Least-Error Acoustic-Source Localization

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19 September 2003

Abstract

We describe a new method for using time-of-arrival data from an arbitrary number of sensors to localize a discrete acoustic source—generating a radially symmetric wavefront. This method predicts the spatial and temporal coordinates of the source by minimizing the sum of the absolute values of the differences between the squares of the theoretical and actual times of arrival. Whenever the errors in the data are unbiased and random, the larger the number of sensors, the greater the expected accuracy of localization. We investigate the properties of our method through its application. First, we demonstrate the improvement of accuracy with the number of sensors. Then, for four sensors, we make direct comparison with Time-Difference-Of-Arrival (TDOA) localizations. We also demonstrate the use of our method for localization of multiple, cotemporary sources. This method is eminently suited to implementation on sensor networks with computationally capable nodes.

Keywords: distributed sensor networks, Lagrange multipliers, multiple sources, optimization.

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1 Introduction

Let there be S nodes at positions $\mathbf{v}_s$; $s = 1, 2, \ldots, S$. Each node is equipped with an auditory sensor. Let there be a discrete sound source at position $\mathbf{v}$ and time t. In the ideal, if the source emits at time t, then the time of arrival at node s, $t_s = t + ||\mathbf{v}_s - \mathbf{v}||/c$; $s = 1, 2, \ldots, S$; where c denotes the velocity of sound, taken to be constant; and where $||\mathbf{w}||$ denotes the Euclidean distance. The goal is to predict $\mathbf{v}$ and t based on the t_s's. Thus, we do not entertain more sophisticated sensors, such as those capable of determining the direction to the source [7]. As c is constant, we will subsequently eliminate t's in favor of $d \stackrel{\text{def}}{=} ct$ and $d_s \stackrel{\text{def}}{=} ct_s$; $s = 1, 2, \ldots, S$.

Departures from ideality—e.g. measurement errors and dispersion of velocity—occasion discrepancies between $d_s - d$ and $||\mathbf{v}_s - \mathbf{v}||$; $s = 1, 2, \ldots, S$. We assume that corrections have been applied toward systematic errors, and, therefore, that the average of $d_s - d - ||\mathbf{v}_s - \mathbf{v}||$ equals zero (for suitable replications).

Statistics specifies judicious uses of redundant data for mitigating noisiness, and classical "least-error" methodologies owe their origins to desiderata such as acoustic-source localization. Time-Difference-Of-Arrival (TDOA) approaches may accommodate redundant nodes [4, 11], but these are unlikely to optimize localization. Bayesian methods could also be employed for acoustic-source localization, but these entail substantial computation for their automated implementation, and it could be advantageous to explore alternative approaches.

There is, in fact, a natural application of least-square methods to our desideratum [5]. This application, however, "centers" the analysis on a selected node by differencing coordinates (and data) with respect to the node's, which is untoward, and symmetrized variants gave us pause. Therefore, we developed a novel criterion and a method for its optimization.

Minimizing the sum of the absolute values of the "errors" was advocated by Laplace [2, p. ix]. Here, we take such to be the differences between the theoretical and actual times of arrival. Minimizing the sum of these summands, instead of the sum of the squares of the "errors", has the advantage of better discounting of highly aberrant measurements. For select desiderata, such as ours, it can be simplest, mathematically, to optimize the sum of the absolute values of the "errors"—despite having to abide discontinuous derivatives [10]. This criterion will be seen to yield counterintuitive optimization characteristics—in the to-ing and fro-ing betwixt the data.

When it suffices to seek a solution in two spatial coordinates, our optimizations employ only elementary algebraic methods, reminiscent of those used in TDOA calculations. The increased complexity required for solving problems in three spatial coordinates is noted in the Appendix.

We apply our least-error methodology in Section 3. Here it is demonstrated, by simulation, how prediction accuracy can improve by increasing the number of nodes. We make direct comparison with the TDOA method [8]: using four nodes, both with simulated data and with gunshot data from a live-fire range. We also demonstrate the use of our method for localization of multiple, cotemporary sources.

Our revisitation of this oft studied objective is motivated by the emergence of distributed-sensor-network (DSN) based technologies. The nodes of such networks are endowed with computational capabilities, and they may employ auditory as well as other types of sensors. DSN's could advantageously be used, among other things, for discovering and monitoring mobile agents or weaponry. Our least-error approaches for source localization could be implemented in a straightforward fashion over the nodes of DSN's. The details of such implementations, however, fall outside the scope of this manuscript.

2 Source Localization in Two Spatial Dimensions

In this section we develop methods for source localization in a plane. Let there be given S nodes with their inferred positions denoted by $\mathbf{v}_s = (x_s, y_s)$; $s = 1, 2, \ldots, S$. The model for the source is a cylindrical wave—emanating from $\mathbf{v} = (x, y)$ "at d". In other words, for

$d' > d$, the locus $\mathbf{v}'(d')$ of the wave is given by $d' - d = ||\mathbf{v}'(d') - \mathbf{v}||$. The data recorded at the nodes are the respective d_s's; $s = 1, 2, \ldots, S$, and, from these, one is to obtain $\mathbf{v}$ and d. This problem will be seen to be well posed for $4 \leq S$.

Our measure of the discordance of the data is

$$\Lambda(x, y; d) \stackrel{\text{def}}{=} \sum_{s=1}^{S} w_s \left|(x - x_s)^2 + (y - y_s)^2 - (d - d_s)^2\right|, \tag{1}$$

where the w_s's, introduced for the sake of generality, are nonzero, real constants. It could be advantageous, by analogy to the χ^2 statistic, for these to equal the reciprocal of the mean value of the respective summand: $|(x - x_s)^2 + (y - y_s)^2 - (d - d_s)^2|$ (under an appropriate error model).

It is necessary for the first partial derivatives of Λ to vanish at its global optimum, provided that they are defined there. Recall that the derivative of $|f(x)|$ equals $f'(x)\mathrm{sgn}(f(x))$; $f(x) \neq 0$, with $\mathrm{sgn}(y)$ denoting the signum function (taking values $-1, 0$ and 1 depending on whether $y \in \mathbb{R}$ is less than, equal to or greater than zero, respectively). Therefore, at a stationary point of (1),

$$\begin{aligned}
\frac{\partial \Lambda}{\partial x} &= 2 \sum_{s=1}^{S} w_s (x - x_s) \mathrm{sgn}_s = 0; \\
\frac{\partial \Lambda}{\partial y} &= 2 \sum_{s=1}^{S} w_s (y - y_s) \mathrm{sgn}_s = 0; \\
\frac{\partial \Lambda}{\partial d} &= -2 \sum_{s=1}^{S} w_s (d - d_s) \mathrm{sgn}_s = 0.
\end{aligned}$$

Herein, sgn_s denotes $\mathrm{sgn}((x - x_s)^2 + (y - y_s)^2 - (d - d_s)^2)$; $s = 1, 2, \ldots, S$, and the partial derivatives of Λ are defined at points where no sgn_s vanishes. When this restriction is satisfied, consideration of the second partial derivatives establishes that the stationary point obtained is always a saddle point: never a minimum. Therefore, at the global minimum of Λ, one or more sgn_s must vanish identically and the respective summands of Λ may be omitted. It is easy to establish that at least two sgn_s's must equal zero at a minimum of Λ. Cases with two vanishing sgn_s's and three vanishing sgn_s's are treated separately.

2.1 Two sgn_s's zeroed

One must investigate the stationary points of Λ resulting when every pair of sgn_s's is zeroed—as the global minimum must obtain either at one of these or one of the stationary points described in Section 2.2. Each pair yields constraints upon Λ. Thus, the respective stationary points may be found using two Lagrange multipliers: λ and μ.

Thus, consider the stationary points of

$$\tilde{\Lambda} = \Lambda + \lambda((x - x_\ell)^2 + (y - y_\ell)^2 - (d - d_\ell)^2) + \mu((x - x_m)^2 + (y - y_m)^2 - (d - d_m)^2). \quad (2)$$

Here, ℓ is not equal to m, and, to exhaust this class of stationary points, $\{\ell, m\}$ must range over all unordered pairs of indices from $\{1, 2, \ldots, S\}$. Thus, given none of the remaining sgn_s's vanish,

$$\frac{\partial\tilde{\Lambda}}{\partial x} = 2\sum_{s=1}^{S}(x - x_s)w_s\mathrm{sgn}_s + 2\lambda(x - x_\ell) + 2\mu(x - x_m) = 0;$$

$$\frac{\partial\tilde{\Lambda}}{\partial y} = 2\sum_{s=1}^{S}(y - y_s)w_s\mathrm{sgn}_s + 2\lambda(y - y_\ell) + 2\mu(y - y_m) = 0;$$

$$\frac{\partial\tilde{\Lambda}}{\partial d} = -2\sum_{s=1}^{S}(d - d_s)w_s\mathrm{sgn}_s - 2\lambda(d - d_\ell) - 2\mu(d - d_m) = 0.$$

This system comprises one linear equation in each unknown: x, y and d. λ and μ will subsequently be selected to effect the two constraints $\mathrm{sgn}_\ell = \mathrm{sgn}_m = 0$. Denote $\sum_{s=1}^{S} w_s\mathrm{sgn}_s$ by ξ, $\sum_{s=1}^{S} x_s w_s\mathrm{sgn}_s$ by ξ_x, $\sum_{s=1}^{S} y_s w_s\mathrm{sgn}_s$ by ξ_y and $\sum_{s=1}^{S} d_s w_s\mathrm{sgn}_s$ by ξ_d. Then the foregoing three equations yield, respectively,

$$x = \frac{\xi_x + \lambda x_\ell + \mu x_m}{\xi + \lambda + \mu},\ y = \frac{\xi_y + \lambda y_\ell + \mu y_m}{\xi + \lambda + \mu} \text{ and } d = \frac{\xi_d + \lambda d_\ell + \mu d_m}{\xi + \lambda + \mu}.$$

Furthermore, the two constraints yield two quadratic equations: one in λ and the other in μ :

$$(\xi_x - \xi x_m + \lambda(x_\ell - x_m))^2 + (\xi_y - \xi y_m + \lambda(y_\ell - y_m))^2 = (\xi_d - \xi d_m + \lambda(d_\ell - d_m))^2; \qquad (3)$$

$$(\xi_x - \xi x_\ell + \mu(x_m - x_\ell))^2 + (\xi_y - \xi y_\ell + \mu(y_m - y_\ell))^2 = (\xi_d - \xi d_\ell + \mu(d_m - d_\ell))^2. \qquad (4)$$

Note that if ℓ were interchanged with m and λ were interchanged with μ, then this pair of equations persists, corroboration for the indexing of these stationary points by the unordered pairs $\{\ell, m\}$. Using the quadratic formula, it is easily seen that the roots of (3) and (4) share some key attributes:

$$\lambda_\pm = (-b_\lambda \pm \sqrt{D})/2a, \text{ and}$$

$$\mu_\pm = (-b_\mu \pm \sqrt{D})/2a,$$

where D denotes the discriminant[1], common to the two quadratic equations; where a is the coefficient of the quadratic term, also common to the two equations; and where b_λ and b_μ denote the respective coefficients of the linear terms. Note that $a = (x_\ell - x_m)^2 + (y_\ell - y_m)^2 - (d_\ell - d_m)^2$; were a to vanish, then there would be only one acceptable λ and μ. If $D < 0$, then the Lagrange multipliers would be complex and the corresponding points would not be stationary points [9, Theorem 2.6].

[1] $D/4 = ((x_\ell - x_m)(\xi_d - \xi d_m) - (d_\ell - d_m)(\xi_x - \xi x_m))^2 + ((y_\ell - y_m)(\xi_d - \xi d_m) - (d_\ell - d_m)(\xi_y - \xi y_m))^2 - ((x_\ell - x_m)(\xi_y - \xi y_m) - (y_\ell - y_m)(\xi_x - \xi x_m))^2$.

For a stationary point (x, y, d) to be admissible, its indeterminates must be finite. When the denominator $\xi + \lambda + \mu$ vanishes, it follows that not all of the indeterminates may be finite. Consider, for example, the respective linear system:

$$\begin{array}{ccccccc} \xi_x & + & \lambda x_\ell & + & \mu x_m & = & 0 \\ \xi_y & + & \lambda y_\ell & + & \mu y_m & = & 0 \\ \xi_d & + & \lambda d_\ell & + & \mu d_m & = & 0 \end{array} \tag{5}$$

According to the fundamental theorem on overdetermined linear systems [3, Vol. 7, p. 52], this system will admit a solution (λ, μ) if and only if

$$\text{Rank}\begin{pmatrix} x_\ell & x_m \\ y_\ell & y_m \\ d_\ell & d_m \end{pmatrix} = \text{Rank}\begin{pmatrix} x_\ell & x_m & \xi_x \\ y_\ell & y_m & \xi_y \\ d_\ell & d_m & \xi_d \end{pmatrix}.$$

Given the random errors under consideration, the rank of the left-hand matrix will almost always be smaller than the rank of the right-hand matrix. Assuming these ranks are unequal, the system (5) has no solution. Therefore, when the denominator $\xi + \lambda + \mu$ vanishes, at least one of x, y and d will almost always be infinite, and such solutions may be ignored because they cannot yield a minimum value of Λ.

If additional sgn_s's were to vanish at one of the foregoing stationary points, then such points would be inadmissible (and should be discarded). Appropriate treatment for the case of three vanishing sgn_s's is described next.

2.2 Three sgn_s's zeroed (2-d)

Note that, with noisy data, it should almost never be the case that more than three sgn_s's jointly vanish, and it is safe to ignore such eventualities. Here, the use of Lagrange multipliers leads to a system whose stationary points are not easily found, as will be seen in the Appendix. Thus, direct elimination is preferable. One may directly solve the quadratic system:

$$(x - x_\ell)^2 + (y - y_\ell)^2 = (d - d_\ell)^2; \tag{6}$$

$$(x - x_m)^2 + (y - y_m)^2 = (d - d_m)^2; \tag{7}$$

$$(x - x_n)^2 + (y - y_n)^2 = (d - d_n)^2, \tag{8}$$

where, ℓ, m and n are distinct elements of $\{1, 2, \ldots, S\}$. Taking the differences (6) – (7) and (6) – (8) yields two linear equations in x, y and d. This allows, say, y and d to be expressed in the following forms:

$$y = \alpha x + \beta; \tag{9}$$

$$d = \gamma x + \delta, \tag{10}$$

where, from application of Cramer's rule (or otherwise),

$$\alpha = \frac{\frac{x_\ell - x_m}{d_\ell - d_m} - \frac{x_\ell - x_n}{d_\ell - d_n}}{\frac{y_\ell - y_n}{d_\ell - d_n} - \frac{y_\ell - y_m}{d_\ell - d_m}},$$

$$\beta = \frac{\psi_m - \psi_n}{\frac{y_\ell - y_n}{d_\ell - d_n} - \frac{y_\ell - y_m}{d_\ell - d_m}},$$

$$\gamma = \frac{x_\ell - x_m}{d_\ell - d_m} + \alpha \frac{y_\ell - y_m}{d_\ell - d_m},$$

$$\delta = \beta \frac{y_\ell - y_m}{d_\ell - d_m} + \psi_m,$$

with

$$\psi_j = \frac{x_j^2 - x_\ell^2 + y_j^2 - y_\ell^2 + d_\ell^2 - d_j^2}{2(d_\ell - d_j)};\ j \in \{m, n\}.$$

One may substitute the limiting behavior for vanishing denominators. (To write a computer program which yields accurate solutions for all values of the parameters is feasible, but it would require some care or the capabilities of symbolic programming). Substituting (9) and (10) into (6) yields a quadratic equation in x :

$$ax^2 + bx + c = 0, \tag{11}$$

with

$$a = 1 + \alpha^2 - \gamma^2,$$

$$b = -2(x_\ell - \alpha(\beta - y_\ell) + \gamma(\delta - d_\ell)) \text{ and}$$

$$c = x_\ell^2 + (\beta - y_\ell)^2 - (\delta - d_\ell)^2.$$

When there are two real roots of (11) and when $S = 3$, these solutions may not, in general, be distinguished, even though only one solution should approximate the source. Therefore, source localization requires either a different approach or more than three nodes. This could inspire higher regard for the nervous systems of many creatures, such as bats, endowed with only two auditory sensors—but adept at echo-location.

2.3 Global minimization of Λ

Global minimization of Λ involves testing all pairs and triples of nodes—by zeroing their sgn_s's. Respective stationary points are generated, and the minimum value of Λ over these stationary points is its global minimum.

With two zeroed sgn_s's one must range over the $\binom{S}{2}$ pairs of indices. For each pair, one must consider the choices for the nonzero sgn_s's: $\mathcal{O}(2^S)$. For each pair and each choice, the candidate stationary solutions are generated. These may be admitted if (i) their λ and $\mu \in \mathbb{R}$

and (ii) their sgn_s's reproduce the chosen sgn_s's. Thus, the computation of the stationary points for each pair of vanishing sgn_s's effectively involves the entire set of nodes.

With three zeroed sgn_s's, the solution procedure is more straightforward. One must range over the $\binom{S}{3}$ triples of indices: $\mathcal{O}(S^3)$. For each triple, one determines the roots of (11). Then, using (9) and (10), one may evaluate (1), including additional nodes as desired.

3 Applications of Least-Error Methodology in Two Dimensions

3.1 Reduction of the Error with Increasing Numbers of nodes

Consider a square of unit side with vertices (0,0), (0,1), (1,0) and (1,1) and a cylindrical-wave source at its center, (1/2,1/2), emitting at $d = 0$. S nodes are independently placed, uniformly and randomly, in the square. A random error, uniform on [-.01,.01], is then added to the exact value of d_s, independently; $s = 1, 2, \ldots, S$. Then, (1) is minimized, with all w_s's$= 1$. Table 1 contains the average root-mean-square error $\left\langle \sqrt{(x-1/2)^2 + (y-1/2)^2 + d^2} \right\rangle$ for $4 \leq S \leq 16$. Note that the average root-mean-square error is (empirically) roughly proportional to $S^{-1/2}$ asymptotically. For the aforementioned reasons, we did not implement sophisticated TDOA procedures designed to use redundant measurements to reduce measurement errors [4, 11].

TABLE 1 HERE

3.2 Comparisons with TDOA

The Time-Difference-Of-Arrival (TDOA) method [1, 4] may be used with two dimensional data from four nodes ($S = 4$). We implemented the approach of [8, pp. 4-5]. (In summary, by effecting the differences with respect to the equation for one node, the four equations yield three. Two linear equations only involving the two spatial variables may be derived from these. The value of d follows by averaging over the four equations). Note that this approach always yields a unique solution, but that it is not designed for data which contains various errors.

We next compare the accuracy of TDOA solutions versus those given by the least-error methodology, applied to one data set simulated as described above. Table 2 contains some error "percentiles" for the two computations. For example, for the TDOA calculation, 75% of the time, the root-mean-square error, $\left\langle \sqrt{||\mathbf{v} - \mathbf{v}_0||^2 + (d - d_0)^2} \right\rangle$, was less than 0.035 whereas the corresponding value for the least-error method was 0.0175. (The nought-subscripted entities denote the "true values" and the non-subscripted entities the predicted values). Due to a "heavy tail" in the TDOA results, attributable to occultation, the respective mean error could not be reliably estimated, even in a large number of trials (10^6). Table 2 illustrates that TDOA's typical performance is also inferior to the least-error methodology.

TABLE 2 HERE

We also implemented and tested a prototype TDOA acoustic-source location network. For this test, six acoustic-sensor nodes were arranged in two groups of three. The system was fielded at the Los Alamos National Laboratory live-fire range.

Each node contained an acoustic sensor, a microprocessor, a (inexpensive) GPS receiver, and an RF transceiver for communication. Together, they formed a sensor network capable of autonomous gunshot location. The on-board GPS receivers provided both a common timebase for the network and positions for the individual nodes. The dominant error comprised by this system derived from inaccuracies in these positions, whose RMS error was roughly 10 m and which was strongly correlated across nodes.

Each node performed a threshold detection of sound impulses, recording the arrival time of the sound. The arrival times were then propagated around the network using a flooding-style communications protocol. The first node to acquire four times of arrival (typically its own measurement and three others) calculated the sound source position, using the aforementioned TDOA algorithm.

The communications traffic from the network (time-of-arrival measurements and calculated sound locations) was captured by a laptop computer, outfitted with a receiver, and stored in a log file. Because six measurement nodes were in use, many times it was not clear which group of four underlay a particular localization. Furthermore, the communications algorithm incorporated in the TDOA network would typically suppress the transmission of the fourth time-of-arrival measurement: the one originating at the node that performed the TDOA calculation. Thus, to compare the TDOA method to our least-error method, we first identified TDOA localizations which could unambiguously be matched to a set of three measurements; then, the fourth measurement was reconstructed. The velocity of sound was taken to equal 343 m/s.

These sets of four unambiguous measurements were then used as input for the least-error method. Figure 1a shows the results of the autonomous TDOA calculation.

FIGURE IA HERE

Two shots were located correctly by the TDOA system, but this approach predicted that the majority of the shots occurred 10 m to 30 m South of their true positions. (The full data set from which this subset was culled exhibits somewhat better performance. The selection of substandard results was not deliberate but may reflect a bias in the selection of unambiguous events.) Errors entering into the TDOA calculation may be due to a combination of GPS position error, individual acoustic threshold settings, and dispersion of velocity.

In Figure 1b, we present the predictions made by the least-error method on the same data and plotted in an identical fashion.

FIGURE 1B HERE

Almost all (eight of nine) of the gunshots are located to within 3 m by our least-error method.

3.3 Multiple Sources

Recall that with sinusoidally varying, plane-wave sources, various optimization methods yield the parameter estimates for all sources based upon measurements at the nodes [6]. With cylindrical-wave "point sources", i.e. emitting at a single point in space and time, the challenges are greater, and novel methods are required for parameter estimation.

The least-error approach engenders an elementary means of discerning multiple, cotemporary "point sources", as follows. Were several signals received at the nodes, then one could perform the least-error analysis for each combination thereof, using one signal from each node. Thus, if there were n_i signals detected at the ith node, then the number of entailed least-error calculations would be $\prod_i n_i$. The minimum values of Λ obtained provide discrimination between true and artifactual sources, as it is reasonable to expect that fictitious combinations will accrue large discordances.

To illustrate these considerations, we modified the simulations of Section 3.1 to include three sources: at $d_0 = 0$ and at $(1/4, 1/4)$, $(1/2, 1/2)$ and $(3/4, 3/4)$. Noise was added as above. Only those *below-threshold* combinations of signals, whose minimum $\Lambda < 0.03$, were retained. (In other settings, different thresholds would be appropriate; this choice pertains to the noise included in our simulations). The results of our simulations are as follows.

TABLE 3 HERE

For virtually all configurations of the nodes, all three sources are detected (and are detected from multiple signal combinations). Note that having sources emit at substantially different d's would decrease the false positive rates. In Figure 2, the spatial coordinates of the below-threshold results for $S = 8$ are depicted.

FIGURE 2 HERE

In practice, "time-slicing", with overlapping time intervals, could be used to reduce the values of n_i. The length of the interval would establish "neighborhoods" for nodes collaborating in source localization.

4 Discussion

The examples of Section 3 illustrate the advantageousness of our least-error method over the simplest, established methods.

A novel aspect of our method is that some of the sgn_s's must vanish at the minimum. When fewer than the maximum number of sgn_s's vanish (three and four in two and three dimensions, respectively), our method has complexity exponential in the number of nodes whose data are to enter. On the other hand, for the maximum numbers of vanishing sgn_s's, ours is evidently a polynomial algorithm which could readily be distributed over the nodes of a DSN with computational capabilities. For some applications and for (yet to be determined) classes of noise, simply ranging over the latter stationary points and accepting the minimum value of Λ may yield a good or an acceptable approximation to its global minimum. It may someday prove valuable to develop universal approximations for our solutions, to render our methods applicable to arbitrarily large numbers of nodes.

One might expect that an important component of the noise in the d_s's will increase with the Euclidean distance from the source due to dispersion. Using the w_s's to mitigate these seems desirable, but it would greatly complicate the optimization of Λ because it would make the w_s's functions of the variables. For some purposes it may suffice to iterate: solving a sequence of optimizations—each having constant w_s's—based on the inferred $\mathbf{v}$ and d from the previous iteration. Convergence is not necessarily at issue, as even one iteration could afford greater accuracy than the setting of all w_s's equal to unity.

More sophisticated methods are plainly required for localization of a large number of cotemporary sources. One might, for instance, effect the desired importance sampling of the signal combinations by means of the Markov chain Monte Carlo method: transitioning between combinations in accordance with the likelihood ratio for the data. A useful approximation yielding the likelihood of the data might involve the adoption of the inferred least-error source coordinates.

It remains to be specified how, in real time, a suitable collection of sensing nodes is to be "activated" and is to identify the optimal triple or quadruple within it. The optimal value of Λ could indicate the accuracy of a solution, and it could be advantageous for the network to first report an inaccurate solution and to proceed to refine it further.

Acknowledgements

This work was performed with NNSA and Los Alamos National Laboratory Discretionary Research support for Modelling and Testing of Distributed Sensor Networks with Collective Computation, Jared Dreicer, P.I., under the aegis of the University of California, under contract W-7405-ENG-36. We are indebted to Angela Mielke for reading a draft of this manuscript.

5 Appendix: Source Localization in Three Spatial Dimensions

This desideratum is very similar to that treated above, but here one seeks $\mathbf{v} = (x, y, z)$ and d, and one has a spherical-wave source. The global optimum of the respective Λ may occur with two, three or four sgn_s's zeroed. For two sgn_s's zeroed, an analysis analogous to that of Sections 2.1 and 2.3 suffices. For four sgn_s's zeroed, by taking differences, one may eliminate three variables, generating three linear equations. A quadratic equation in the remaining variable yields two candidate solutions, as described in Section 2.2. As above, the cases with more than four sgn_s's zeroed may safely be neglected. Thus, only the case of zeroing three sgn_s's remains, in order to effect the global minimization of Λ.

5.1 Three sgn_s's zeroed (3-d)

Here, using Lagrange multipliers λ, μ and ν, we seek the stationary points of

$$\tilde{\Lambda} = \Lambda + \lambda((x-x_\ell)^2+(y-y_\ell)^2+(z-z_\ell)^2-(d-d_\ell)^2)+\mu((x-x_m)^2+(y-y_m)^2+(z-z_m)^2-(d-d_m)^2)$$

$$+\nu((x-x_n)^2+(y-y_n)^2+(z-z_n)^2-(d-d_n)^2). \tag{12}$$

The indices ℓ, m and n are distinct, and, to exhaust this class of stationary points, $\{\ell, m, n\}$ must range over all unordered triples of indices from $\{1, 2, \ldots, S\}$. Thus, given none of the remaining sgn_s's vanish, we seek the stationary points as the solutions of the system

$$\frac{\partial \tilde{\Lambda}}{\partial x} = 2\sum_{s=1}^{S}(x-x_s)w_s\mathrm{sgn}_s + 2\lambda(x-x_\ell) + 2\mu(x-x_m) + 2\nu(x-x_n) = 0;$$

$$\frac{\partial \tilde{\Lambda}}{\partial y} = 2\sum_{s=1}^{S}(y-y_s)w_s\mathrm{sgn}_s + 2\lambda(y-y_\ell) + 2\mu(y-y_m) + 2\nu(y-y_n) = 0;$$

$$\frac{\partial \tilde{\Lambda}}{\partial z} = 2\sum_{s=1}^{S}(z-z_s)w_s\mathrm{sgn}_s + 2\lambda(z-z_\ell) + 2\mu(z-z_m) + 2\nu(z-z_n) = 0;$$

$$\frac{\partial \tilde{\Lambda}}{\partial d} = -2\sum_{s=1}^{S}(d-d_s)w_s\mathrm{sgn}_s - 2\lambda(d-d_\ell) - 2\mu(d-d_m) - 2\nu(d-d_n) = 0.$$

This system comprises one linear equation in each unknown: x, y, z and d. λ, μ and ν will subsequently be selected to effect the three constraints $\mathrm{sgn}_\ell = \mathrm{sgn}_m = \mathrm{sgn}_n = 0$. As above, denote $\sum_{s=1}^{S} w_s\mathrm{sgn}_s$ by ξ, $\sum_{s=1}^{S} x_s w_s\mathrm{sgn}_s$ by ξ_x, $\sum_{s=1}^{S} y_s w_s\mathrm{sgn}_s$ by ξ_y, $\sum_{s=1}^{S} z_s w_s\mathrm{sgn}_s$ by ξ_z and $\sum_{s=1}^{S} d_s w_s\mathrm{sgn}_s$ by ξ_d. Then the foregoing four equations yield, respectively,

$$x = \frac{\xi_x + \lambda x_\ell + \mu x_m + \nu x_n}{\xi + \lambda + \mu + \nu},\ y = \frac{\xi_y + \lambda y_\ell + \mu y_m + \nu y_n}{\xi + \lambda + \mu + \nu},$$

$$z = \frac{\xi_z + \lambda z_\ell + \mu z_m + \nu z_n}{\xi + \lambda + \mu + \nu} \text{ and } d = \frac{\xi_d + \lambda d_\ell + \mu d_m + \nu d_n}{\xi + \lambda + \mu + \nu}.$$

Furthermore, the three constraints yield three quadratic forms: each in a pair of variables. These are unlikely to be definite, and theory for such systems is rudimentary. Hence, one might proceed by using the quadratic equation to eliminate two of the three variables, using the two equations containing, say, λ to eliminate the other two variables, and seeking the real roots of the remaining equation, yielding an algebraic function of λ.

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Figure Captions

Figure 1a.

TDOA localizations using six nodes (+'s) and four of the times of arrival. Lanes (numbered squares) correspond to shooters; the lanes used were #'s 2,3,6 and 7. The predicted coordinates for the gunshots for firings from each of these lanes are indicated by respective symbols. The dimensions of the axes are in meters, and the x axis is oriented East-West, whereas the y-axis is oriented North-South; these are referenced to an arbitrary origin Northwest of the plot. Note that the point from lane three at (-6959,-862) is actually the superposition of two nearly identical predictions for coordinates of two gunshots from lane three.

Figure 1b.

Least-error localizations, using the same nodes and combinations of four times of arrival as in Figure 1a. The axes are also the same as in Figure 1a.

Figure 2.

The predicted spatial coordinates of the below-threshold least-error solutions, given three cotemporary sources at $d = 0$ and (1/4,1/4), (1/2,1/2) and (3/4,3/4).

S	$\langle$RMS$\rangle$
4	.033
5	.012
6	.0091
7	.0079
8	.0072
9	.0067
10	.0063
11	.0060
12	.00577
13	.00556
14	.00539
15	.00521
16	.00507

Table 1: Average Root-Mean-Square Localization Error

	⟨RMS⟩	25%	50%	75%	95%
TDOA	—	0.008	0.014	0.035	0.218
Least-Error	0.033	0.007	0.0099	0.0175	0.075

Table 2: Root-Mean-Square Localization Error Percentiles; $S = 4$.

S	False Positive Rate
4	0.4
6	0.2
8	0.03

Table 3: The proportions of false positives in the below-threshold results for different numbers of nodes. False positives were defined as those with the minimum, over all sources $(\mathbf{v}_0; d_0)$ of $||\mathbf{v} - \mathbf{v}_0||^2 + (d - d_0)^2 > 0.02$.