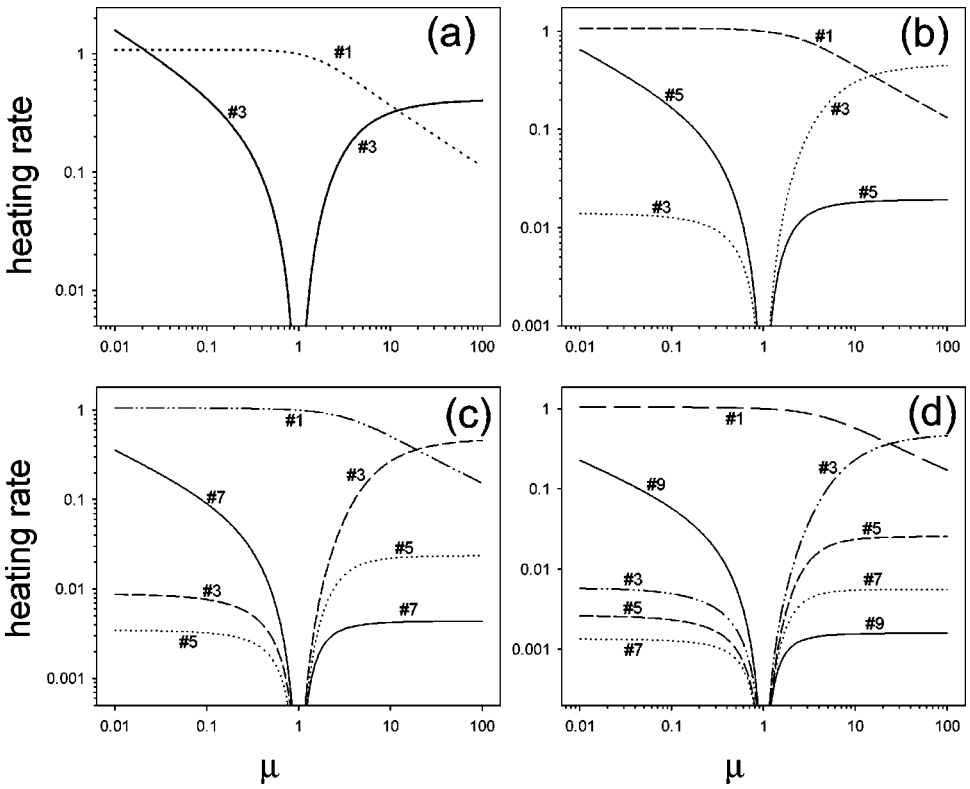

FIG. 4. Normalized heating rates for the axial modes as a function of μ for (a) 3, (b) 5, (c) 7, and (d) 9 ions.

modes) can heat. There are three such modes, one along $\hat{x}$, one along $\hat{y}$, and one along $\hat{z}$. In interpreting Figs. 4 and 5, it is important to recall that the normalized heating rate defined in Eq. (21) is inversely proportional to the mode frequency. For instance, the μ dependence of the heating rate of the highest-frequency transverse mode can be largely ascribed to variations in the mode frequency, rather than to changes in the coupling of the mode to the electric field.

V. PROSPECTS FOR SYMPATHETIC COOLING

Heating reduces logic gate fidelity in two ways. The logic mode itself can be heated, but by choosing a cold mode, we can render this effect negligible. On the other hand, the Rabi frequency of the transition between logic-mode motional states depends on the total wave-packet spread of the ion involved in the transition [3,4]. Heating on modes other than the logic mode can thus lead to unknown, uncontrolled changes in this Rabi frequency, resulting in overdriving or underdriving of the transition. The purpose of sympathetic cooling is to remove this effect by cooling the center ion and thus all hot modes.

In the foregoing, we have chosen to consider only the case of a crystal of an odd number of ions, with the cooling ion at the center. We now see that this is the only case suitable for sympathetic cooling, since only in this case do we find both (a) cold modes for arbitrary μ and (b) isolation of motion of the cold modes from motion of the cooling ion. As long as the crystal is symmetric under inversion in $\vec{z}$, the normal modes must be either symmetric (hot) or antisymmetric (cold) under inversion in $\vec{z}$. If the cooling ion is not at the center of the crystal, the crystal symmetry is broken and cold modes will only appear for particular values of μ. The case of a crystal of an even number of ions, with two cooling ions at the center, will again yield cold modes. However, excitation of any mode will cause the cooling ions to move, so that the cold modes are not well isolated from the sympathetic cooling in this case.

For sympathetic cooling to be useful, we must find a cold mode suitable for use in quantum logic. The cold mode must be spectrally well separated from any other modes in order to maximize gate speed. We can use the lowest-lying cold axial mode as the logic mode for $\mu \lesssim 3$. In this mode, called the breathing mode, the center ion remains fixed and the spacings between ions expand and contract in unison. Unless the trap is operated very close to the instability point of the string configuration, the breathing mode is better separated from its neighbors than are any of the cold transverse modes. For $\mu \gtrsim 3$ any cold mode, either axial or transverse, is nearly degenerate with a hot mode. In this regime one must make a specific calculation of mode frequencies in order to find the best-resolved cold mode. Even so, the cold axial modes are again better separated from their neighbors than are the cold transverse modes, except for ϵ very close to $\epsilon_s(\mu)$. It seems best to select a cold axial mode as the logic mode in most cases.

By selecting our laser-beam geometry appropriately, we can ensure that the Rabi frequency of the motional transition on the axial mode used for logic depends chiefly on the spread of the ion wave packet along *z*. In this case, heating of the axial modes will affect logic-gate fidelity, but heating of the transverse modes will have little effect. If the mass of the central ion is nearly the same as that of the others ($\mu \approx 1$), only the lowest axial mode will heat significantly, and we

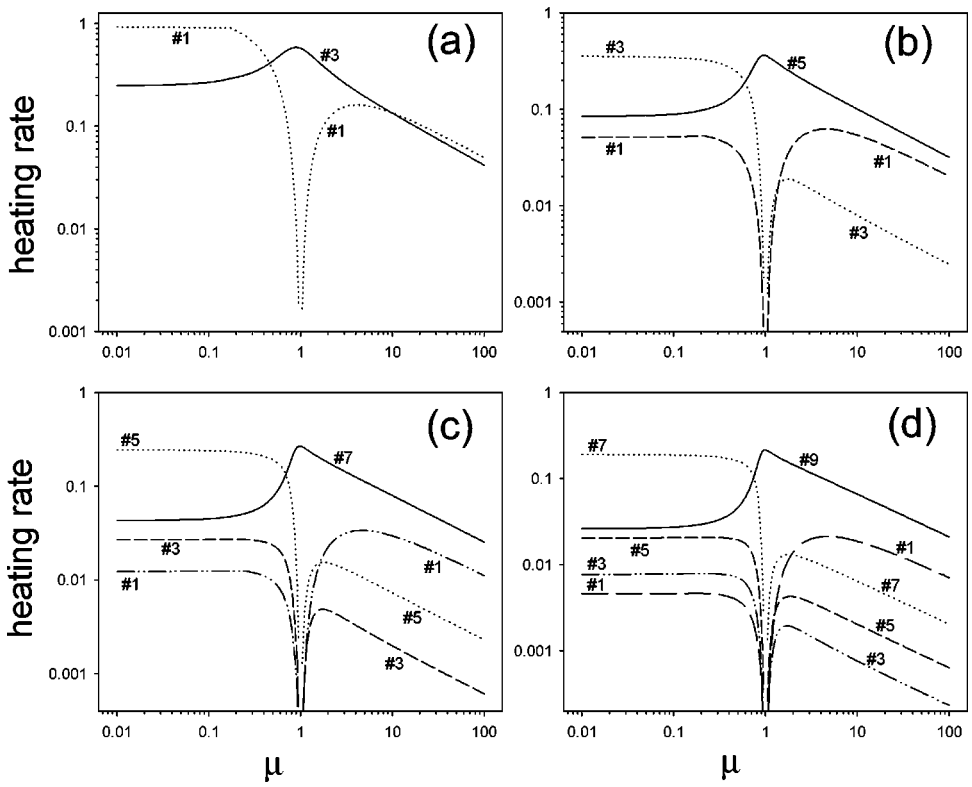

FIG. 5. Normalized heating rates for the transverse modes as a function of μ with $\epsilon = 1.1\epsilon_0(\mu)$ for (a) 3, (b) 5, (c) 7, and (d) 9 ions.

can continuously cool this mode by cooling only the central ion, ensuring that all ions remain in the Lamb-Dicke limit [3]. If μ is not near 1, we must cool all $(N+1)/2$ hot modes (again by addressing the central ion) to keep all ions in the Lamb-Dicke limit.

The analysis above indicates that, all other things being equal, we are best off if our substituted ion is identical to, or is an isotope of, the logic ions. However, sympathetic cooling can still be useful if the two ion species have different masses. For example, we can consider sympathetic cooling using the species $^9Be^+$ and $^{24}Mg^+$. Linear traps constructed at NIST have demonstrated axial secular frequencies of over 10 MHz for single trapped $^9Be^+$ ions. For three ions with $^{24}Mg^+$ as the central ion, $\omega_z(Be^+) = 2\pi \times 10$ MHz yields a spacing of 1.6 MHz between the cold axial breathing mode and its nearest neighbor. If we reverse the roles of the ions $[\omega_z(Mg^+) = 2\pi \times 10$ MHz], the spacing increases to 6.2 MHz. The transverse modes are much harder to resolve from each other. For three ions with $^{24}Mg^+$ in the center, we require $\omega_{r0}(Be^+) = 2\pi \times 27.6$ MHz to obtain $\epsilon = 1.1\epsilon_s$, and the spacing between the cold transverse zigzag mode and its nearest neighbor is only 560 kHz. Reversing the roles of the ions, we find $\epsilon = 1.1\epsilon_s$ at $\omega_{r0}(Mg^+) = 2\pi \times 14.7$ MHz with a spacing of 1.1 MHz. For this combination of ion species, the cold axial breathing mode seems most appropriate for logic. For a string of 3 or 5 ions, sympathetic cooling would require driving transitions on 2 or 3 axial-mode sidebands, respectively. From this example we see that sympathetic cooling can be useful even for ion mass ratios of nearly 3 to 1.

VI. CONCLUSION

We have investigated a particular sympathetic cooling scheme for the case of an ion string confined in a linear RF trap. We have numerically calculated the mode frequencies of the axial and transverse modes as functions of the mass ratio μ and trap anisotropy ϵ for 3, 5, 7, and 9 ions. We have also calculated the heating rates of these modes relative to the heating rate of a single ion, assuming that the heating is driven by a uniform stochastic electric field. The results indicate that the scheme is feasible for many choices of ion species if we use a cold axial mode as the logic mode. The optimal implementation of the scheme employs two ion species of nearly equal mass. However, a demonstration of sympathetic cooling using $^9Be^+$ and $^{24}Mg^+$ appears well within the reach of current experimental technique.

ACKNOWLEDGMENTS

This research was supported by NSA, ONR, and ARO.

[1] A. Steane, Appl. Phys. B: Lasers Opt. **64**, 623 (1997).
[2] J.I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).
[3] D.J. Wineland *et al.*, J. Res. Natl. Inst. Stand. Technol. **103**, 259 (1998).
[4] B.E. King *et al.*, Phys. Rev. Lett. **81**, 1525 (1998).
[5] S.K. Lamoreaux, Phys. Rev. A **56**, 4970 (1997).
[6] D.F.V. James, Phys. Rev. Lett. **81**, 317 (1998).
[7] C. Henkel and M. Wilkins, e-print quant-ph/9902009.
[8] D.J. Larson *et al.*, Phys. Rev. Lett. **57**, 70 (1986).
[9] We assume that the cooling transitions of the center ion are

well resolved from all relevant transitions of the logic ions.
[10] M.G. Raizen *et al.*, Phys. Rev. A **45**, 6493 (1992).
[11] H. Walther, Adv. At., Mol., Opt. Phys. **31**, 137 (1993).
[12] M. Drewsen *et al.*, Phys. Rev. Lett. **81**, 2878 (1998).
[13] T.B. Mitchell *et al.*, Science **282**, 1290 (1998).
[14] D.F.V. James, Appl. Phys. B: Lasers Opt. **66**, 181 (1998).
[15] P. K. Ghosh, *Ion Traps* (Oxford, Clarendon, 1995).
[16] R. Rafac *et al.*, Proc. Natl. Acad. Sci. USA **88**, 483 (1991).
[17] J.P. Schiffer, Phys. Rev. Lett. **70**, 818 (1993).
[18] D.J. Wineland *et al.*, Fortschr. Phys. **46**, 363 (1998).