APPENDIX 6.3

ON THE USE OF INTERNAL STATE VARIABLES IN THERMOVISCOPLASTIC CONSTITUTIVE EQUATIONS

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ABSTRACT

The use of internal state variables in modeling of inelastic solids is gaining widespread usage in current research. Therefore, it is useful to construct a well-defined framework for internal state variable models which is based in continuum mechanics. The objective of this paper is to review and clarify the general theory of internal state variables and to apply it to inelastic metals currently in use in high temperature environments. In this process, certain constraints and clarifications will be made regarding internal state variables.

It will be shown that the Helmholtz free energy can be utilized to construct constitutive equations which are appropriate for metallic superalloys. Furthermore, internal state variables will be shown to represent locally averaged measures of dislocation arrangement, dislocation density, and intergranular gracture. Finally, the internal state variable model will be demonstrated to be a suitable framework for comparison of several currently proposed models for metals and can therefore be used to exhibit history dependence, nonlinearity, and rate as well as temperature sensitivity.

INTRODUCTION

The prediction of inelastic behavior of structural materials at elevated temperature is a problem of great importance which has accordingly been given a great deal of interest by the research community in recent years. These materials exhibit substantial complexity in their thermomechanical constitution. In fact, so complex is their material response that it could be argued that without useful *a priori* information, experimental characterization is futile. The purpose of this paper is to show how the thermodynamics with internal state variables can be utilized to emplace certain constraints on the allowable form of thermomechanical constitutive equations, thus providing some limited insight regarding experimental requirements.

Historically, there have been two distinct approaches to the modelling of inelastic materials: 1) the functional theory [1], in which all dependent variables are assumed to depend on the entire history of independent variables; and 2) the internal state variable (ISV) approach [2], wherein history dependence is postulated to appear implicitly in a set of internal state variables. Lubliner [3] has shown that in most circumstances ISV models can be considered to be special cases of functional models. For experimental as well as analytic reasons numerous recently proposed models for the classes of materials

discussed herein have been proposed in ISV form. Therefore, in this paper the ISV method will be reviewed as well as clarified and it will be shown that this general framework is useful in modeling metals at elevated temperature.

The paper begins with a review of ISV theory, and this is supplemented with a section describing the procedure for constructing macroscopically averaged internal state variables. These concepts are then applied to metals at elevated temperatures. Finally, applications to boundary value problem solving techniques are discussed.

REVIEW OF THE INTERNAL STATE VARIABLE (ISV) APPROACH

The concept of internal state variables, sometimes called hidden variables, was apparently first utilized in thermodynamics by Onsager [4,5] and numerous applications are recorded since the second world war [2,6-14]. Although not originally described for application to solids, the approach which will be discussed herein is due to Coleman and Gurtin [2].

In the theory of internal state variables applied to solids the following state variables are required in order to fully characterize the state of the body at all points x_j and at all times t:*

1) the displacement field $u_i = u_i(x_k,t)$; (1)

2) the stress tensor $\sigma_{ij} = \sigma_{ij}(x_k,t)$; (2)

3) the body force per unit mass $f_i = f_i(x_k,t)$; (3)

4) the internal energy per unit mass $u = u(x_k,t)$; (4)

5) the heat supply per unit mass $r = r(x_k,t)$; (5)

6) the entropy per unit mass $s = s(x_k,t)$; (6)

7) the absolute temperature $T = T(x_k,t)$; (7)

8) the heat flux vector $q_i = q_i(x_k,t)$; (8)

and

9) $\alpha^k_{ij} = \alpha^k_{ij}(x_m,t)$, $k = 1, 2, \ldots, n$; (9)

where α^k_{ij} are a set of n internal state variables which are necessary to account for inelastic material behavior. Although they are listed here as second order tensors, they may be tensors of other rank as well [15].

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* For convenience, only infinitesimal deformations will be considered here, although the general theory applies to finite deformations as well.

The method of Coleman and Noll [16] may be used to obtain the spatial and time distribution of the body force f_i and heat supply r from the conservation of linear momentum and energy, respectively, assuming the displacements u_i and the temperature T are specified independent variables. Subsequently, it is hypothesized that constitutive equations of state may be constructed for the state variables described in (1) through (8) in terms of u_i and T and their spatial derivatives:

$$\sigma_{ij}(x_k,t) = \sigma_{ij}(\varepsilon_{mn}(x_k,t),\ T(x_k,t),\ g_m(x_k,t),\ \alpha^p_{mn}(x_k,t)) \quad ; \quad (10)$$

$$u(x_k,t) = u(\varepsilon_{mn}(x_k,t),\ T(x_k,t),\ g_m(x_k,t),\ \alpha^p_{mn}(x_k,t)) \quad ; \quad (11)$$

$$s(x_k,t) = s(\varepsilon_{mn}(x_k,t),\ T(x_k,t),\ g_m(x_k,t),\ \alpha^p_{mn}(x_k,t)) \quad ; \quad (12) \text{ and}$$

$$q_i(x_k,t) = q_i(\varepsilon_{mn}(x_k,t),\ T(x_k,t),\ g_m(x_k,t),\ \alpha^p_{mn}(x_k,t)) \quad ; \quad (13)$$

where g_m is the spacial temperature gradient $T_{,m}$ and

$$\varepsilon_{ij} \equiv \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad . \quad (14)$$

The form of equations (11) through (13) implies that all constitutive equations are evaluated in the specified state (x_k,t). For this reason σ_{ij}, u, s, and q_i are termed observable state variables since they can be determined from equations of state even though there is implicit history dependence via the internal state variables α^p_{mn}, which are defined to be of the form:

$$\dot{\alpha}^k_{ij} \equiv \Omega^k_{ij}(\varepsilon_{mn},\ T,\ g_m,\ \alpha^\ell_{mn}) \quad ; \quad (15)$$

where time and spacial dependence have been dropped for notational convenience. If equations (15) are at all times integrable in time, then the following form is equivalent to (15):

$$\alpha^k_{ij}(x_m,t) = \int_{-\infty}^{t} \Omega^k_{ij}(x_m,t')dt' \quad ; \quad (16)$$

where t is the time of interest and t' is a dummy variable of integration. Therefore, it is apparent that α^k_{ij} are not directly observable at any time and must therefore be considered to be hidden or internal.

Although the above framework has been shown to be applicable to rate dependent crystalline solids [17,18], it is often misconstrued that the absence of explicit strain-rate dependence renders the model inappropriate for use in viscoplasticity theories. It is alternatively hypothesized that

$$\sigma_{ij} = \sigma_{ij}(\varepsilon_{mn},\ \dot{\varepsilon}_{mn},\ T,\ g_m,\ \alpha^\ell_{mn}) \quad (17)$$

is an appropriate form of thermomechanical constitutive equations (10). Although metals at elevated temperature certainly exhibit strain-rate dependence, there are several reasons why equations (17) are less desirable than equations (10). First, equations (17) are not actually equations of state since the inclusion of strain rate implies knowledge is required at some time other than the current time t. Secondly, as demonstrated in discussions of

materials similar to (17) but without internal state variables [19], very little useful information will come from thermodynamic constraints. Finally, explicit strain rate dependence is actually redundant for the materials discussed herein, as will be shown later. Therefore, although this is certainly a semantical issue, equations (10) through (13) and (15) are utilized as the constitutive model in the balance of this paper.

It should also be pointed out that internal state variable growth laws (15) could contain explicit strain-rate dependence:

$$\dot{\alpha}_{ij} = \Omega^{k}_{ij}(\varepsilon_{mn}, \dot{\varepsilon}_{mn}, T, g_m, \alpha^{\ell}_{mn}) \quad , (18)$$

as in the example of a rate independent elastic-plastic material, in which equations of the above form are linear in strain rate:

$$\dot{\alpha}^{\ell}_{ij} = \Omega^{k}_{ijpq}(\varepsilon_{mn}, T, g_m, \alpha^{\ell}_{mn})\dot{\varepsilon}_{pq} \quad . (19)$$

Such a form, although not excluded by the principle of equipresence [20], is only necessary in the circumstance wherein specific rate independence is required, as can be demonstrated by direct substitution of (19) into (16). Furthermore, although the thermodynamic constraints will vary somewhat when (19) are utilized [21,22], the results will be quite similar to those described below.

On the basis of the Coleman-Mizel procedure [23] it can be shown that satisfaction of the first and second laws of thermodynamics for the class of materials detailed above will lead to the following conclusions:

$$h \equiv u - Ts = h(\varepsilon_{mn}, T, \alpha^{k}_{mn}) \quad ; (20)$$

where h is the specific Helmholtz free energy;

$$\sigma_{k\ell} = \rho \frac{\partial h}{\partial \varepsilon_{k\ell}} \quad ; (21)$$

$$s = -\frac{\partial h}{\partial T} \quad ; (22)$$

and

$$q_i = -k_{ij} g_j + 0(g_i) \quad . (23)$$

Equations (21) should not be interpreted as defining as hyperelastic material since the Helmholtz free energy, described by (20), is dependent on the internal state and therefore path dependent.

Although not directly related to our problem, it is useful to note that the path dependence of the Helmholtz free energy precludes the usefulness of equations (21) in Rice's J-integral for fracture mechanics [24]. However, in the case wherein the loading path is radial:

$$\varepsilon_{ij} = k_{ij}\bar{\varepsilon} \; ; \; \alpha^{\ell}_{ij} = k^{\ell}_{ij}\bar{\varepsilon} \; ; \; \bar{\varepsilon} \equiv \sqrt{\varepsilon_{ij}\varepsilon_{ij}} \quad , (24)$$

where k_{ij} and k^{2}_{ij} are constant coefficients, then it is well known that equations (15) are directly integrable so that the free energy can be described by

$$h = h(\varepsilon_{mn}, T, \alpha^{i}_{pq}) = h(\varepsilon_{mn}, T, \alpha^{i}_{pq}(\varepsilon_{mn})) = h(\varepsilon_{mn}, T) \quad . \quad (25)$$

Thus, for the case of proportional loading only, the constitutive equations are derivable directly from a potential function and the J-integral method is applicable.

THE LOCAL AVERAGING PROCESS

Constitutive equations (10) through (13) and (15) are theoretically pointwise in nature; that is, they are applicable to fixed infinitesimal material points. However, practically speaking, there is no way to construct experiments on material points since at the microscopic level the continuum assumption becomes invalid. Rather, it is considered acceptable to construct constitutive equations by subjecting local specimens to surface deformations (or tractions) which lead to spacially homogeneous stresses and strains so that some local average of the pointwise observable state variables can be determined directly from the effects on the boundaries of the specimens.

As shown in Fig. 1, the scale of the smallest dimension of a local specimen is generally constructed so as to be at least an order of magnitude larger than the scale of the largest material inhomogeneity. This sizing helps preserve the continuum assumption while at the same time averaging out the effects of point defects such as crystal lattice dislocations. Conversely, the scale of the largest dimension of a typical specimen should be as small as possible compared to the scale of the global boundary value problem of interest. This constraint is necessary in order to preserve the notion that constitutive equations are indeed pointwise in nature, but it is pragmatic in that it is a simple matter of economy.

The local rather than pointwise constitutive equations that result from experimentation are assumed to be of the same form as pointwise equations (10) through (13) and (15). For example, in the uniaxial test described in Fig. 1 it is customary to define

$$\bar{\sigma}_{11} \equiv \frac{1}{A} \int_{B_1} \sigma_{11} dx_2 dx_3 \quad , \quad (26)$$

$$\bar{\varepsilon}_{11} \equiv \frac{1}{L} \int_{L} \varepsilon_{11} dx_1 \quad , \quad (27)$$

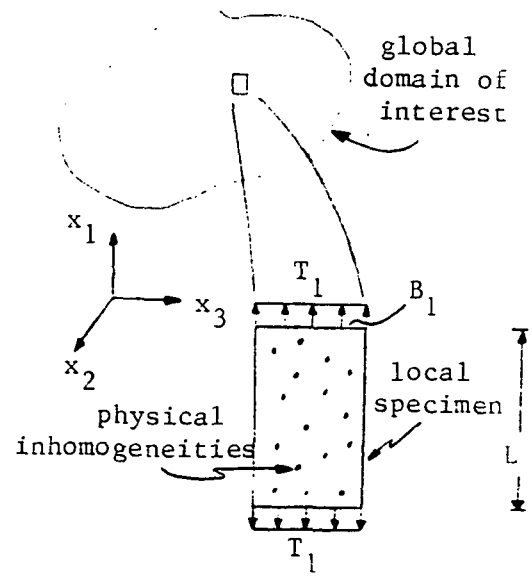

Figure 1

and

$$\bar{T} \equiv T(a_1, a_2, a_3) \quad , (28)$$

where L is the local specimen gage length, A is the cross-sectional area in the x_2-x_3 plane, and (a_1, a_2, a_3) is some arbitrary point on the surface of the specimen. Utilizing these quantities, it is then hypothesized that

$$\sigma_{11}(\varepsilon_{11}, T, \alpha^{\ell}_{mn}) \equiv \bar{\sigma}_{11}(\bar{\varepsilon}_{11}, \bar{T}, \bar{\alpha}^{\ell}_{mn}) \quad , (29)$$

where

$$\bar{\alpha}^{\ell}_{mn} \equiv \frac{1}{V} \int_V \alpha^{\ell}_{mn} \, dx_1 \, dx_2 \, dx_3 \quad , (30)$$

and all quantities with bars represent the locally measured state variables.

Although equations (29) represent an often used way of relating pointwise equations to experimental results, the local averaging process is nevertheless fraught with shortcomings since definitions (26) through (28) all reresent nonunique relations between pointwise state variables σ_{ij}, ε_{ij}, T, α^{k}_{ij} and their locally defined counterparts $\bar{\sigma}_{ij}$, $\bar{\varepsilon}_{ij}$, $\bar{T}$, and $\bar{\alpha}^{k}_{ij}$. There are in fact an infinite number of distributions $\alpha^{\ell}_{mn}(x_1, x_2, x_3)$ which will result in identical values of $\bar{\alpha}^{\ell}_{mn}$. However, assuming that the scale of inhomogeneities is small and that the distribution of α^{ℓ}_{mn} is random the specimen will be statistically homogeneous and the relation between $\bar{\alpha}^{\ell}_{mn}$ and α^{ℓ}_{mn} will be reasonably one to one.

For example, suppose that during some monotonically increasing local strain history $\bar{\varepsilon}_{11}$ a particular internal state variable α^{1}_{11} such as a single dislocation arrangement is governed on a pointwise basis by the almost discontinuous behavior shown in Fig. 2. Suppose further that the time t at which the internal state begins to change is determined by the pointwise stress state. Then the number of dislocation rearrangements occurring in the local specimen as a function of time might be distributed as shown in Fig. 3. If the local specimen is large compared to the scale of the dislocation, and there are numerous dislocation rearrangements, as is usually the case in testing of metals, then the peak of the curve shown in Fig. 3 will be several orders of magnitude greater than unity. It follows from equations (30) that the locally averaged value of the internal state variable represented in Fig. 2 will be as qualitatively shown in Fig. 4.

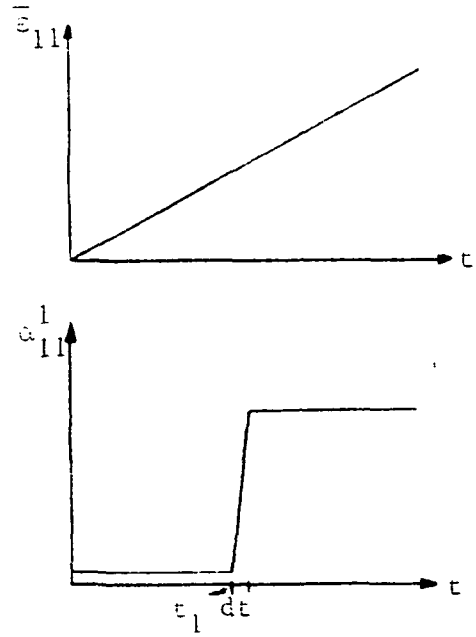

Figure 2

APPLICATION TO METAL CONSTITUTION

In order to describe the class of metals discussed herein, the free energy maybe expanded in terms of the elastic strain tensor ε^E_{kl} and the temperature T in a second order Taylor series expansion as follows:

$$\varepsilon^E_{ij} \equiv \varepsilon_{ij} - \varepsilon^I_{ij} - \varepsilon^T_{ij} \quad , (31)$$

where ε^T_{ij} is the thermal strain tensor and ε^I_{ij} is the inelastic strain tensor, considered to be an internal state variable [17,18,22,25], and

$$h = h_R + \frac{1}{2\rho}\varepsilon^E_{ij}D_{ijkl}\varepsilon^E_{kl} - \frac{C_v}{2T}(T - T_R) \quad , (32)$$

where the subscript R refers to quantities in the reference state, D_{ijkl} is the linear elastic modulus tensor, and $C_v \equiv -T\left(\frac{\partial^2 h}{\partial T^2}\right)$ is the specific heat at constant elastic strain. Substitution of equation (32) into (21) will result in

$$\sigma_{kl} = D_{klmn}(\varepsilon_{mn} - \varepsilon^I_{mn} - \varepsilon^T_{mn}) \quad . (33)$$

The above equations, together with internal state variable growth laws (15), will be shown to be a suitable framework for comparison of all of the models to be discussed herein.

Internal State Variables in Metals

It is now generally agreed in the literature that in single crystals there are two locally averages internal state variables: the back stress ($\alpha_{1_{ij}}$) representing dislocation arrangement; and the drag stress (α_2) representing dislocation density; where the bars have been dropped for convenience and the superscript has been converted to a subscript in order to avoid the confusion which would arise if a state variable were raised to some power. For obvious reasons the back stress is a second order tensor, whereas the drag stress is a scalar. In specimens composed of multiple crystals it is generally agreed that a third internal state variable loosely termed damage ($\alpha_{3_{ij}}$) is necessary in order to account for intergranular mechanisms such as grain boundary sliding and microvoid growth and coalescence that may occur at high temperature and/or large strain. Although damage is obviously a directionally related quantity and therefore tensorial in nature, it is difficult to distinguish phenomenologically between damage and drag stress since both are primarily stiffness reducing mechanisms.

Within the thermodynamic framework described earlier it is also possible to define the inelastic strain tensor to be an internal state variable. However, this interpretation is not generally utilized within the materials

literature. It is hypothesized that the rate of growth of the internal state variables does not depend on the inelastic strain tensor so that

$$\dot{\alpha}^k_{ij} = \Omega^k_{ij}(\varepsilon_{mn}, T, g_m, \alpha_{1_{mn}}, \alpha_2, \alpha_{3_{mn}}) \quad . \quad (34)$$

Due to the form of equations (34) it is said that since the inelastic strain tensor does not appear on the right hand side it is not an internal state variable. However, within the framework defined herein, it is still possible to construct an internal state variable growth law of the form

$$\dot{\varepsilon}^I_{ij} \equiv \Omega^I_{ij}(\varepsilon_{mn}, T, g_m, \alpha_{1_{mn}}, \alpha_2, \alpha_{3_{mn}}) \quad , \quad (35)$$

which is precisely in agreement with definitions (15).

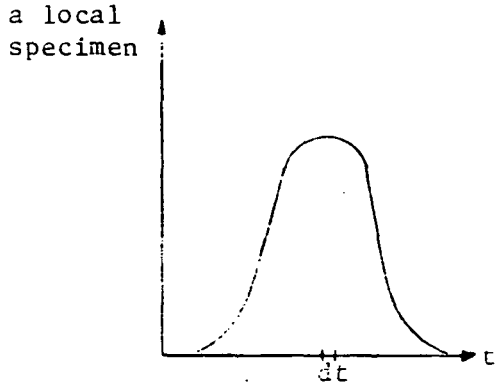

Figure 3

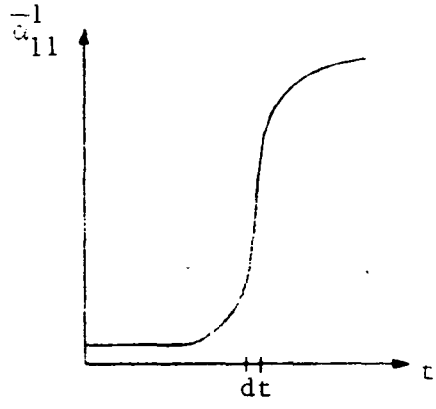

Figure 4

In order to qualitatively verify the supposition that the inelastic strain tensor can be regarded to be an ISV, consider the example of a uniaxial bar subjected to applied displacements such that the end tractions will be evenly distributed. It is customary to deduce the inelastic strain in an experiment of this type by utilizing the output from a load cell to determine the stress and then making use of equations (33) to determine the elastic strain. This result and the total strain measured by an extensometer are then substituted into equations (31) to determine the inelastic strain. Nevertheless, this does not imply that the inelastic strain tensor in an observable state variable. This result can be arrived at only in constitutive experiments such as uniaxial bar tests in which the stress and strain tensors are spacially homogeneous. In heterogeneous boundary value problems, only two state variables may be input (temperature and either stress or strain), and for this case equations (31) and (33) must be supplemented with an ISV growth law of the form of equations (35) in order to determine the inelastic strain tensor. Therefore, in the context of the current thermodynamic framework the inelastic strain tensor may be interpreted to be an ISV.

A Framework for Current Metals Models

In order to establish that current models can be constructed from equations (33), consider the standard solid shown in Fig. 5. The governing differential equation for this analog is

$$\sigma + \frac{\eta_M}{E_M}\dot{\sigma} = E_\infty \varepsilon + \eta_M \left[1 + \frac{E_\infty}{E_M}\right] \dot{\varepsilon} \quad , \quad (36)$$

where by convention the stress is denoted σ and the strain is ε. Equation (36) may be written in the following equivalent form

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{[E_M+E_\infty]} + \frac{E_M}{\eta_M}\frac{[\sigma-E_\infty\varepsilon]}{[E_M+E_\infty]} \quad . \tag{37}$$

In accordance with the instantaneous linear elastic behavior of metals, it is assumed that

$$E_M+E_\infty \equiv E = \text{Young's modulus} = \text{constant}, \tag{38}$$

so that it is clear that equation (37) can be integrated in time to give the following stress formulation

$$\varepsilon(t_1) = \frac{\sigma(t_1)}{E} + \varepsilon^I(t_1) \quad . \tag{39}$$

where ε^I is the inelastic strain, defined by

$$\varepsilon^I(t_1) = \int_{-\infty}^{t_1} \frac{E_M}{E}\frac{[\sigma-E_\infty\varepsilon]}{\eta_M}\,dt \quad . \tag{40}$$

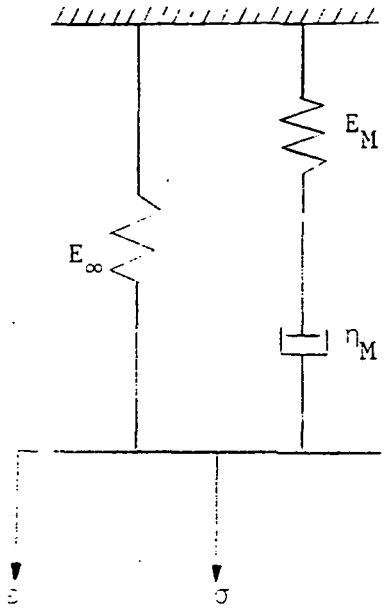

Figure 5

Equation (39) may be solved for the stress and substituted into equation (40) so that it is clear that equation (40) is in accordance with ISV growth laws (16). Further, it can be seen from the standard solid analog in Fig. 5 that since $\sigma-E_\infty\varepsilon$ represents the stress in the Maxwell element, ε^I is not observable, so that ε^I satisfies the two conditions required for it to be an internal state variable.

Equation (39) may be written equivalently in the following strain formulation:

$$\sigma(t_1) = E[\varepsilon(t_1) - \varepsilon^I(t_1)] \quad , \tag{41}$$

which is an equation of state compatible with constitutive equations (10) as well as equations (33). Since no other internal state variables are present in this equation, and also, no additional internal state variables are present in growth law (40) it is apparent that the standard solid analog with constant coefficients E_M, η_M and E_∞ is a single internal state variable model.

It has been noted by several researchers that the standard solid is an appropriate analog for thermoviscoplastic metals if the springs and dashpot are nonlinearized [26,27]. In order to demonstrate this feature, consider a multiaxial extension of equation (36):

$$\sigma_{pq} + K_{pqmn}\,\dot{\sigma}_{mn} = G_{pqmn}\,\varepsilon_{mn} + M_{pqmn}\,\dot{\varepsilon}_{mn} \quad , \quad (42)$$

where by convention the small strain tensor ε_{ij} is used in conjunction with the work conjugate stress tensor σ_{ij}. In order to model metals K_{pqmn}, G_{pqmn}, and M_{pqmn} are required to be nonlinear in some as yet undetermined way. In addition, in accordance with constraint equation (38), it is required that

$$K^{-1}_{ijmn}\,M_{mnkl} = D_{ijkl} \quad , \quad (43)$$

where D_{ijkl} is the linear elastic modulus tensor. Equations (42) may be rewritten in a strain formulation equation of state form as follows:

$$\sigma_{ij} = D_{ijkl}\,[\varepsilon_{kl} - \varepsilon^{I}_{kl}] \quad , \quad (44)$$

where ε^{I}_{kl} is the inelastic strain tensor, defined by

$$\varepsilon^{I}_{ij} \equiv \int_{-\infty}^{t_1} M^{-1}_{ijpq}\,[\sigma_{pq} - G_{pqmn}\varepsilon_{mn}]\,dt \quad . \quad (45)$$

Substituting equations (43) and (44) into equations (45) will result in

$$\varepsilon^{I}_{ij} = \int_{-\infty}^{t_1} \{K^{-1}_{ijmn}[\varepsilon_{mn} - \varepsilon^{I}_{mn}] - M^{-1}_{ijpq}\,G_{pqmn}\varepsilon_{mn}\}\,dt \quad , \quad (46)$$

so that equations (46) are in accordance with growth laws (16). The number of internal state variables contained in the model will depend on the degree of nonlinearity proposed in the nonlinear tensors K_{pqmn}, G_{pqmn}, and M_{pqmn}, and this will be discussed in the following section. However, before continuing, it should be pointed out that the constitutive equations developed in this section assume that the elastic and inelastic strain tensors may be linearly decoupled. It has been shown that this assumption is invalid for finite deformation [28]. However, even under finite deformation conditions the inelastic strain is decoupled from the elastic strain in such a way that the inelastic strain tensor may be considered to be an internal state variable.

Current Models for Metals

The framework for metals models discussed in the previous section can be used to describe numerous models currently under development [26,27,29-58]. For example, the microphysically based isothermal model proposed by Krieg, et al., [30] is of the form described by equations (33):

$$\sigma_{ij} = D_{ijkl}(\varepsilon_{kl} - \varepsilon^{I}_{kl}) \quad , \quad (47)$$

where

$$\dot{\varepsilon}^{I}_{ij} \equiv \dot{\varepsilon}_o \left\{ \frac{\left[\left(\sigma'_{kl} - \alpha'_{1_{kl}}\right)\left(\sigma'_{kl} - \alpha'_{i_{kl}}\right)\right]^{\frac{1}{2}}}{\alpha_2} \right\}^{m} \frac{\left(\sigma'_{ij} - \alpha'_{1_{ij}}\right)}{\left[\left(\sigma'_{pq} - \alpha'_{1_{pq}}\right)\left(\sigma'_{pq} - \alpha'_{1_{pq}}\right)\right]^{\frac{1}{2}}} , \quad (48)$$

and $\dot{\varepsilon}_o$ and m are material constants, and σ'_{ij} is the deviatoric stress tensor and $\alpha'_{1_{ij}}$ is the deviatoric component of the back stress tensor. Since equations (48) contain the stress tensor, substituting equations (33) into (48) will result in equations consistent eith growth laws (15). In addition, Krieg, et al., give the back stress and drag stress to be, respectively,

$$\dot{\alpha}'_{1_{ij}} = A_\alpha \dot{\varepsilon}^{I}_{ij} - r_\alpha \frac{\alpha'_{1_{ij}}}{\left(\alpha'_{1_{pq}} \alpha'_{1_{pq}}\right)^{\frac{1}{2}}} , \quad (49)$$

and

$$\dot{\alpha}_2 = A_R \left(\dot{\varepsilon}^{I}_{ij} \dot{\varepsilon}^{I}_{ij}\right)^{\frac{1}{2}} - r_R , \quad (50)$$

where A_α and A_R are hardening constants, and r_α and r_R are recovery functions of temperature and internal state variables. It can be seen that since ISV growth laws (49) and (50) are consistent with equations (15), the model proposed by Krieg, et al., contains three internal state variables: the inelastic strain tensor, the back stress tensor, and the drag stress tensor.

Furthermore, classical plasticity theories can be described by the general form

$$\sigma_{ij} = D_{ijmn} \left(\varepsilon_{mn} - \varepsilon^{I}_{mn}\right) , \quad (51)$$

where

$$\dot{\varepsilon}^{I}_{ij} = \dot{\lambda} \frac{\partial F}{\partial \sigma_{ij}} . \quad (52)$$

$\dot{\lambda}$ is a scalar valued function of state, and F is a scalar valued state function for inelastic behaviour often taken to be the yield function. If F is described by the von Mises yield criterion [53], given by

$$F(\sigma_{ij} - \alpha_{1_{ij}}) = \tfrac{1}{2}(\sigma_{ij} - \alpha_{1_{ij}})(\sigma_{ij} - \alpha_{1_{ij}}) = k^2 , \quad (53)$$

where $\alpha_{1_{ij}}$ is a tensor describing the yield surface center in stress space and k is a constant representing the yield surface size, then equations (52) can be written as

$$\dot{\varepsilon}_{ij} = \dot{\lambda}(\sigma_{ij} - \alpha_{1_{ij}}) \quad , (54)$$

resulting in a kinematic hardening model with constant yield surface size. Substitution of equations (51) into the above will yield a result consistent with rate independent ISV growth laws (19).

Furthermore, if the yield surface translation is derived from the Ziegler modification [60] of the Prager work hardening rule [61], it may be described by

$$\dot{\alpha}_{1_{ij}} = \dot{\mu}(\sigma_{ij} - \alpha_{1_{ij}}) \quad , (55)$$

where $\dot{\mu}$ is a scalar valued function of state. By use of equations (51), equations (55) can also be shown to be consistent with equations (19). Therefore, a classical plasticity-based kinematic hardening model contains two internal state variables: the inelastic strain tensor and the yield surface translation tensor representing the back stress.

In order to further illustrate the applicability of equations (33), (35) and (15) to current models for metals, ten of these models have been cast in uniaxial form in Table 1, wherein it is shown that although the framework for each model is identical (Valanis' model is in simplified form), the ISV growth laws vary widely both in number and form.

CONCLUSION

The main content of this paper has been to review and clarify the continuum and thermodynamics based internal state variable model for application to thermoviscoplastic metals. In this process the following points have been made:

1) the definition of an internal state variable utilized in this model has been clarified;

2) internal state variables in metals represent local averages of dislocation arrangement, dislocation density, and intergranular damage,

3) in the context of the ISV definition given here, inelastic strain may also be interpreted as an internal state variable;

4) the path dependent Helmholtz free energy may be expanded in a second order expansion in elastic strain and temperature in order to obtain a stress-strain equation of state;

5) rate dependence enters the constitutive equations implicitly via the inelastic strain, as demonstrated by the nonlinear standard solid analog; and

6) a three-dimensional generalization of the standard solid may be used as a means of comparison of the general form of several currently proposed models.

Further ramifications of the ISV model discussed are also of importance, although not detailed herein. For example, this model may be utilized to construct a coupled heat conduction equation which may be utilized to predict heat generation in thermoviscoplastic metals [62]. Furthermore, the concept of internal state variables may be utilized to construct models for the mechanical constitution of composites with damage [63,15,65,66].

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TABLE 1. COMPARISON OF UNIAXIAL MODELS

Theory	Stress-Strain Relaxation	Internal State Variable Growth Laws	Comments	Material Parameters
Cernocky and Krempl	(T1) $\sigma=E[\epsilon-\epsilon^I-\epsilon^T]$	(T2) $\dot{\epsilon}^I=\frac{\sigma-G}{Ek}$	1. $G=G(\epsilon,T)$ is obtained from extrapolation of relaxation data. 2. k is curve-fit to $k=R_0 e^{R_1 e^{-\left[\frac{\lvert\sigma-G\rvert}{R_2}\right]^{R_3}}}$	E, R_0, R_1, R_2, R_3
Krieg, Swearengen, and Rohde	(T3) $\sigma=E[\epsilon-\epsilon^I]$	(T4)[1] $\dot{\epsilon}^I=c_1\left[\frac{\lvert\sigma-a_1\rvert}{a_2}\right]^{c_2}\operatorname{sgn}(\sigma-a_1)$ (T5)[1,2] $\dot{a}_1=c_3\dot{\epsilon}^I-c_4 a_1^2[e^{c_5 a_1^2}-1]\operatorname{sgn}(a_1)$ (T6)[2] $\dot{a}_2=c_6\lvert\dot{\epsilon}^I\rvert-c_7[a_2-a_{2_0}]^n$		$E, c_1, c_2, c_3, c_4, c_5, c_6, c_7, a_{2_0}, n$
Bodner et al.	(T7) $\sigma=E[\epsilon-\epsilon^I]$	(T8)[1] $\dot{\epsilon}^I=\frac{2}{\sqrt{3}}D_0 e^{-\left[\frac{n+1}{2n}\right]\left[\frac{\sigma}{a_2}\right]^{2n}}\operatorname{sgn}(\sigma)$ (T9) $\dot{a}_2=[Z_1-a_2]\dot{W}_p-AZ_1\left[\frac{a_2-Z_I}{Z_1}\right]^r$	1. $\dot{W}_p=\sigma\dot{\epsilon}^I$	$E, D_0, n, m, Z_1, Z_I, A, r$
Walker	(T10) $\sigma=E[\epsilon-\epsilon^I]$	(T11)[1] $\dot{\epsilon}^I=\left[\frac{\lvert\sigma-a_1\rvert}{a_2}\right]^n\operatorname{sgn}(\sigma-a_2)$ (T12)[2] $\dot{a}_1=[n_1+n_2]\dot{\epsilon}^I-[a_1-a_{1_0}-n_1\epsilon^I][\lvert\dot{\epsilon}^I\rvert\frac{\partial}{\partial R}([n_3+n_4R][\ln(\frac{n_5R}{1+n_6R}+1)])+n_7\lvert a_1-a_{1_0}\rvert^{m-1}]$ (T13) $\dot{a}_2=n_8\lvert\dot{\epsilon}^I\rvert-n_9\lvert\dot{\epsilon}^I\rvert a_2-n_{10}[a_2-a_{2_0}]^q$	1. R is the cumulative inelastic strain: $R=\int^t\frac{\partial\epsilon^I}{\partial\tau}d\tau$. 2. The growth law for a_2, eq.(T13), is not presently used in the model; a_2 is assumed to be a constant.	$E, n, n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9, n_{10}, m, q, a_{1_0}, a_{2_0}$
Miller	(T14) $\sigma=E[\epsilon-\epsilon^I-\epsilon^T]$	(T15)[1] $\dot{\epsilon}^I=B\theta'\left[\sinh(\frac{\lvert\sigma-a_1\rvert}{a_2})1.5\right]^n\operatorname{sgn}(\sigma-a_1)$ (T16)[1,2] $\dot{a}_1=H_1\dot{\epsilon}^I-H_1B\theta'[\sinh(A_1\lvert a_1\rvert)]^n\operatorname{sgn}(a_1)$ (T17)[2] $\dot{a}_2=H_2\lvert\dot{\epsilon}^I\rvert[C_2+\lvert a_1\rvert-\frac{A_2}{A_1}a_2^3]-H_2C_2B\theta'[\sinh(A_2a_2^3)]^n$	1. $\theta'=e^{-\frac{Q}{RT}}$ for $T>0.6\,T_m$ $\theta'=e^{-\left[\frac{Q}{0.6RT_m}\right]\left[\ln(\frac{0.6T_m}{T})+1\right]}$ for $T\leq 0.6\,T_m$ T_m is the melting temp. k is the gas constant.	$E, B, n, H_1, A_1, H_2, C_2, A_2, Q$

Cescotto and Leckie	(T18) $\sigma=E[\epsilon-\epsilon^I-\epsilon^T]$	(T19)[1] $\dot{\epsilon}^I=f(\lvert\frac{\sigma-a_1}{a_2}\rvert)\ \mathrm{sgn}(\sigma-a_1)$ (T20)[2] $\dot{a}_1=\frac{3}{2}h_a\dot{\epsilon}^I - r_a a_1$ (T21) $\dot{a}_2=h_\kappa - r_\kappa$	1. f, h_a, r_a, h_κ, and r_κ are experimentally determined functions.	E
Hart	(T22) $\sigma=E[\epsilon-\epsilon^I-\epsilon^T]$	(T23)[1] $\dot{\epsilon}^I=\dot{a}^*\left[\frac{2}{3}\right]^{M/2}\left[\frac{\lvert\sigma-a_1\rvert}{\mu}\right]^M \mathrm{sgn}(\sigma-a_1)$ (T24)[2] $\dot{a}_1=\frac{3}{2}\mu\dot{\epsilon}^I - \frac{\mu\left[\frac{a_2'}{G}\right]^m f e^{-\frac{Q}{RT}}}{\left[\ln\left(\frac{3a_2'}{2\lvert a_1\rvert}\right)\right]^{1/\lambda}}$ (T25) $\dot{a}_2'=c\left[\frac{2}{3}\right]^{k/2} f e^{-\frac{Q}{RT}}\left[\frac{\sigma}{a_2'}\right]^k \frac{a_2'}{\left[\ln\left(\frac{a_2'}{\sqrt{\frac{2}{3}}\,a_1}\right)\right]^{1/\lambda}}$	1. The drag stress is taken to be a constant, μ, hence there is no a_2 as in other models. There is, however, a third internal state variable, termed a_2'. 2. T is the absolute temp. R is the gas constant.	E, $\dot{a}^*$, M, μ, G, m, f, Q, k, λ, c
Robinson	(T26) $\sigma=E[\epsilon-\epsilon^I-\epsilon^T]$	(T27) $\dot{\epsilon}^I=\frac{1}{2\mu}\left[\frac{1}{\sqrt{3}}\left\lvert\frac{\sigma-a_1}{K}\right\rvert\right]^{n-1}[\sigma-a_1]$ (T28)[2] $\dot{a}_1=\frac{2\mu H}{\left[\frac{1}{\sqrt{3}}\left\lvert\frac{a_1}{K}\right\rvert\right]^\beta}\dot{\epsilon}^I - R\left[\frac{1}{\sqrt{3}}\left\lvert\frac{a_1}{K}\right\rvert\right]^{n-\beta-1} a_1$	1. G_0 is the initial value of $\frac{a_1^2}{3K^2}$.	E, μ, K, n, β, H, R, G_0
Valanis	(T29) $\sigma=E[\epsilon-\epsilon^I-\epsilon^T]$	(T30) $\dot{\epsilon}^I=k_1 f_1(\sigma,\epsilon)\dot{\epsilon} + k_2 f_2(\sigma,\epsilon)$	1. Represents simplified form of Valanis' model.	E, k_1, k_2, f_1, f_2
Allen and Haisler	(T31) $\sigma=E[\epsilon-\epsilon^I-\epsilon^T]$	(T32) $\dot{\epsilon}^I=\dot{\lambda}[\sigma-a_1] + \dot{g}[\sigma-a_1]$ (T33) $\dot{a}_1=\dot{\mu}[\sigma-a_1]$ (T34) $\dot{a}_2=f(\dot{\epsilon}^{I^2})$	1. Considerable curve-fitting and interpolation of stress-strain and creep data required to obtain material parameters $\dot{\lambda}$, $\dot{g}$, and $\dot{\mu}$.	E, $\dot{\lambda}$, $\dot{g}$, $\dot{\mu}$, f

1 $\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

2 $\dot{\epsilon}^I$ can be substituted directly into growth law for a_1 and a_2 to obtain a form consistent with internal state variable growth laws (14).

Note: parentheses ( ) imply "function of", whereas brackets [ ] imply multiplication.