5. Relation between flow and mass fields

This section is an introduction to the dynamics of *large-scale* motions in the extra-tropical atmosphere and ocean. Important simplifications are possible on large space and time scales.

The main goal is to make potential vorticity (PV) a useful dynamical tool. The ordinary vertical vorticity ζ fully determines the rotational motion in the horizontal plane, but it is not conserved.

In order to determine the flow from the distribution of PV, which is conserved, one needs a second relationship between "motion" and "mass". The various choices, called "balance approximations", eliminate gravity-inertia waves as a side-effect. The horizontal divergence involved in gravity waves becomes a "slave" to the geostrophically balanced motion.

5.1 Scale analysis

For simplicity, we proceed from the *shallow-water system*:

$$du/dt = fv - gh_x \tag{5.1}$$

$$dv/dt = -fu - gh_y \tag{5.2}$$

$$dh/dt = -(H+h)(u_x + v_y)\,. \tag{5.3}$$

Here *h* refers to the displacement of the free surface from its resting level *H*. In this system, "mass" refers directly to $h(x, y, t)$. The Coriolis effect is included.

Consider "long waves" in the mid-latitude troposphere, with global wavenumber less than 10 (wavenumbers 1, 2 or 3 are "ultra-long waves"). We will look for the leading-order behavior of disturbances with this scale, using some additional estimates:

L --- horizontal length scale $\rightarrow$	10^6 m
H ---depth scale $\rightarrow$	10^4 m
V --- wind speed $\rightarrow$	10 m/s
c --- phase speed $\rightarrow$	10 m/s
f_0 --- angular frequency of earth $\rightarrow$	10^{-4} s^{-1}
g --- gravitational constant $\rightarrow$	10 m/s^2
a --- radius of earth $\rightarrow$	10^7 m

The first four scales are appropriate for the extratropical atmosphere. For the ocean, the horizontal length scale is an order of magnitude smaller, while the velocity scales are usually two orders of magnitude smaller.

We normalize coordinates and dependent variables as: $u = V\tilde{u}$, *etc*., where $\tilde{u}$ is the non-dimensional variable. The time scale is a derived parameter, $T = L/c$. The *x*-momentum equation 5.1 becomes

$$\frac{V^2}{L}\left(\frac{c}{V}\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{v}\tilde{u}_{\tilde{y}}\right) + \frac{g\delta}{L}\tilde{h}_{\tilde{x}} - f_0 V\frac{f}{f_0}\tilde{v} = 0, \tag{5.4}$$

where δ has been introduced as the scale for *h*.

Now divide 5.4 by $f_0 V$ to reach

$$\mathrm{Ro}\left(\frac{c}{V}\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{v}\tilde{u}_{\tilde{y}}\right) + A\tilde{h}_{\tilde{x}} - \frac{f}{f_0}\tilde{v} = 0. \tag{5.5}$$

Here Ro = $V/(f_0 L)$, the “Rossby number”, and $A = \delta g/(f_0 LV)$. From the dimensional estimates, Ro $\approx 10^{-1}$ (even smaller for the ocean away from boundary currents). Therefore, the last two terms in 5.5 are in approximate balance and we should set $A = 1$ by making $\delta = f_0 LV/g$. This says that, at the scales of interest, *most of the pressure gradient is not involved in changing the velocity.*

Taylor-expanding the Coriolis parameter produces $f/f_0 \approx 1 + B(\tilde{y} - \tilde{y}_0)$, where $B \equiv (L/a)\cot\theta_0$ and θ_0 and y_0 are the latitude and *y*-position at which $f = f_0$. The expansion can be justified only if $B \ll 1$. Substituting dimensional estimates for the atmosphere -- or oceanic boundary currents and eddies -- we get $B \approx 1/10$. To proceed formally, we expand all the variables in the manner of

$$\tilde{u} = \tilde{u}_0 + \mathrm{Ro}\tilde{u}_1 + \mathrm{Ro}^2\tilde{u}_2 + \ldots \tag{5.6}$$

and group together terms in each equation according to their power of Ro. With a little luck, this will yield a set of simplified forecast equations with nice properties such as an energy principle.

5.2 The geostrophic wind

If *B* is of the same order as Ro, the lowest-order terms (those with *no* factor of Ro) in 5.5 are

$$\partial\tilde{h}_0/\partial\tilde{x} - \tilde{v}_0 = 0. \tag{5.7}$$

The corresponding result from the y-momentum equation is

$$\partial\tilde{h}_0/\partial\tilde{y} + \tilde{u}_0 = 0. \tag{5.8}$$

The system at this order is purely diagnostic. The lowest-order horizontal velocity in 5.7-5.8 can be expressed as $\tilde{\boldsymbol{V}}_0 = \hat{z} \times \tilde{\nabla}\tilde{h}_0$, or with dimensions restored,

$$\boldsymbol{V}_g = \left(\frac{g}{f_0}\right)\hat{z} \times \nabla h. \tag{5.9}$$

The subscript "g" is introduced for "geostrophic". This is actually an approximation of the geostrophic velocity based on $B \ll 1$ (otherwise the Coriolis parameter is not constant). In this model, $\boldsymbol{V}_g$ is non-divergent, with streamfunction $\psi = (g/f_0)h$. The geostrophic vorticity can be expressed as $\tilde{\zeta}_0 = \tilde{\nabla}^2\tilde{h}_0$ using the 2-D Laplacian, or $\zeta_g = \nabla^2\psi$.

The non-dimensional form of 5.3, using $c = V$, is

$$\frac{\delta}{H}(\tilde{h}_{\tilde{t}} + \tilde{u}\tilde{h}_{\tilde{x}} + \tilde{v}\tilde{h}_{\tilde{y}}) = -\left(1 + \frac{\delta}{H}\tilde{h}\right)(\tilde{u}_{\tilde{x}} + \tilde{v}_{\tilde{y}}). \tag{5.10}$$

Since $\tilde{D}$ on the rhs is at most O(Ro) but $\tilde{h}_0$ is order-unity, we conclude that δ/H is at most of order Ro. The standard scaling for w is based on mass continuity, $u_x + v_y = -w/(H+h)$, rather than the kinematic free-surface condition, $dh/dt = w$. Thus, $W = V(H/L)$ is standard. With this choice, the vanishing of the horizontal divergence at lowest order then means that

$$\tilde{w}_0 = 0 \tag{5.11}$$

and $\tilde{w} = \mathrm{Ro}\tilde{w}_1 + \ldots$.

In spherical coordinates, the lowest-order velocity is

$$u_g = -\left(\frac{g}{f_0}\right)\frac{\partial h}{a\partial\theta} \quad \text{and} \quad v_g = \left(\frac{g}{f_0}\right)\frac{\partial h}{a\cos\theta\partial\lambda}, \tag{5.12}$$

which, of course, is still non-divergent.

5.3 Quasi-geostrophic shallow-water model

For a time-dependent system, now consider the first-order (one factor of Ro) terms in the momentum equation:

$$\frac{d_0}{d\tilde{t}}\tilde{\boldsymbol{V}}_0 = -\hat{\boldsymbol{z}} \times \tilde{\boldsymbol{V}}_1 - \tilde{\nabla}\tilde{h}_1 - b(\tilde{y} - \tilde{y}_0)\hat{\boldsymbol{z}} \times \tilde{\boldsymbol{V}}_0, \tag{5.13}$$

where $d_0/d\tilde{t} \equiv \partial/\partial\tilde{t} + \tilde{\boldsymbol{V}}_0 \cdot \tilde{\nabla}$ and $b = B/\mathrm{Ro}$. After eliminating $\tilde{h}_1$ by cross-differentiating components, we have

$$\frac{d_0}{d\tilde{t}}\tilde{\zeta}_0 = \tilde{w}_1 - b\tilde{v}_0. \tag{5.14}$$

Here we have also introduced the first-order terms of mass continuity: $\tilde{w}_1 = -\tilde{\nabla} \cdot \tilde{\boldsymbol{V}}_1$. Mass continuity also allows us to write the leading-order part of 5.10 as

$$\frac{d_0\tilde{h}_0}{d\tilde{t}} = \mathrm{Ri}\mathrm{Ro}^2\tilde{w}_1, \tag{5.15}$$

with $\mathrm{Ri} \equiv gH/V^2$, the "Richardson number". Since $\tilde{\zeta}_0$ and $\tilde{\boldsymbol{V}}_0$ each have a known diagnostic relationship to $\tilde{h}_0$, 5.14 and 5.15 form a closed, time-dependent system for $\tilde{h}_0$ and $\tilde{w}_1$. This is the so-called "quasi-geostrophic" system.

Eliminating $\tilde{w}_1$ in 5.14-5.15 yields the quasi-geostrophic, shallow-water form of Ertel's potential vorticity equation:

$$\frac{d_0}{d\tilde{t}}(\tilde{\zeta}_0 + b\tilde{y} - r^2\tilde{h}_0) = 0, \tag{5.16}$$

where $r^2 \equiv \mathrm{Ri}^{-1}\mathrm{Ro}^{-2}$. This statement expresses conservation, following the geostrophic motion, of the quasi-geostrophic potential vorticity, $\tilde{q}_0 \equiv \tilde{f} + \tilde{\zeta}_0 - r^2\tilde{h}_0$, where $\tilde{f} = \mathrm{Ro}^{-1} + b(\tilde{y} - \tilde{y}_0)$, the linearly varying planetary vorticity. The dimensional form of $\tilde{q}_0$, in units of vorticity, is

$$q_g = f_0 + \beta y + \left(\nabla^2 - \frac{f_0^2}{gH}\right)\psi, \tag{5.17}$$

where $\psi = \dfrac{g}{f_0}h$. The conservation statement 5.16 may be written

$$\frac{d_g q_g}{dt} = \partial q_g/\partial t + J(\psi, q_g) = 0, \tag{5.18}$$

with $d_g/dt \equiv \partial/\partial t + \boldsymbol{V}_g \cdot \nabla$ and $J(A, B) \equiv A_x B_y - A_y B_x$. Since 5.18 involves a single unknown, ψ, the goal of finding a closed dynamical system involving PV has been achieved. A system in which ψ can be obtained from q alone, as in 5.17, is said to possess an "invertibility principle".

From 5.16, the ratio of the "mass" perturbation to the relative vorticity is of the order of r^2

in the expression for potential vorticity. This is also the ratio between available potential energy and kinetic energy in a balanced model. The nondimensional parameter r^2, called the "Burger number", may be written $r^2 = L^2/L_R^2$ by defining $L_R \equiv \sqrt{gH}/f_0$, the so-called Rossby *radius of deformation*. We infer from 5.16 that when the disturbance is small compared to the radius of deformation, it is mainly a velocity perturbation, and absolute vorticity is approximately conserved. When the disturbance scales are large compared to the Rossby radius, the height field is nearly conserved or else balanced mainly by changes in *planetary* vorticity.

As long as the divergence is O(Ro), the large-scale "limit" is still subject to the condition $L \approx L_R$. However, another balanced (*i.e.*, nearly geostrophic) system is possible for extremely large-scale disturbances if the divergence appears at a lower order, *i.e.*, in the geostrophic velocity. PV conservation in this system is suggested by removing the relative vorticity from (5.16). However, when divergence is O(1), we don't expect mass continuity or the expression for PV to "linearize", because then $\delta \approx H$. The form of q that arises in this situation is mentioned at the end of the section.

Eliminating h_0 from 5.14 and 5.15 is not very productive, but eliminating $\partial/\partial t$ yields a useful diagnostic equation:

$$(\tilde{\nabla}^2 - r^2)\tilde{w}_1 = -r^2\tilde{\boldsymbol{V}}_0 \cdot \tilde{\nabla}(\tilde{\zeta}_0 + b\tilde{y}). \tag{5.19}$$

On larger scales than the deformation radius ($r \gg 1$), the response to vorticity advection is vortex stretching (first term on rhs of 5.14). In this limit, our scaling keeps $\tilde{w}_1$ order-unity even though $\partial\tilde{h}_0/\partial\tilde{t} = 0$. On smaller scales than L_R, the $\mathrm{O}(r^2)$ vertical motion keeps the height in geostrophic balance without significantly stretching the planetary vorticity.

Another type of balanced flow occurs in the presence of external forcing when $b \gg 1$. This assumption implies $L \approx a$ and since the external deformaiton radius L_R is at least as large as a, we have $b \gg r^2$, as well. If the forcing consists of a wind stress, $d\boldsymbol{V}/dt = H^{-1}(\tau^{(x)}, \tau^{(y)})$, as in an ocean basin, then 5.16 becomes

$$\beta v_g = H^{-1}\nabla_\perp \cdot \tau, \tag{5.20}$$

where the rhs is the curl of the wind-stress vector. This relation is known as *Sverdrup balance*. In regions where surface westerlies increase towards the north, *i.e.*, $\nabla_\perp \cdot \tau = -\partial\tau^{(x)}/\partial y < 0$, 5.20 predicts a steady southward drift in the basin. In both hemispheres, Sverdrup drift is equatorward if the surface wind is westerly. Mass is conserved by means of a viscous boundary current along the western side of the basin.

In a more realistic model with vertical structure, τ affects the deep ocean indirectly through a vertical motion dh/dt imposed at the interface between the inviscid deep layer and the viscous surface layer. Thus, we keep $b \gg 1$, but since L_R^2 is now based on the *internal* deformation radius, we can have r^2 large enough for $b \approx r^2$. Then since $\delta/H = r^2\mathrm{Ro}$ is order-unity, mass continuity remains nonlinear: $dh/dt = -(H + h)D$. Moreover, the horizontal divergence D

is primarily associated with *planetary* vortex stretching by the *geostrophic* flow: $D \approx \nabla \cdot \boldsymbol{V}_g = -\beta v_g / f$, where $\beta = df/dy$ (the velocity is divergent at lowest order). We now have $d_g h/dt = (H+h)\beta v_g/f$. This may be rewritten as a statement of conservation, following the geostrophic motion, of the quantity

$$q_{PG} = fH/(H+h). \tag{5.21}$$

Evidently, q_{PG} is the PV for this system, the so-called "planetary geostrophic" model (shallow-water version). As in the quasi-geostrophic large-scale "limit", there is no contribution from the relative vorticity. Since the geostrophic velocity is not expressible in terms of a streamfunction, the invertibility principle must be written $h = H(fq^{-1} - 1)$ or $\boldsymbol{V}_g = (gH/f)\nabla_\perp(fq^{-1})$.

The second inequality used to obtain 5.20 is a "rigid-lid" approximation. It does not exclude the possibility of horizontal pressure gradients at the top, "under the lid". To be consistent with mass continuity, the column-integrated geostrophic divergence under a rigid lid must be cancelled by the ageostrophic divergence. Most of the ageostrophic divergence is confined to the viscous surface layer (see section 5.5). However, because the geostrophic flow diverges on the scale of the earth's radius rather than the forcing scale, the full-column geostrophic mass transport tends to dominate the ageostrophic mass transport (see problem 5.4).

5.4 Thermal wind and baroclinicity vector

In pressure coordinates, the geostrophic wind is

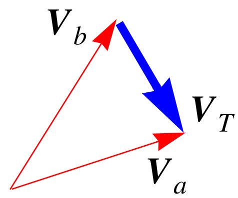
$$\boldsymbol{V}_g = \frac{1}{f_0}\hat{\boldsymbol{z}} \times \nabla\phi, \tag{5.22}$$

with $\phi = gz$ (*cf.* 5.9, in which $\phi = gh$). We postpone the development of PV dynamics for this model to the end of the chapter. For two pressure levels, $p_a < p_b$, define the *thermal wind*:

$$\begin{aligned} \boldsymbol{V}_T &\equiv \boldsymbol{V}_g(p_a) - \boldsymbol{V}_g(p_b) \\ &= \frac{1}{f_0}\hat{\boldsymbol{z}} \times \nabla(\phi_a - \phi_b). \end{aligned} \tag{5.23}$$

The difference $\phi_a - \phi_b$ is called the *thickness*. For an ideal gas, the hydrostatic relation gives $\phi_a - \phi_b = \int_a^b RT \mathrm{d}\log p$, so that

$$\boldsymbol{V}_T = \frac{R}{f_0}\hat{\boldsymbol{z}} \times \nabla\int_a^b T \mathrm{d}\log p. \tag{5.24}$$



The "thickness" is a streamfunction for the geostrophic vertical shear vector, $\boldsymbol{V}_T$. In the northern hemisphere, the thermal wind is directed with cold on the left and warm on the right. In the diagram below, T represents the log-p average in the layer where $\boldsymbol{V}_T$ is defined. In this case, $\boldsymbol{V}_g$ is becoming more westerly and less southerly with height.

The baroclinicity vector can be expressed in terms of the geostrophic wind and the thermal wind as follows.

Horizontal component of the baroclinicity vector:

$$\begin{aligned} \boldsymbol{S}_h &= -\hat{z} \times \frac{\partial}{\partial z}(\alpha \nabla_h p) \\ &= -f_0 \frac{\partial \boldsymbol{V}_g}{\partial z} = \rho f_0 g \frac{\partial \boldsymbol{V}_g}{\partial p}. \end{aligned} \tag{5.25}$$

(Subscript "h" refers to horizontal derivatives at constant z.) The rhs represents forcing of circulation about a horizontal axis. If velocity is entirely geostrophic, new circulation all goes into the planetary part of C_a (material circuits are being tilted relative to the equatorial plane by the shear). This allows the flow to be steady.

Vertical component of the baroclinicity vector:

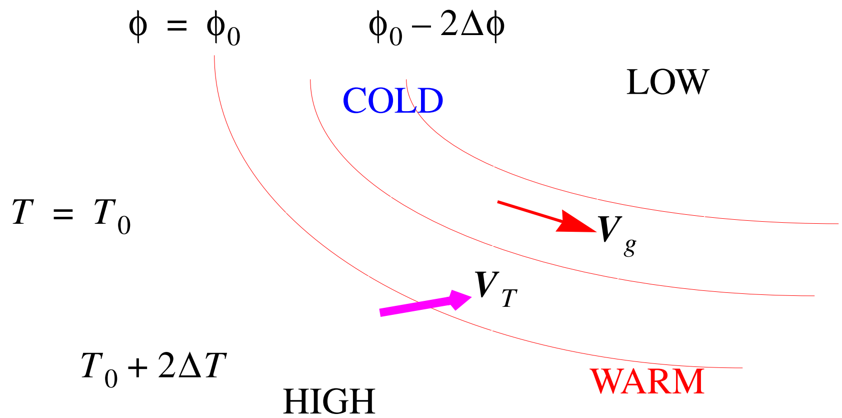
$$\begin{aligned} S_3 &= -\hat{z} \cdot (\nabla_h \alpha \times \nabla_h p) \\ &= \hat{z} \cdot \nabla_p \phi \times \nabla_p \log T \\ &= \frac{f_0^2}{g} \hat{z} \cdot \boldsymbol{V}_g \times \frac{\partial \boldsymbol{V}_g}{\partial z}. \end{aligned} \tag{5.26}$$

Here we have used the hydrostatic relation to introduce $\partial \boldsymbol{V}_g / \partial z$. Simplest cases:

(1) Barotropic: $\partial \boldsymbol{V}_g / \partial z = 0$. Then $\boldsymbol{S}_h = 0$ and $S_3 = 0$.

(2) Equivalent barotropic: $\partial \boldsymbol{V}_g / \partial z$ parallel to $\boldsymbol{V}_g$. Then $S_3 = 0$.

In the diagram below, we assume $f_0 > 0$. The circulation tendency (non-Boussinesq) is cyclonic, that is, $S_3 > 0$.



5.5 Ekman layers

The lowest-order system has no predictive value, but we can use it to analyze balanced flows. We will consider a balance involving the Coriolis force, pressure-gradient force and vertical momentum diffusion -- called Ekman balance.

We first need to summarize *mixing-length theory:* The velocity has variability down to a minimumm "resolved" time scale. Shorter-time-scale motions ("turbulent eddies") produce momentum forcing, ***F***, via unresolved momentum fluxes or "Reynolds stresses". For example, in the x-equation,

$$F_x = -\frac{1}{\rho}\nabla \cdot (\rho \overline{u'\boldsymbol{V}'}), \tag{5.27}$$

where the prime refers to the short-time-scale fluctuations and the overbar means time averaging.

Most of the turbulent eddy flux convergence near a horizontal boundary is in the vertical:

$$\boldsymbol{F} \approx -(1/\rho)\partial(\rho \overline{\boldsymbol{V}'w'})/\partial z. \tag{5.28}$$

We estimate the anomaly $\boldsymbol{V}'$ by assuming momentum transport over a typical vertical displacement l'; we estimate w' by assuming unit aspect ratio for the turbulent eddies. Thus,

$$\boldsymbol{V}' = -l'\frac{\partial \boldsymbol{V}}{\partial z} \quad \text{and} \quad w'/l' = \left|\frac{\partial \boldsymbol{V}}{\partial z}\right| \tag{5.29}$$

This leads to

$$\boldsymbol{F} = \frac{1}{\rho}\frac{\partial}{\partial z}\left(A\frac{\partial \boldsymbol{V}}{\partial z}\right), \tag{5.30}$$

where $A = \rho \overline{l'^2}|\partial \boldsymbol{V}/\partial z|$, a scalar quantity which should depend only on distance to the surface.

Now we have the Ekman-layer equations for u and v:

$$-f_0(v - v_g) = \frac{1}{\rho}(Au_z)_z. \tag{5.31}$$

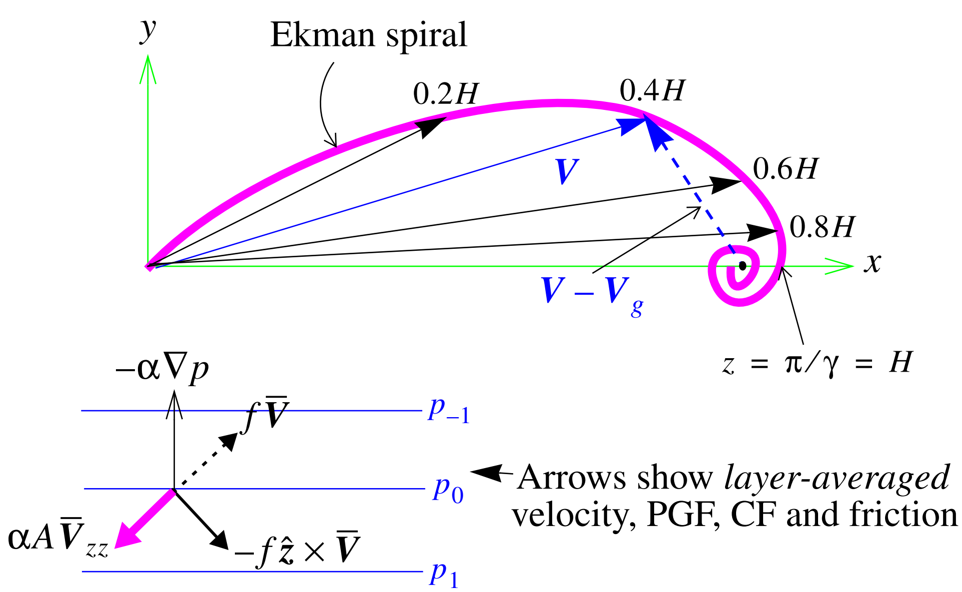
$$f_0(u - u_g) = \frac{1}{\rho}(Av_z)_z. \tag{5.32}$$

These are written in z-coordinates and with the “turbulent eddy diffusion” included as forcing. There is still no predictive content.

To get a solution, assume that nothing but u and v varies with z in the Ekman layer. In the case where $\boldsymbol{V}$ vanishes at the surface (“no slip” at $z = 0$), the solution is

$$\boldsymbol{V} = \boldsymbol{V}_g - \exp(-\gamma z)\mathbf{R}(-\gamma z)\boldsymbol{V}_g \tag{5.33}$$

where $\gamma = \sqrt{f_0\rho/(2A)}$ and $\mathbf{R}(\alpha)$ is the matrix that rotates vectors counterclockwise through an angle α. The upward-decaying solution is chosen so that $\boldsymbol{V}$ matches on to $\boldsymbol{V}_g$ at $z = \infty$.



The deviation $\boldsymbol{V} - \boldsymbol{V}_g$ has a cross-isobaric component (the ageostrophic "Ekman drift") that is mainly to the left of $\boldsymbol{V}_g$: friction weakens the rotational constraint and allows the pressure gradient to do some work (against the friction).

The total ageostrophic mass-flux is

$$\boldsymbol{M} = \rho\int_0^\infty (\boldsymbol{V} - \boldsymbol{V}_g)dz = -\rho\frac{\mathbf{R}(-\pi/4)\boldsymbol{V}_g}{\sqrt{2}\gamma}$$
$$= -\frac{\rho}{2\gamma}(\boldsymbol{V}_g - \hat{z}\times\boldsymbol{V}_g). \tag{5.34}$$

Hence, from an integral of the mass continuity equation, the vertical mass flux out of the Ekman layer is

$$\rho w(\infty) = -\nabla\cdot\boldsymbol{M} = \frac{\rho}{2\gamma}\zeta_g, \tag{5.35}$$

since $\boldsymbol{V}_g$ is non-divergent. This is used as a *lower* boundary condition for the frictionless flow far above the boundary.

5.6 Thermodynamic scaling for the atmosphere

Next we begin developing the PV dynamics for a hydrostatic, *continuously stratified* atmosphere. Scaling the continuity equation in pressure coordinates, we get

$$\frac{V}{L}(\tilde{u}_{\tilde{x}} + \tilde{v}_{\tilde{y}}) + \frac{W}{P}\tilde{\omega}_{\tilde{p}} = 0, \tag{5.36}$$

where W is the scale for ω. The variables are non-dimensional and no bigger than order-unity. Consequently, we should use $W = PV/L$. However, as in the shallow-water model, the non-divergence of the geostrophic velocity will make the resulting $\tilde{\omega}$ of order Ro.

Since we have $\partial\phi/\partial p = -\alpha$ from hydrostatic balance, conservation of entropy should be rewritten in terms of α:

$$\frac{d}{dt}\alpha + \left(\frac{c_v}{c_p}\right)\frac{\alpha}{p}\omega = \frac{Q}{c_pT}\alpha, \tag{5.37}$$

where $Q = c_pTd\log s/dt$, the diabatic heating. With further use of the hydrostatic relation, this can be put in the form

$$\frac{d_h}{dt}\alpha - \frac{N^2}{g^2}\alpha^2\omega = \frac{Q}{c_p T}\alpha, \tag{5.38}$$

where d_h/dt is the material derivative *excluding* vertical advection. Recall that $N^2 = g\partial\log\theta/\partial z$.

We use hydrostatic balance to scale the specific volume: $A = \Phi/P$ for the horizontal variations of α and $A_0 = gH/P$ for the horizontally-averaged α. Note that $A/A_0 = \delta/H$, where $\delta = \Phi/g$ is the scale for the vertical displacements. Also let N_0^2 be a typical value of N^2. Then 5.38 becomes

$$\mathrm{Ro}\frac{d_h\tilde{\alpha}}{d\tilde{t}} - \mathrm{Ri}\mathrm{Ro}^2\tilde{S}\tilde{\omega} = \mathrm{Ro}\kappa\frac{\tilde{Q}}{\tilde{p}}, \tag{5.39}$$

with $\kappa \equiv R/c_p$. The variables are non-dimensional [$\tilde{Q} = Q/(f_0V^2)$; $\tilde{S} = N^2\alpha^2/(N_0^2A_0^2)$] and the new non-dimensional parameter is

$$\mathrm{Ri} = N_0^2H^2/V^2, \tag{5.40}$$

the Richardson number.

Over the troposphere, θ varies by about 10%. Hence $N_0^2 \approx 10^{-4}\mathrm{s}^{-2}$ and $\mathrm{Ri} = 10^2 \approx \mathrm{Ro}^{-2}$. Since $\tilde{\omega}$ is of order Ro, the terms on the lhs of 5.39 are of the same order. The first-order equation is

$$\frac{d_0\tilde{\alpha}_0}{d\tilde{t}} - \tilde{S}_0(\tilde{p})\tilde{\omega}_1 = \kappa\frac{\tilde{Q}}{\tilde{p}}, \tag{5.41}$$

where $\tilde{S}_0$ is the horizontal average of $\mathrm{Ri}\mathrm{Ro}^2\tilde{S}$. Departures from the horizontal average are neglected in the vertical advection because geostrophy keeps the vertical displacements of order Ro compared to H and therefore $A/A_0 \sim \mathrm{O}(\mathrm{Ro})$. As in the shallow-water model, the time derivative uses only the lowest-order (geostrophic) velocity: $d_0/d\tilde{t} = \partial/\partial\tilde{t} + \tilde{\boldsymbol{V}}_0 \cdot \tilde{\nabla}$, where $\tilde{\boldsymbol{V}}_0 = (-\tilde{\phi}_{0\tilde{y}}, \tilde{\phi}_{0\tilde{x}})$

The result in 5.41 demonstrates that the vertical thermodynamic structure of the extratropical troposphere is not controlled entirely by diabatic forcing and small-scale processes (e.g., convection), but by a combination of such effects and internal large-scale dynamics. Without the diabatic forcing, 5.41 is analogous to 5.15.

5.7 Quasi-geostrophic baroclinic model in isobaric coordinates

The first-order (one factor of Ro) terms in the momentum equation are:

$$\frac{d_0}{d\tilde{t}}\tilde{\boldsymbol{V}}_0 = -\hat{\boldsymbol{z}} \times \tilde{\boldsymbol{V}}_1 - \nabla\tilde{\phi}_1 - b(\tilde{y} - \tilde{y}_0)\hat{\boldsymbol{z}} \times \tilde{\boldsymbol{V}}_0, \tag{5.42}$$

where $b = \frac{\beta}{\mathrm{Ro}}$. Eliminate ϕ_1 by cross-differentiating components:

$$\frac{d_0}{d\tilde{t}}\tilde{\zeta}_0 = \frac{\partial\tilde{\omega}_1}{\partial\tilde{p}} - b\tilde{v}_0 . \tag{5.43}$$

Since $\tilde{\zeta}_0$ and $\tilde{\boldsymbol{V}}_0$ have a known relationship to $\tilde{\phi}_0$, 5.43 together with 5.41 and the hydrostatic relation, $\tilde{\phi}_{0\tilde{p}} = -\tilde{\alpha}_0$, forms a closed, time-dependent system for $\tilde{\phi}_0$ and $\tilde{\omega}_1$. This is the quasi-geostrophic system for a continuously stratified fluid.

The dimensional form of the quasi-geostrophic system (5.43 and 5.41), without forcing:

$$\frac{\partial\nabla^2\phi}{\partial t} + u_g\frac{\partial\nabla^2\phi}{\partial x} + v_g\left(f_0\beta + \frac{\partial\nabla^2\phi}{\partial y}\right) - f_0^2\frac{\partial\omega}{\partial p} = 0 . \tag{5.44}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial p}\right) + u_g\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial p}\right) + v_g\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial p}\right) + S\omega = 0 . \tag{5.45}$$

Recall that ϕ is the deviation from a horizontally averaged state, $\bar{\phi}(p)$, and that $S(p) = -\bar{\alpha}d\log\bar{\theta}/dp$, the mean “static stability” of the atmosphere.

(1) Eliminating ω yields a version of Ertel’s potential vorticity equation:

$$\frac{d_g}{dt}\left[\nabla^2\phi + f_0^2\frac{\partial}{\partial p}\left(\frac{1}{S}\frac{\partial\phi}{\partial p}\right)\right] = -\beta\frac{\partial\phi}{\partial x} . \tag{5.46}$$

This expresses conservation, following the geostrophic motion, of the “pseudo-potential vorticity”,

$$q_g \equiv f_0 + \beta y + \zeta_g + f_0\frac{\partial}{\partial p}\left(\frac{1}{S}\frac{\partial\phi}{\partial p}\right) . \tag{5.47}$$

This quantity is subtly different from the “potential vorticity” in that the static stability S is differentiated. Otherwise, there would be linearized *vertical* advection in 5.46. Using 5.46-5.47 makes the vertical velocity technically extraneous in a quasi-geostrophic forecast. Like 5.16, the result 5.46 involves a single unknown, implying a closed dynamical system.

(2) Eliminating $\partial/\partial t$ yields the diagnostic equation:

$$\left(\nabla^2 + \frac{f_0^2}{S}\frac{\partial^2}{\partial p^2}\right)\omega = \frac{f_0}{S}\frac{\partial}{\partial p}[\boldsymbol{V}_g \cdot \nabla(\zeta_g + \beta y)] + \frac{1}{S}\nabla^2\left[\boldsymbol{V}_g \cdot \nabla\left(-\frac{\partial\phi}{\partial p}\right)\right] . \tag{5.48}$$

This is analogous to 5.20 and is known as the *omega equation*.

Problems

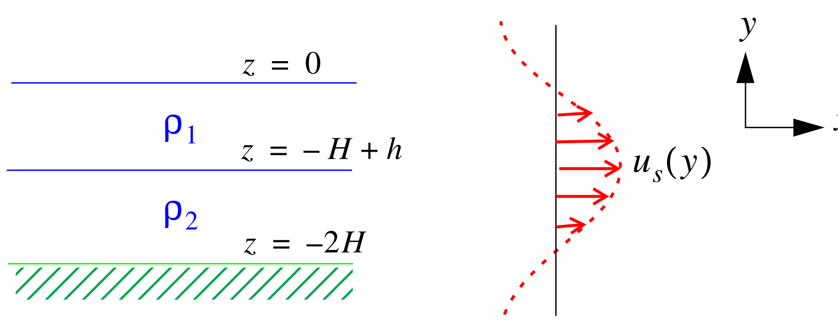
5.1 Show that in a hydrostastic, Boussinesq, planetary-geostrophic model, the potential vorticity is $q_{PG} = f\partial\rho/\partial z$, analogous to 5.21. What is the invertibility principle in this model?

5.2 A certain quasi-geostrophic shallow-water flow on an *f*-plane has $\boldsymbol{V}_g = U\hat{\boldsymbol{x}} + V\sin(kx)\hat{\boldsymbol{y}}$ at time $t = 0$, where k, U and V are constants.

a) Obtain the height field as a function of time by substituting $k(x - ct)$ for the phase. How does the phase speed c behave in the limits $kL_R \ll 1$ and $kL_R \gg 1$.

b) In the same two limits, describe the particle trajectories, including the vertical displacements at the free surface.

5.3 Consider a 2-layer ocean in a rotating frame with $f = \text{const}$, $\beta = 0$. The density ρ_1 of the upper layer is uniform and smaller than that of the lower layer ρ_2. The surface of the ocean is always flat, and the interface between the layers is flat *initially*. Both layers have an initial depth H.



a) The upper layer is adjusted to a steady wind stress such that $u(z = 0) = u_s(y)$. Find the vertical profile of u and v in this layer ($-H < z < 0$) subject to the Ekman equations:

$$-f_0 v = K u_{zz}$$

$$f_0 u = K v_{zz}.$$

Take K to be constant and assume that the total velocity decays away from the surface.

b) Let $u_s = U\cos[l(y - y_0)]$. Using mass continuity, deduce the vertical velocity at the interface. Assume that H is much greater than the depth of the Ekman layer. Use this value to obtain the height perturbation h of the interface as a function of time and latitude.

c) Assuming that the lower layer is not directly subject to frictional forcing, write down the *y*-momentum equation for the lower layer in terms of the height gradient $\partial h/\partial y$. Assume hydrostatic balance.

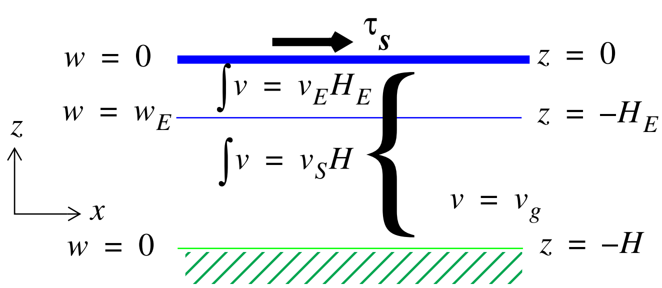
d) Assuming that geostrophic adjustment is fast compared to adjustments in h, obtain the zonal velocity $u(y)$ in the lower layer as a function of time. As a function of the meridional scale of the forcing, how long does it take for this lower-layer flow to become as strong as u_s?

5.4 Consider a homogeneous ocean over a flat bottom forced by a purely zonal wind stress $\vec{\tau}(x, y, z = 0) = \tau_s(y)\hat{\boldsymbol{x}}$. Make the rigid-lid approximation, $w = 0$ at $z = 0$.

a) Show that the Ekman "pumping" is

$$w_E = -\frac{d\tau_s/dy}{f} + \frac{\beta}{f^2}\tilde{\tau}_s,$$

where $\tilde{\tau}_s \equiv \tau_s - f v_g H_E$ and $z = -H_E$ is the depth where the zonal stress, $\tau(z)$, vanishes.



b) Show that this pumping is dynamically consistent with the residual flow in the inviscid interior $(-H < z < -H_E)$ given by

$$v_g(H - H_E) = v_S H - v_E H_E,$$

where $v_S H \equiv -\beta^{-1}(d\tau_s/dy)$, the Sverdrup mass flux, and $v_E H_E \equiv -f^{-1}\tilde{\tau}_s$, the Ekman mass flux.

c) Finally, show that the geostrophic mass flux, $v_g H$, far exceeds the ageostrophic mass flux if the earth's radius a far exceeds the scale L of the forcing.

5.5 A perturbation in the geopotential field on a β-plane ($f = f_0 + \beta y$) is given as

$$\phi = \frac{f_0}{k} V \sin(kx)\sin(ly)\left[1 - \cos\left(\frac{\pi}{2}\frac{p_0 - p}{p_0 - p_1}\right)\right], \; p_1 < p < p_0,$$

where f_0 and V are constants and (k, l) is a horizontal wavenumber vector (total wavenumber $K = \sqrt{k^2 + l^2}$).

a) Assume that the static stability, $S = -\bar{\alpha}\frac{d\log\bar{\theta}}{dp}$, is constant. Find an expression for K in the case where the pseudo-potential vorticity equals $f_0 + \beta y$ at mid-depth, $p = \frac{1}{2}(p_1 + p_0)$. What is $L \equiv K^{-1}$ specifically when $S = 0.1\alpha_0/\Delta p$, assuming $\Delta p = p_0 - p_1 = 10^5$ and $\alpha_0 = 1$ in m.k.s. units? Use $f_0 = 10^{-4}\mathrm{s}^{-1}$ and $V = 10\mathrm{ms}^{-1}$.

b) In the absence of any other flow, how will this disturbance evolve, according to the quasi-geostrophic model?

c) From the "omega equation", determine the quasi-geostrophic divergent circulation, assuming $\omega = 0$ at both $p = p_0$ and $p = p_1$. In what parts of the disturbance would you most expect precipitation to occur?

d) Using the answers to (b) and (c), determine how the amplitude of vertical particle displacements depends on β. Hint: (b) provides a time scale and (c) provides a velocity scale.