REPORT No. 473

STRENGTH TESTS OF THIN-WALLED DURALUMIN CYLINDERS IN COMPRESSION

By Eugene E. Lundquist

SUMMARY

This report is the second of a series presenting the results of strength tests of thin-walled duralumin cylinders and truncated cones of circular and elliptic section. It contains the results obtained from compression tests on 45 thin-walled duralumin cylinders of circular section with ends clamped to rigid bulkheads. In addition to the tests on duralumin cylinders, there are included the results of numerous tests on rubber, celluloid, steel, and brass cylinders obtained from various sources.

The results of all tests are presented in nondimensional form and are discussed in connection with existing theory. In the theoretical discussion, it is shown that the buckling of the walls of a thin-walled cylinder in compression can be correlated with the buckling of flat plates under edge compression in an elastic medium, and that perhaps many solutions for problems in the buckling of plates can, with the proper factors, be applied to similar problems in the buckling of cylinders and curved sheets.

INTRODUCTION

In a stressed-skin or monocoque structure, the strength and stability of the curved skin are closely related to the strength and stability of the walls of a thin-walled cylinder, not only for compression but for other types of loading as well. The National Advisory Committee for Aeronautics, in cooperation with the Army Air Corps; the Bureau of Aeronautics, Navy Department; the Bureau of Standards; and the Aeronautics Branch of the Department of Commerce, made on extensive series of tests on thin-walled cylinders and on truncated cones of circular and elliptic section at Langley Field, Va. In these tests the absolute and relative dimensions of the specimens were varied in order to study the types of failure and to establish useful quantitative data in the following loading conditions: Torsion, compression, bending, and combined loading.

The fist report of this series (reference 1) presents the results obtained in the torsion (pure shear) tests on cylinders of circular section. The present report is the second of the series and presents the results obtained in the compression tests on cylinders of circular section.

In addition to the results of the N.A.C.A. compression tests on duralumin cylinders, there are presented, through the courtesy of Dr. L. H. Donnell of the California Institute of Technology, the unpublished results of 40 compression tests on steel and brass cylinders. There are also included the results of numerous compression tests on rubber, celluloid, and steel cylinders reported in references 2, 3, and 4. The latter two of these references came to the attention of the author after the experiments of the present report had been completed. It is suggested that they be read in conjunction with the present report. The first is largely theoretical; the second, experimental.

By way of introduction to the detailed discussion of the test data herein presented, it may be said that secondary, or local, failure in thin-walled cylinders of circular section under uniform compression seems to have been first investigated by Lilly (1905–07). Since that time Timoshenko, Lorenz, Southwell, and others have studied the problem theoretically. The first theoretical studies were confined to the case of deformation symmetrical with respect to the axis (fig. 1). Later the theory was extended to include the case of deformation not symmetrical with respect to the axis (figs. 2 and 6), but the results of the extensions did not always lead to the same conclusions.

Perhaps the best known of the early treatises on the subject of the stability of thin-walled cylinders in compression is that by Southwell (reference 5), but unfortunately his interpretation of the general equation is incomplete. In reference 2, Robertson shows that Southwell's final equations are invalid for certain cases, and he derives new equations for these cases. In the present report it is shown that Southwell's general equation contains the equations for the buckling of flat and curved plates subjected to edge compression.[1]

Until Robertson made the tests reported in reference 2, there seem to have been no comprehensive tests made to verify the theory. Robertson found that failure occurred by the formation of a multilobed wrinkle

[1] In reference 3 Flügge has presented the general equation in graphical form and has recognized that the buckling of flat plates is a special case of the buckling of a cylinder of infinite radius. Timoshenko also seems to have recognized the same fact in some of his early work, 1914–16.

40768—34——38

pattern, as predicted by his modification of Southwell's final equations, but that both the number of wrinkles, or lobes, forming in the circumference of the cylinder and the stress at failure were less than the theoretical values. The stress at failure was slightly less than the critical stress for the 2-lobed failure predicted by Southwell.

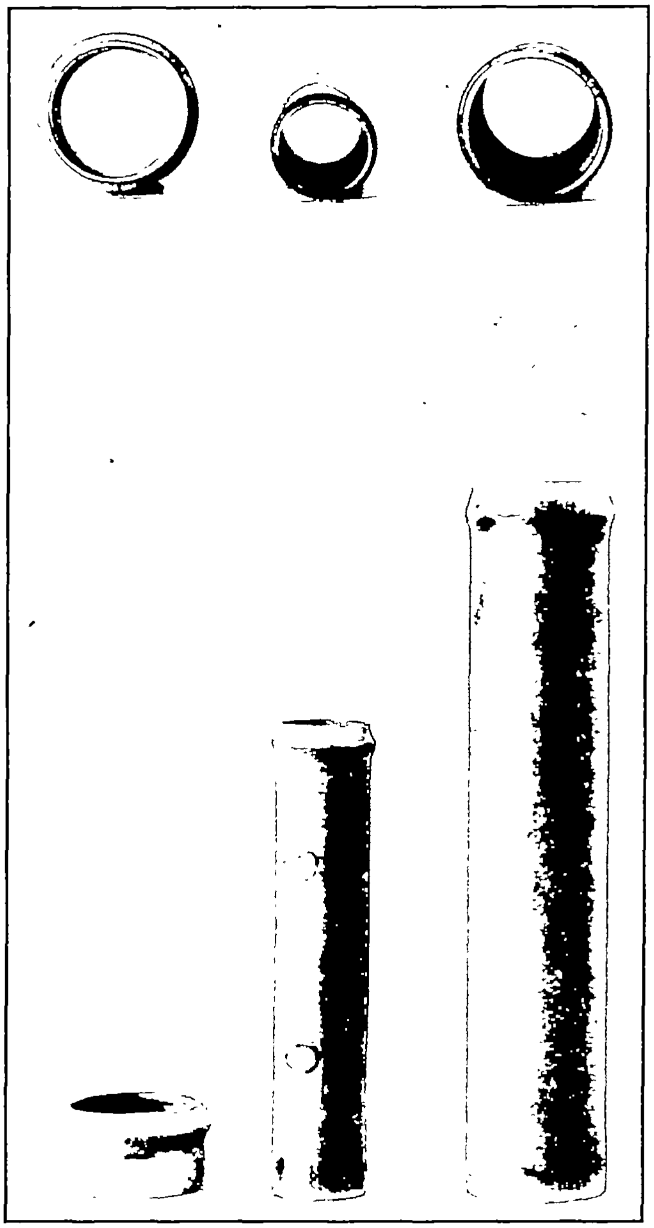


FIGURE 1.—Buckling symmetrical with respect to the axis. (Tubes tested by the Bureau of Standards.)

The results of the N.A.C.A tests herein reported confirm, quantitatively, the results obtained by Robertson. Consequently, in the present report detailed consideration is given to the lack of agreement between theory and experiment.

As an introduction to a study of the strength and behavior of curved sheet and stiffener combinations, a general discussion of the buckling of flat and curved plates under edge compression is given in an appendix.

TESTS ON DURALUMIN CYLINDERS

Material.—The duralumin (Al. Co. of Am. 17ST) used in the N.A.C.A. tests was obtained from the manufacturer in sheet form with nominal thicknesses of 0.011, 0.016, and 0.022 inch. The properties of the material, as determined by the Bureau of Standards from specimens selected at random, are given in table I. Typical stress-strain curves taken parallel and normal to the direction of rolling are given in figures 3 and 4, respectively.

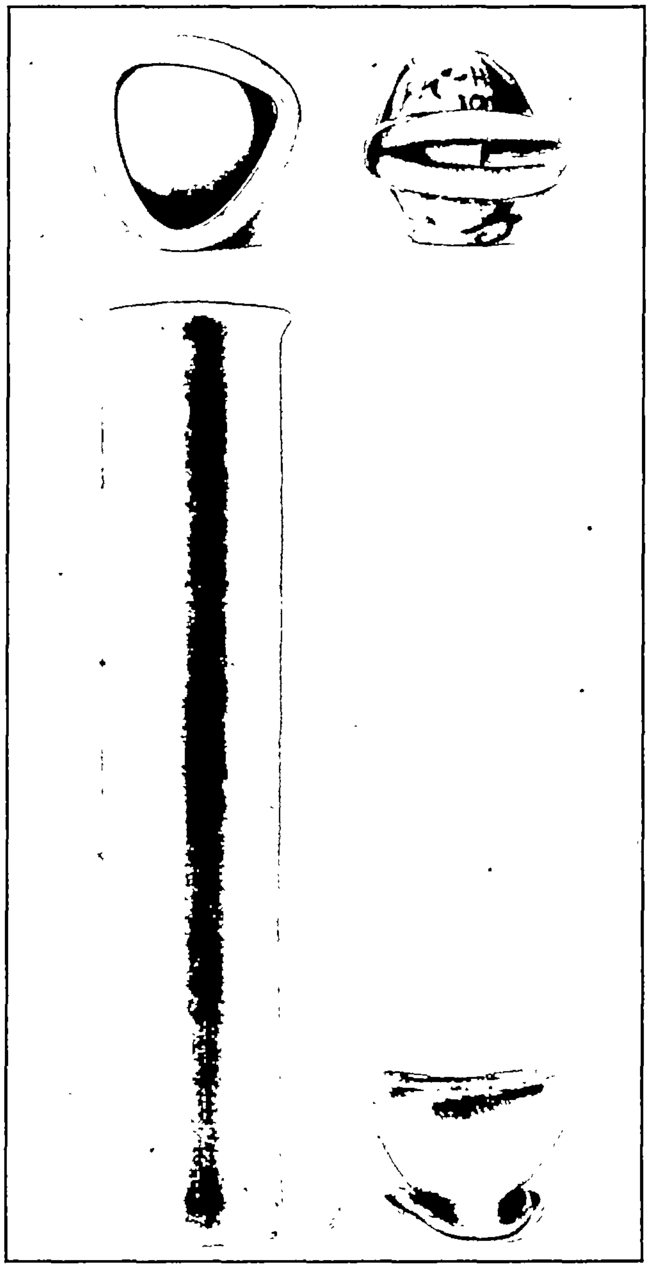


FIGURE 2.—Buckling not symmetrical with respect to the axis. (Tubes tested by the Bureau of Standards.)

In table I and figures 3 and 4, it will be observed that the modulus of elasticity is substantially the same in the two directions of the sheet but that the ultimate strength and yield point are considerably lower normal to than parallel to the direction of rolling. However, as all the cylinders tested failed at stresses considerably below the yield-point stress, the difference in the strength properties in the two directions has no bearing on the results.

Specimens.—The test specimens consisted of right circular cylinders of 7.5- and 15.0-inch radius with lengths ranging from 3.6 to 22.5 inches. The cylinders were constructed in the following manner: First, a duralumin sheet was cut to the dimensions of the developed surface. The sheet was then wrapped about and clamped to the end bulkheads. (See figs. 5 and 6.) With the cylinder thus assembled, a butt strap 1 inch wide and of the same thickness as the sheet was fitted, drilled, and bolted in place to close the seam. In the assembly of the specimen, care was taken to avoid having either a looseness of the skin (soft spots), or wrinkles in the walls when finally constructed.

The end bulkheads, to which the loads were applied, were each constructed of two steel plates one fourth inch thick, separated by a plywood core 1½ inches thick for the bulkheads of 7.5-inch radius, and 3½ inches thick for the bulkheads of 15.0-inch radius. These parts were bolted together and turned to the specified outside diameter. Steel bands approximately one fourth inch thick were used to clamp the duralumin sheet to the bulkheads. These bands were bored to the same diameter as the bulkheads.

Apparatus and method.—The thickness of each sheet was measured to an estimated precision of ±0.0003 inch at a large number of stations, by means of a dial gage mounted in a special jig. In general, the variation in thickness throughout a given sheet was not more than 2 percent of the average thickness. The average thicknesses of the sheets were used in all calculations of radius/thickness ratio and stress.

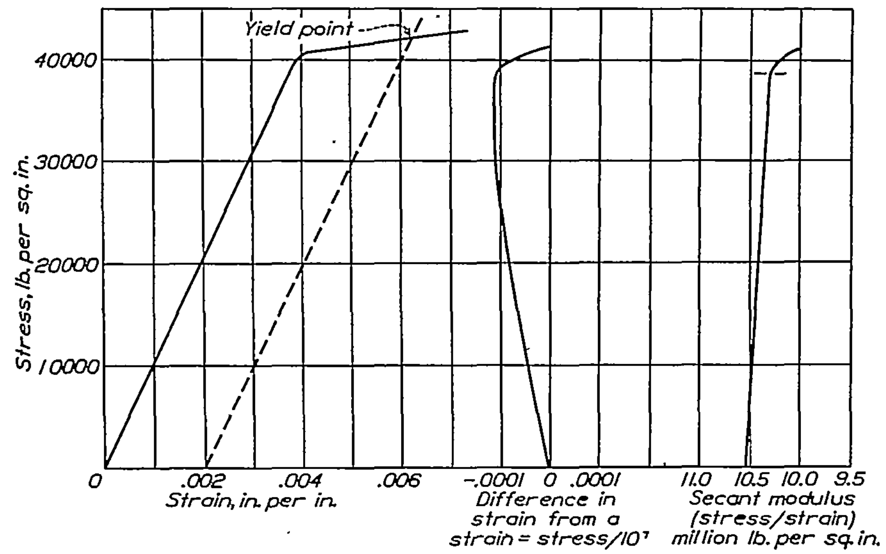


FIGURE 3.—Stress-strain diagram for sheet duralumin 0.011 inch thick, parallel to the direction of rolling.

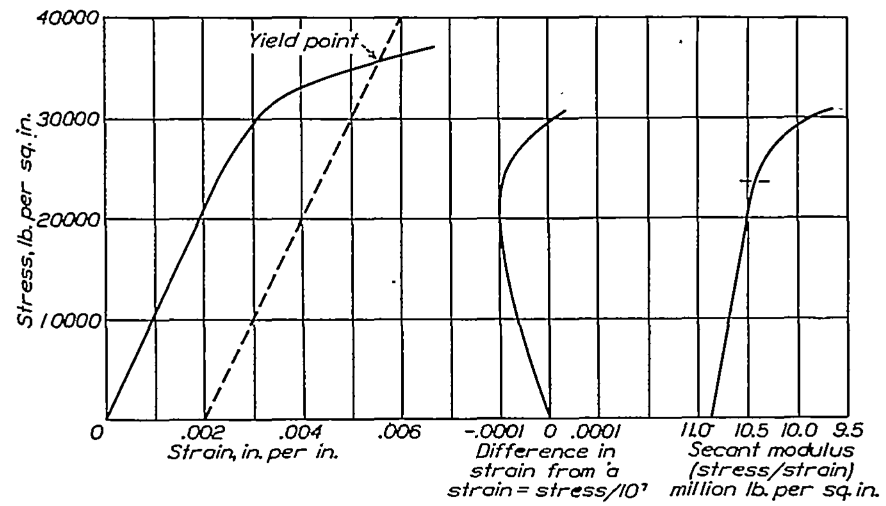


FIGURE 4.—Stress-strain diagram for sheet duralumin 0.011 inch thick, normal to the direction of rolling.

A photograph of the loading apparatus used in the compression tests is shown in figure 5. Loads were applied by the jack in increments of about 1 percent of the estimated load at failure. At first wrinkling, which usually occurred prior to failure, diamond-shaped wrinkles began to form and grew steadily in size and sometimes in number with increase in load until failure occurred by a sudden formation of wrinkles in several circumferential rows. (See fig. 6.) Failure was always accompanied by a loud report and by a reduction in load, which continued with deformation of the cylinder after failure. In all the tests, 5 to 10 minutes elapsed from the time that load was first applied to the specimen until failure occurred.

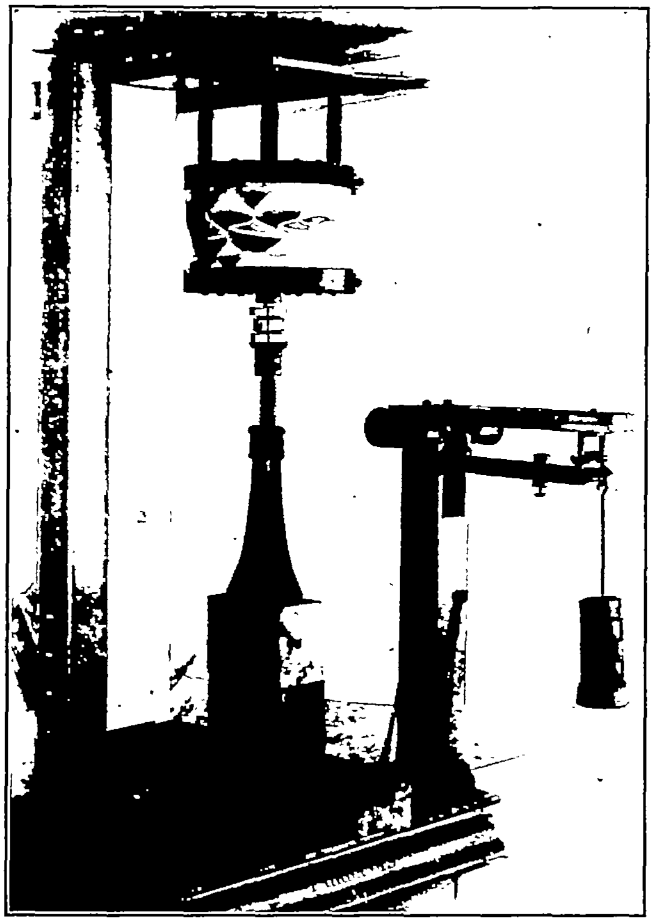


FIGURE 5.—Loading apparatus used in N.A.C.A. compression tests.

With the cylinder loaded at the axis, the 1-inch butt strap at the seam caused a slight eccentricity on all specimens. The effect of this eccentricity was small, however, as there was no tendency for failure to occur consistently on one side of the cylinder.

TESTS ON RUBBER, CELLULOID, STEEL, AND BRASS CYLINDERS

In the compression tests made by Donnell, brass and steel shim stock were used. The sheet that formed the walls of the cylinder was first cut to size and then wrapped about a mandrel and soldered at the seam. In order to stiffen the ends for bearing against the heads of the testing machine, a light metal ring was soldered in place at each end of the cylinder. All the cylinders were tested in a special machine constructed at the California Institute of Technology. For more complete information concerning the material, specimens, and method of testing, the reader is referred to a report by Donnell on the strength of cylinders in torsion (reference 6). The compression cylinders were constructed in the same manner as the torsion cylinders and were tested in the same machine as the medium-length torsion specimens.

For detailed information concerning the tests on rubber, celluloid, and steel cylinders, the reader is referred to the original sources (references 2, 3, and 4). It should be mentioned, however, that the size and type of specimen and the method of testing differed greatly among the various groups of cylinders tested and that some of these differences were responsible for differences in the strengths obtained. These factors are considered in the discussion of the results.

The results of all the test data considered in this report are given in tables II to VI, inclusive, and in figures 7, 8, and 9.

THEORY AND DISCUSSION OF RESULTS

BRIEF RÉSUMÉ OF GENERAL THEORY

By use of the theory of thin shells as applied to a cylinder of infinite length, Southwell derived an equation (equation 98 of reference 5) relating the critical stress, the properties of the material, and the phenomenon of failure. This equation as given by Prescott in a more simplified form (equation (17.126) of reference 7, p. 553) is

$$-q^2\frac{S}{E}[(k^2+q^2)^2+k^2+2q^2+2\sigma q^2] + A\begin{bmatrix}(k^2+q^2)^4+k^4+3k^2q^2+2(1-\sigma)q^4\\ -2k^6-7k^4q^2-(7+\sigma-2\sigma^2)k^2q^4-\sigma q^6\end{bmatrix} + q^4=0 \tag{1}$$

where

$A=\dfrac{t^2}{12(1-\sigma^2)r^2}$

$q=\dfrac{2\pi r}{\lambda_a}$

$k=\dfrac{2\pi r}{\lambda_c}$, an integer

r, radius of cylinder

t, thickness of cylinder wall

σ, Poisson's ratio

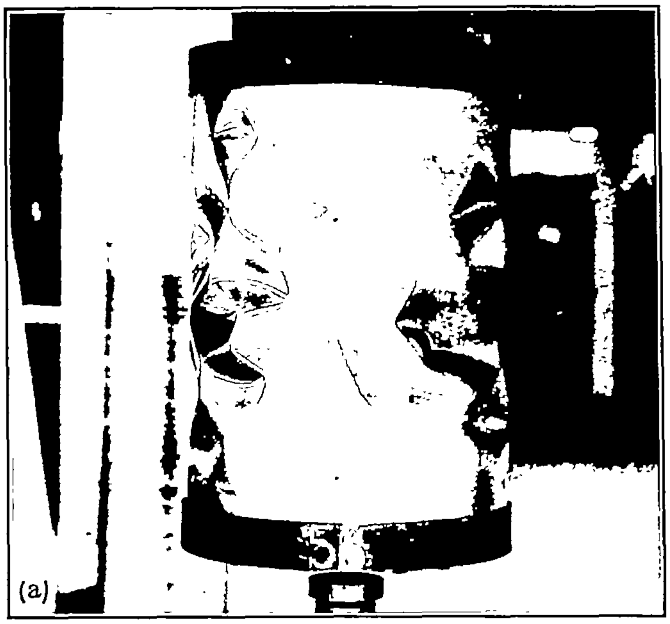
E, modulus of elasticity

S, critical stress

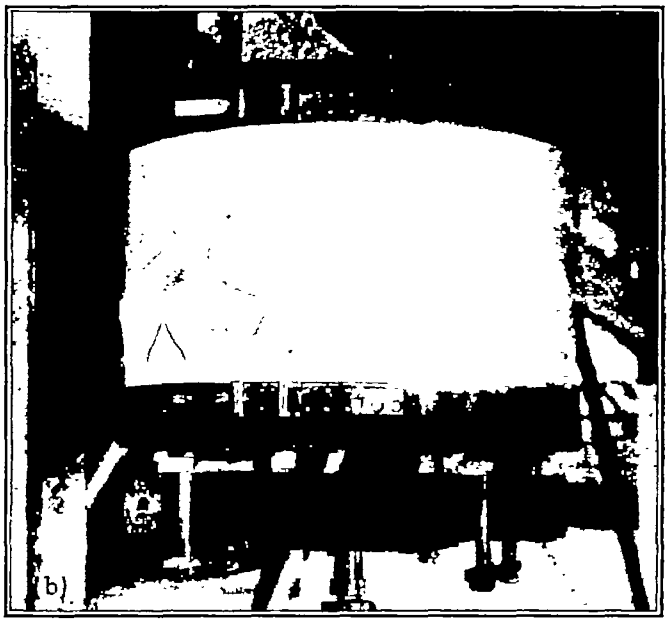
λ_a and λ_c, wave lengths of the wrinkles in the direction of the axis and circumference of the cylinder, respectively.

In his interpretation of the general equation, Southwell reasoned that for a lobed type of failure ($k>1$) q must be small if S/E is to have a value possible in practice. Upon the basis of this assumption, Southwell wrote as an approximation for equation (1):

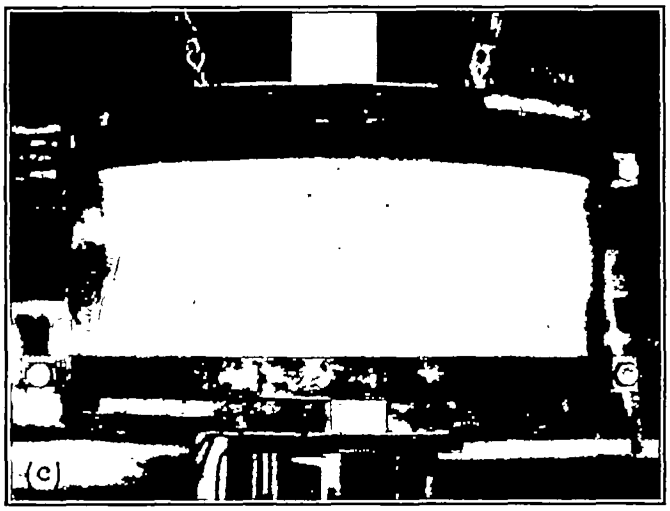
$$\frac{S}{E}=\frac{q^2}{k^2(k^2+1)}+\frac{Ak^2(k^2-1)^2}{(k^2+1)q^2} \tag{2}$$



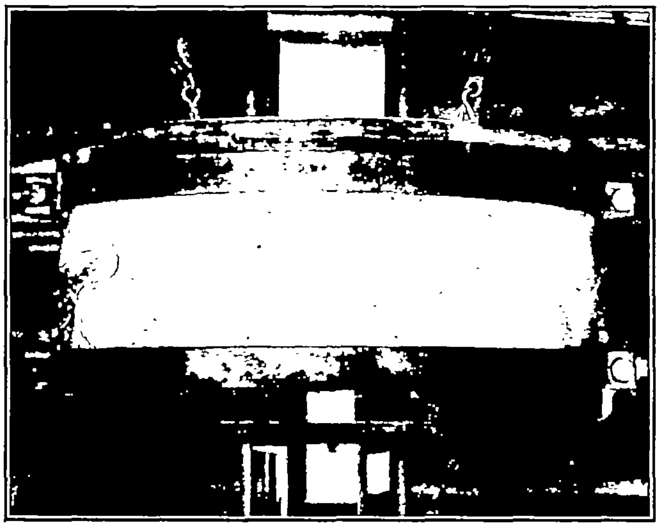
(a) $r=7.5$ in.; $\frac{l}{r}=2.50$; $\frac{r}{t}=646$



(b) $r=15.0$ in.; $\frac{l}{r}=1.00$; $\frac{r}{t}=920$



(c) $r=15.0$ in.; $\frac{l}{r}=0.63$; $\frac{r}{t}=1{,}415$



(d) $r=15.0$ in.; $\frac{l}{r}=0.50$; $\frac{r}{t}=711$

FIGURE 6.—Cylinders after failure, N.A.C.A. tests on duralumin cylinders.

In reference 2, Robertson shows Southwell's reasoning to be in error, and that for a lobed type of failure, a possible value of S/E may occur with a large value of q. Upon the assumption that q is large, Robertson wrote as an approximation for equation (1):

$$\frac{S}{E}=\frac{1}{\alpha^2}+A\alpha^2 \tag{3}$$

where

$$\alpha=\frac{k^2+q^2}{q}$$

In references 2 and 5 it is not clear as to which absolute values of q may be regarded as small and which as large. An inspection of equation (1) reveals that any given value of q may be regarded as small or large, depending upon the value of k. From equation (3) it is evident that Robertson regarded values of q approximately equal to k as large. It would therefore seem better to say that equation (2) is an approximation for equation (1) when q/k is small, and that equation (3) is an approximation for it when q/k is not small. In any event it is desirable to examine the accuracy and limitations of equations (2) and (3).

If $q=\epsilon k$ and $\sigma=0.3$, the quantities in the brackets in the first and second terms of equation (1), together with the Southwell and Robertson approximations for them, became

C, correct values

First term $[k^4(1+\epsilon^2)^2+k^2(1+2.6\epsilon^2)]$

Second term $\begin{bmatrix} k^8(1+\epsilon^2)^4+k^4(1+3\epsilon^2+1.4\epsilon^4) \\ -k^6(2+7\epsilon^2+7.1\epsilon^4+0.3\epsilon^6) \end{bmatrix}$

S, Southwell approximation ($\epsilon=\frac{q}{k}$, small)

First term $[k^4+k^2]$

Second term $[k^8+k^4-2k^6]$

R, Robertson approximation ($\epsilon=\frac{q}{k}$, not small)

First term $[k^4(1+\epsilon^2)^2]$

Second term $[k^8(1+\epsilon^2)^4]$

In table VII numerical values of the preceding quantities are tabulated for values of ϵ ranging from 0.01 to 10 and values of k from 1 to 25, the case $k=0$ being omitted in the table because when $k=0$, q is infinitely large with respect to k. Robertson's approximation, which includes only the highest powers of q in each term of equation (1), is therefore valid for $k=0$.[2]

From table VII, it is evident that for each value of ϵ there is a value of k that cannot be exceeded if q/k is to be regarded as small. For values of k given above the horizontal lines in table VII, Southwell's approximation (q/k small) is preferable to Robertson's approximation (q/k not small). For values of k given below the horizontal lines, Robertson's approximation is equally as good as, or preferable to, Southwell's.

PRIMARY FAILURE OR COLUMN ACTION (k=1)

According to the theory (reference 5 or 7), the case $k=1$ corresponds to a buckling of the cylinder as a whole in a manner similar to primary failure in columns. This type of failure is characteristic only of a long slender tube where the wave length λ_a is large compared to the circumference ($2\pi r$). It therefore follows that for $k=1$, both q and the ratio q/k must be small and Southwell's approximation applies. Thus, putting $k=1$ in equation (2) leads to the Euler column formula,

$$\frac{S}{E}=\frac{q^2}{2}=\frac{1}{2}\left(\frac{2\pi r}{\lambda_a}\right)^2$$

Since $\frac{\lambda_a}{2}=l$, the length of a pin-ended column, and $\frac{r}{\sqrt{2}}=\rho$, the radius of gyration of a thin-walled tube,

$$S=\frac{\pi^2 E}{\left(\frac{l}{\rho}\right)^2}$$

In terms of the total load this latter equation becomes

$$P=\frac{\pi^2 EI}{l^2}$$

[2] The statement that Robertson's approximation is valid for $k=0$, is made on the assumption that $q>1$. In the end it can be shown that when $k=0$ all practical values of q are greater than 1.

SECONDARY FAILURE OR LOCAL BUCKLING

[All values of k other than $k=1$]

All values of k other than 1 correspond to failure by local buckling of the cylinder walls. Figure 1 shows the type of failure for $k=0$. Figures 2 and 6 show failures that correspond to $k>1$.

It will be recalled that equations (2) and (3) are merely approximations for the general equation relating the critical stress, the properties of the material, and the phenomenon of failure. For a test cylinder that is free to buckle into any wrinkle pattern, instability will occur at the minimum value of the critical stress. Thus differentiating S with respect to q as the independent variable in equation (2), it is found that Southwell's approximation gives:

$$\frac{S_{min}}{E}=\frac{k^2-1}{k^2+1}\,2\sqrt{A}=\frac{k^2-1}{k^2+1}\sqrt{\frac{1}{3\,(1-\sigma^2)}}\,\frac{t}{r} \qquad (4)$$

when

$$q^4=k^4\,(k^2-1)^2\,A=\frac{k^4\,(k^2-1)^2\,t^2}{12\,(1-\sigma^2)\,r^2} \qquad (5)$$

In a similar manner, differentiating S with respect to α in equation (3), the corresponding equations for Robertson's approximation are:

$$\frac{S_{min}}{E}=2\sqrt{A}=\sqrt{\frac{1}{3\,(1-\sigma^2)}}\,\frac{t}{r} \qquad (6)$$

when

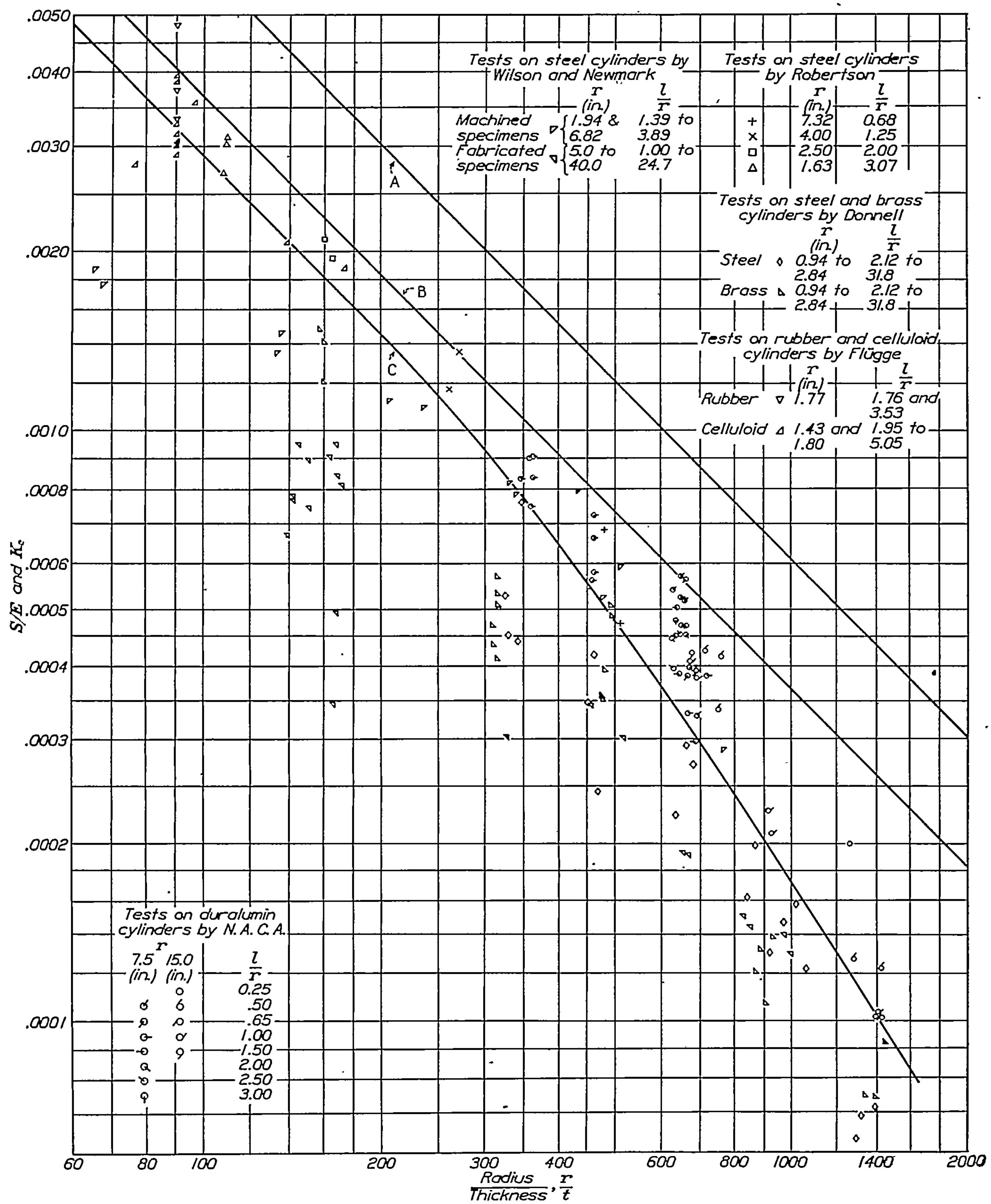
$$\alpha=\frac{k^2+q^2}{q}=\sqrt[4]{\frac{1}{A}}=\sqrt[4]{12\,(1-\sigma^2)}\sqrt{\frac{r}{t}} \qquad (7)$$

It will be noted from equations (4) and (6) that when q/k is small, the critical stress is $\frac{k^2-1}{k^2+1}$ times the critical stress when q/k is not small. As the ratio $\frac{k^2-1}{k^2+1}$ is always less than unity for $k>1$, the critical stress for the formation of lobes as given by Southwell is always less than the critical stress given by Robertson. Consequently, failure should occur in the manner predicted by Southwell unless the dimensions of the test cylinder are such that it is physically impossible for q/k to be small.

For a cylinder of infinite length in which failure of the Euler type is prevented it is certainly possible for q/k to be small. In such a cylinder, buckling should occur at a stress equal to the smallest value of S_{min} given by equation (4) or when $k=2$, the minimum value of k for Southwell's approximation (table VII). For $k=2$, the ratio $\frac{k^2-1}{k^2+1}=0.6$, and equation (4) becomes

$$\frac{S_{min}}{E}=0.6\sqrt{\frac{1}{3\,(1-\sigma^2)}}\,\frac{t}{r} \qquad (8)$$

From Southwell's treatment of the buckling of a cylinder of infinite length, a section of length $\lambda_a/2$ may be regarded as equivalent to a cylinder of finite



Curve A, Robertson cylinders; graph of equation (6) with $\sigma = 0.3$
Curve B, Cylinders of infinite length, graph of equation (8) with $\sigma = 0.3$
Curve C, Lower limit of test data for the most perfect test specimens

*FIGURE 7—Logarithmic plot of S/E and K_c against r/t.

length with hinged ends. Thus for a test cylinder with hinged ends and of such length that the Euler type of failure does not occur, $\lambda_a/2$ may approach the length of the cylinder and q may be as small as $\pi r/l$, where l is the length of the cylinder. If the value of k in equation (5) that corresponds to this value of q is greater than the value of k given by the horizontal line in table VII for the ratio of q/k obtained from equation (5), it is physically impossible for q/k to be small, as Southwell assumed; therefore Robertson's equations based on q/k not small, apply. In this report a cylinder of such dimensions that it is impossible for q/k to be small is designated a "Robertson" cylinder, in contrast to a "Southwell" cylinder in which it is possible for q/k to be small.

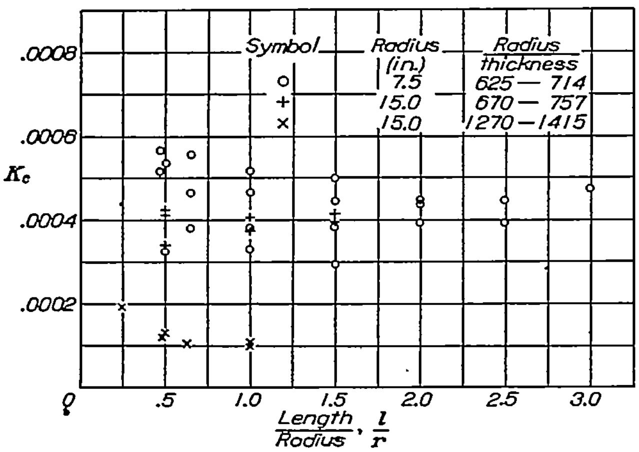


FIGURE 8.—Plot of K_c against length/radius ratio, N.A.C.A. tests on duralumin cylinders.

Although Southwell's theory does not apply to a cylinder with ends clamped to rigid bulkheads, it is reasonable to assume that for such a cylinder the effective value of $\lambda_a/2$ is less than l. Consequently, if it be assumed that $\lambda_a/2=l$ and the test cylinder under consideration comes within the range of Robertson's approximation as outlined in the preceding paragraph, it may certainly be classed as a Robertson cylinder.

For the duralumin cylinders tested by the N.A.C.A. and the steel cylinders tested by Robertson and reported in reference 2, the lengths ranged from approximately $0.25r$ to $3.0r$. Consequently, the smallest value of q that could occur in these cylinders is $\frac{\pi r}{3r}$ or 1, approximately. From figure 10, where equation (5) is presented in graphical form, it is found that for $q=1$ and $\frac{r}{t}=100$, $k=4$, approximately, and hence $\frac{q}{k}=\frac{1}{4}$, approximately. Since 4 is below the horizontal lines in table VII for $\frac{q}{k}=0.2$ and 0.4, it is impossible for $\frac{q}{k}$ to be small in this cylinder; hence, it is classed as a Robertson cylinder. In a similar manner, it can be shown that all the test cylinders of these two groups are classed as Robertson cylinders and, so far as stressed-skin or monocoque structures are concerned, the radius, thickness, and spacing of bulkheads are such that any part of the structure is always a part of a Robertson cylinder. Consequently, the significance of Robertson's equations will be discussed in considerable detail.

CLOSE RELATION BETWEEN THE BUCKLING OF A ROBERTSON CYLINDER AND THE BUCKLING OF A FLAT PLATE

Multiplication of the right-hand side of equation (6) by $\alpha^2\sqrt{A}$ (which is equal to unity, by equation (7)) gives

$$\frac{S_{min}}{E}=2A\left[\frac{k^2+q^2}{q}\right]^2$$

from which

$$S_{min}=\frac{8\pi^2Et^2}{12(1-\sigma^2)\lambda_c^2}\left[\frac{\lambda_a}{\lambda_c}+\frac{\lambda_c}{\lambda_a}\right]^2$$

On putting $\lambda_a=2a$ and $\lambda_c=2b$

$$S_{min}=\frac{2\pi^2Et^2}{12(1-\sigma^2)b^2}\left[\frac{a}{b}+\frac{b}{a}\right]^2 \qquad (9)$$

Inspection of equation (9) reveals that the critical stress for a Robertson cylinder is twice the critical stress for a flat plate that buckles to form waves or wrinkles of the same size as form in the cylinder (reference 8). In reference 9 it is shown that the critical load or stress for a long strut that buckles in an elastic medium [3] is also twice the critical load or stress for the same type of buckling without the lateral support of the medium. Consequently, by analogy, the buckling of a Robertson cylinder may be regarded as equivalent to the buckling of a flat plate under edge compression in an elastic medium.

PHENOMENON OF FAILURE

It will be noted that for a Robertson cylinder, equation (7) does not define a particular wrinkle pattern, but rather a family of wrinkle patterns. For $k=0$ ($\lambda_c=\infty$), the walls of the cylinder form circumferential bulges or corrugations (fig. 1). For $k\geqq 2$, diamond-shaped or wavelike wrinkles of various dimensions form. (See figs. 2 and 6.) If equation (7) is solved for q it will be found that two values of q are associated with each value of k, one smaller and the other larger than k (reference 2, p. 598). In order to determine which of the many wrinkle patterns described by equation (7) is most likely to occur, it is necessary to examine the condition of buckling for each.

Although equation (7) shows that buckling may occur by the formation of circumferential corrugations, this type of failure is not likely to occur in preference

[3] An elastic medium is assumed to provide lateral resistance to buckling. The resistance is distributed along the length of the column and is proportional to the lateral deflection.

to the formation of diamond-shaped or wavelike wrinkles. When buckling begins, circumferential tensions and compressions that are set up in the crests and troughs of the corrugations reinforce the longitudinally stressed elements and prevent complete failure until the yield point of the material is reached or exceeded. In fact, it is doubtful if this type of failure, known in the theoretical literature by the name "deformation symmetrical with respect to the axis", has ever occurred except in tubes of small radius/thickness ratio where the stresses have equaled or exceeded the yield point. (See reference 2.)

If failure occurs by the formation of diamond-shaped or wavelike wrinkles, it might be expected, by analogy to the buckling of flat plates, that the wave lengths of the buckles in the direction of the axis and circumference of the cylinder will be equal. Differentiation of k with respect to q in equation (7) shows that the maximum

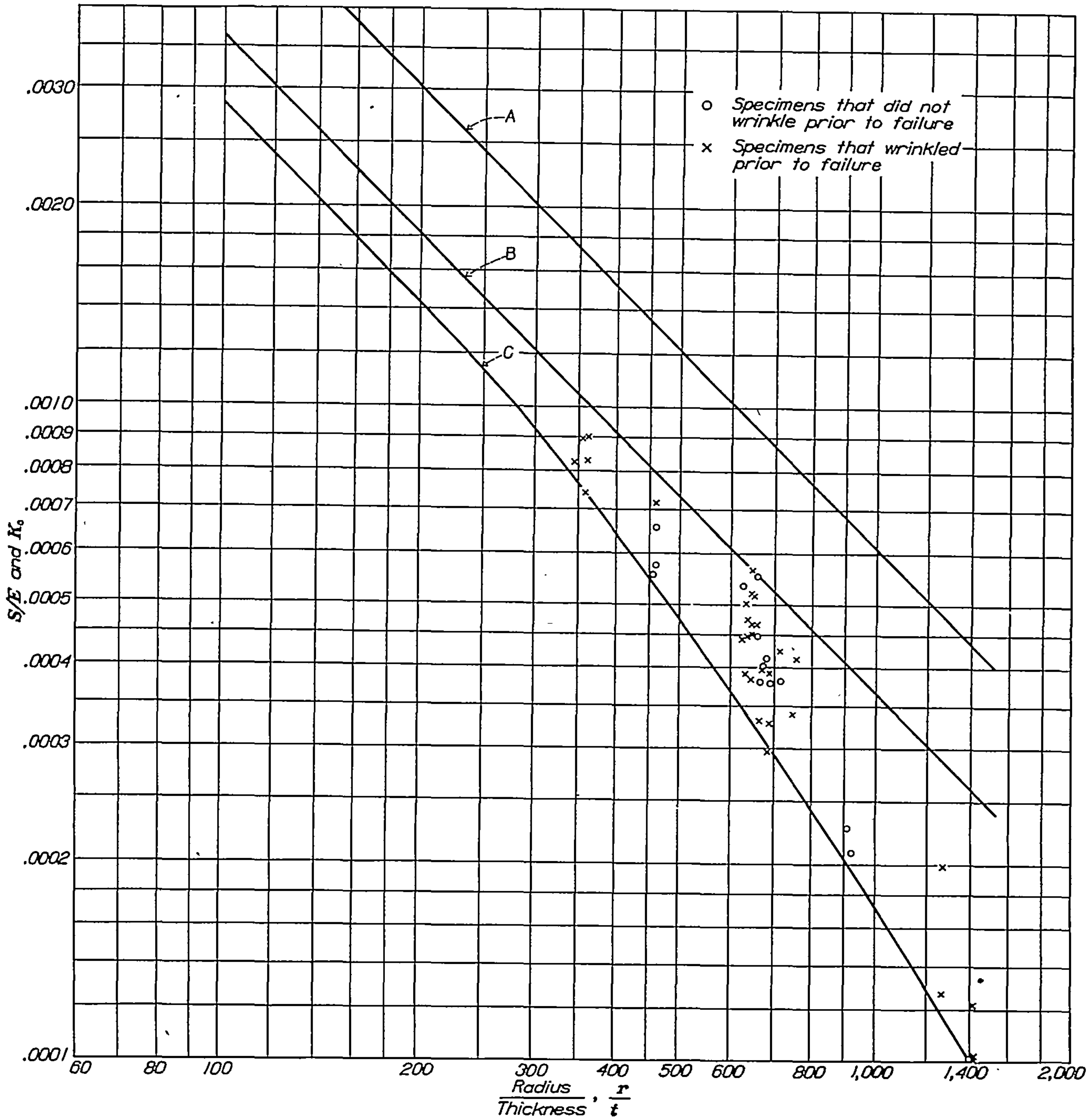


FIGURE 9.—Effect of wrinkling prior to failure on the strength of thin-walled cylinders in compression. Tests by N.A.C.A. Curves A, B, and C obtained from figure 7.

value of k is obtained when $k=q$ or $\lambda_a=\lambda_c$. Thus, if $\lambda_a=\lambda_c$, equation (7) gives

$$k_{max}=\frac{1}{2}\sqrt[4]{\frac{I}{A}}=\sqrt[4]{\frac{3}{4}(1-\sigma^2)}\sqrt{\frac{r}{t}} \qquad (10)$$

Examination of the graphical presentation of the general equation relating the critical stress with the properties of the material and the phenomenon of failure as given in figure 16 of reference 3 shows a gradual reduction of S_{min} and k with a decrease in q. Thus for both Southwell and Robertson cylinders it might be expected that the number of wrinkles will be given by equations (5) and (7), respectively, if q has the smallest ($\lambda_a/2$ the largest) value consistent with the length and end conditions of the test cylinder.

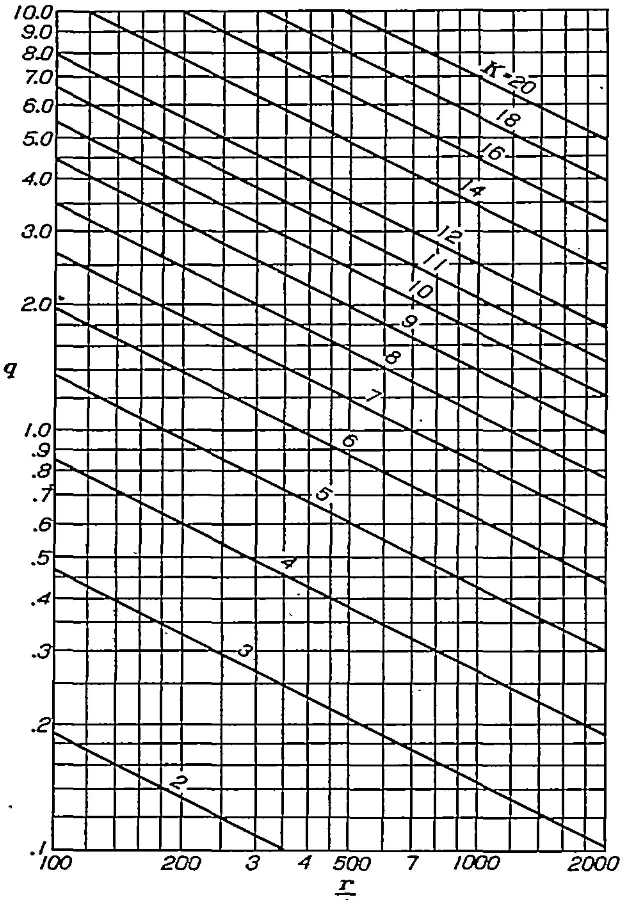


FIGURE 10.—Graphical solution of equation (5),

$$q^4=\frac{k^4(k^2-1)^2t^2}{12(1-\sigma^2)r^2}$$

(curves drawn for $\sigma=0.3$)

From the photographs of cylinders after failure (fig. 6), it is evident that λ_a tends to be equal to λ_c. However, upon reference to tables II to VI inclusive, where calculated and experimental values of k are tabulated for comparison, it is observed that the number of wrinkles that form in the circumference of the cylinder is always less than k_{max}. Except for short cylinders ($l/r<1.0$), the experimental values are generally greater than the smallest values of k calculated when assuming $\lambda_a/2=l/2$ and $\lambda_a/2=l$ in equation (5) or (7). For cylinders of a given radius/thickness ratio, the experimental values of k vary inversely with l/r and approach a constant value for the larger values of l/r. These observed facts are explained as follows:

When buckling occurs by the formation of diamond-shaped or wavelike wrinkles, the walls of the cylinder subdivide into a series of curved plates of length a and width b simply supported along the adjacent edges. According to the equations of the deflected surface (equation (56), reference 5; or equation (17.116), reference 7), these edges do not move out of the plane of the cylinder wall. However, when buckling begins there is no positive support provided to prevent a displacement normal to the cylinder wall. Consequently, the edges of the curved plates behave like Euler columns and buckle, with the result that the cylinder collapses. If the type of buckling predicted by the theory actually took place so that the edges of the curved plates remained in the plane of the cylinder wall, a collapse of the cylinder would not occur and the curved plates would buckle in a manner similar to the corresponding flat plate simply supported at the edges. The load carried by the cylinder would then increase to a maximum value that is dependent upon the properties of the material in the same manner as the load for a flat plate increases to a maximum. (See references 8 and 10.)

From the preceding description of the behavior of a test cylinder during the process of failure, it is evident that the wrinkle pattern assumed by the cylinder when it collapses is dependent upon a type of buckling not described by the equations of the theory. It is, therefore, to be expected that the number of wrinkles that actually form may differ from the theoretical number, although the two values may be related in some manner.

For example, in a few of the N.A.C.A. tests where preliminary wrinkles formed prior to failure, the wave lengths were very nearly equal to $\frac{2\pi r}{k_{max}}$. However, upon continued loading the wrinkles grew steadily in size until failure occurred.

In another case a tube ($r=0.607$ in.; $r/t=15.7$; $l/r=10.6$) was observed that developed what would correspond to a 2-lobed deformation of the Southwell type, but failure occurred at one end by the formation of three or four lobes of the Robertson type.

STRESS AT FAILURE IN TEST CYLINDERS

In figure 7 the results of all the tests are plotted logarithmically, S/E against r/t, together with the graphs of equations (6) and (8). It will be observed from this figure that except for a few of the tests made by Flügge, all of the tests made by Flügge, Robertson, and the N.A.C.A. plot between curves B and C. It will also be observed that for these tests, the stress at failure approaches as a maximum the critical stress for a cylinder of infinite length, and that the results of the tests scatter widely and depart from the theoretical straight line at large values of the radius/thickness ratio. This departure from the theoretical stress relation is caused by imperfections and eccentricities in the elements of the cylinder which, relatively speaking, increase with increase in the radius/thickness ratio.

The results of the tests on machined cylinders by Wilson and Newmark (reference 4) plotted in figure 7 are in agreement with the results discussed in the preceding paragraph, except for radius/thickness ratios less than 235 where the stresses obviously approached or exceeded the proportional limit and yield point of the material. In fact, there were additional tests made by Wilson and Newmark on machined specimens, the results of which are not included in the present report because the stresses developed were so high that yielding of the material must have occurred.

The results of the tests made by Wilson and Newmark on fabricated steel cylinders and the results of the tests made by Donnell on steel and brass cylinders plot below curve C in figure 7. In the tests by Wilson and Newmark the cylinders were loaded directly between the heads of a testing machine. The ends of the cylinder were held circular by means of either a steel angle or a wood ring, but contact with the heads of the machine was made by the shell alone. This fact, together with the fact that fabricated cylinders with riveted or welded seams are inherently less perfect than machined cylinders, may possibly account for the reduced strength of the Wilson and Newmark cylinders.

In the case of Donnell's tests, the reduced strength may possibly have been caused by imperfect cylinders. Obviously, in the construction of cylinders with shim stock of the small dimensions used by Donnell, greater skill is required to obtain a given degree of perfection than would be required for larger cylinders of thicker material.

Upon reference to the tests on the large fabricated cylinders of series E in table VI, it will be observed that a change from welded to riveted seams has but little effect upon the stress at failure.

It should be mentioned here that the fabricated cylinders tested by Wilson and Newmark were built of structural material such as might be used in bridges, buildings, or ships, whereas the cylinders tested by Donnell were built of material much thinner than that used even in aircraft structures. As a consequence, the test data should have a wide range of usefulness in various fields of engineering, particularly in view of the fact that the results are in agreement when plotted nondimensionally.

In view of the wide differences between the results of the various tests, it is concluded that for design purposes the compressive stress at failure is best given by an equation of the most general form

$$S_c = K_c E \tag{11}$$

where K_c is a nondimensional coefficient that varies with the dimensions and imperfections of the cylinder. Except for very short cylinders the radius and thickness as expressed by the ratio r/t are the only dimensions of the cylinder that need be considered because the large effect of slight imperfections and eccentricities in the elements of the cylinder completely overshadow the small effect of length. (See fig. 8.) Consequently, the designer who uses these data must estimate the degree of perfectness of his particular design by a proper choice of the value of K_c from figure 7. In general, design values for K_c will be less than the values given by curve C which represents the lower limit of the results obtained with the most perfect laboratory test specimens.

The results of the N.A.C.A. tests plotted in figure 7 are replotted in figure 9, where distinction is made regarding those points representing cylinders in which wrinkling occurred prior to failure. From this figure it is concluded that preliminary wrinkling did not apparently reduce the stress at failure.

Why the critical stress for the Robertson cylinders tested approached as a maximum, but did not exceed, the critical stress for a cylinder of infinite length, is difficult to explain. It may be that the type of buckling for a collapse of the cylinder is associated with the lower stress, or that when the stress given by equation (8) is reached a condition of instability is obtained and the cylinder deforms slightly as a part of a cylinder of infinite length. Then, since the cylinder is so sensitive to slight imperfections and eccentricities, failure is precipitated at the critical stress for a cylinder of infinite length, and the wrinkles assume a pattern consistent with the dimensions of the cylinder. This latter explanation of the inability to develop stresses greater than the critical stress for a cylinder of infinite length seems reasonable because it is impossible to construct a mathematically perfect cylinder. Thus, a slight deformation of the test cylinder may be expected at the critical stress for a cylinder of infinite length, and the upper limit of perfectness is established by this fact.

EFFECT OF END BULKHEADS

In view of the fact that the stress at failure is independent of length ($l/r > 0.5$, fig. 8) and approaches but does not exceed the critical stress for a cylinder of infinite length (fig. 7), it is concluded that the end bulkheads contribute little, if any, toward the stability of the cylinder walls. This conclusion appears reasonable so long as the length of the cylinder is greater than several times the smallest value of $\lambda_a/2$ likely to be associated with the type of failure characteristic of the cylinder under consideration. For the test cylinders considered in this report, the smallest value of $\lambda_a/2$ is not likely to exceed the value corresponding to the condition $k=q$ in equation (7) or

$$\frac{\lambda_a}{2}=\frac{\pi r}{k_{max}}=\frac{\pi r}{\sqrt[4]{\frac{3}{4}(1-\sigma^2)}\sqrt{r/t}} \qquad (12)$$

In table VIII values of $\lambda_a/2r$ obtained from equation (12) are tabulated for comparison with the smallest length/radius ratios of the test cylinders considered in this report. An examination of this table in connection with figures 7 and 8 shows that for practical purposes, if l/r is greater than three to five times the value of $\lambda_a/2r$ obtained from equation (12), bulkheads have no effect upon the stress at failure.

EFFECT OF LONGITUDINAL STIFFENERS

From consideration of the effect of bulkheads on the strength of thin-walled cylinders in compression, it follows by parallel reasoning that longitudinal stiffeners will contribute little toward the stability of the cylinder walls if spaced at a distance greater than several times the smallest value of $\lambda_c/2$ likely to be associated with elastic buckling of the walls. Since the smallest values of $\lambda_c/2$ are obtained when $k=q$ in equation (7), this minimum value of $\lambda_c/2$ is equal to the value of $\lambda_a/2$ given by equation (12). With longitudinal stiffeners, however, failure as characterized by a collapse of the cylinder is delayed until the stiffeners fail. The ultimate load supported by a cylinder with longitudinal stiffeners may therefore greatly exceed the load at which buckling begins.

CONCLUSIONS

1. For thin-walled cylinders in which failure occurs by elastic buckling of the walls, the stress at failure (collapse of the cylinder) is best given by an equation of the form

$$S_c=K_cE$$

where K_c is a nondimensional coefficient that varies with the dimensions and imperfections of the cylinder. Except for very short cylinders, the radius and thickness as expressed by the ratio r/t are the only dimensions that need be considered.

2. Wrinkling prior to failure does not apparently reduce the stress at failure.

3. For large fabricated cylinders, a change from welded to riveted seams has but little effect upon the stress at failure.

4. After the cylinder has failed, the wave lengths of the wrinkles in the direction of the axis and circumference are equal. The number of wrinkles that form in the circumference varies inversely with the length/radius ratio and for a given radius/thickness ratio seems to approach a constant value at the larger values of l/r.

5. The compressive stress at failure is independent of length so long as the length of the cylinder is greater than three to five times the value of $\lambda_a/2$ given by the equation

$$\frac{\lambda_a}{2}=\frac{\pi r}{k_{max}}=\frac{\pi r}{0.91\sqrt{r/t}}$$

In the preceding conclusions, a cylinder whose length is less than from three to five times the value of $\lambda_a/2$ given by the above equation is designated "a very short cylinder."

6. Bulkheads, or transverse stiffeners, to be effective in preventing failure by elastic buckling of the walls, hence in strengthening the walls, should be spaced at a distance less than from three to five times the value of $\lambda_a/2$ given by the above equation. The same conclusion applies for longitudinal stiffeners, except that failure as characterized by a collapse of the cylinder would be delayed until the stiffeners failed.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics.
Langley Field, Va., *June 10, 1933.*

APPENDIX

GENERAL DISCUSSION OF THE BUCKLING OF FLAT AND CURVED PLATES UNDER EDGE COMPRESSION

EQUATIONS FOR THE BUCKLING OF A FLAT PLATE UNDER EDGE COMPRESSION IN AN ELASTIC MEDIUM

The critical stress for a flat plate under edge compression without the support of an elastic medium is equal to the critical stress for a pin-ended plate column of the same thickness with a length equal to

$$\frac{b}{\frac{a}{b}+\frac{b}{a}}$$

where b is the half-wave length of a buckle normal to the direction of loading, and a is the half-wave length of the same buckle parallel to the direction of loading. Equation (11) of reference 9 gives the critical load for a column under compression in an elastic medium. Consequently, if

$$\frac{b}{\frac{a}{b}+\frac{b}{a}}$$

is substituted for l in this equation and both sides divided by bt, the area of a strip of plate of width b, the following general expression is obtained for the critical stress in a flat plate under edge compression in an elastic medium:

$$S=\frac{P_{cr}}{bt}=\frac{E'I\pi^2}{bt\beta^2}\left[1+\frac{K\beta^4}{E'I\pi^4}\right]$$
$$=\frac{\pi^2Et^2}{12(1-\sigma^2)\beta^2}\left[1+\frac{12(1-\sigma^2)K\beta^4}{Ebt^3\pi^4}\right] \qquad (13)$$

where

P_{cr}, critical load for a strip of plate of width b, pounds.

t, thickness of plate, inches.

S, critical stress, pounds per square inch.

$E'=\frac{E}{1-\sigma^2}$, bending modulus for a plate as contrasted with E, the bending modulus of a beam or column, pounds per square inch.

$I=\frac{1}{12}bt^3$, moment of inertia of a strip of plate of width b, inches^4.

K, modulus of the elastic medium taken in such a way that $\frac{K}{b}$ times the deflection represents the reaction of the medium per unit area of the plate, pounds per square inch.

σ, Poisson's ratio.

$\beta=\frac{b}{\frac{a}{b}+\frac{b}{a}}$.

If the plate is large and free to buckle in any manner, β will assume such a value that S is a minimum. Consequently, differentiation of S with respect to β in equation (13) gives

$$S_{min}=2\sqrt{\frac{KEt}{12(1-\sigma^2)b}} \qquad (14)$$

when

$$\beta=\sqrt[4]{\frac{Ebt^3\pi^4}{12(1-\sigma^2)K}} \qquad (15)$$

EQUIVALENT ELASTIC MEDIUM FOR A ROBERTSON CYLINDER

When buckling begins in a Robertson cylinder, the effect of curvature is such as to set up forces that oppose buckling. For small deflections, these forces are proportional to the deflections, hence they may be regarded as analogous to the lateral reactions of an elastic medium in the previous discussion of the buckling of flat plates. The modulus of an equivalent elastic medium for a Robertson cylinder may therefore be obtained by equating S_{min} in equations (6) and (14) and solving for K. Thus,

$$\sqrt{\frac{1}{3(1-\sigma^2)}}\frac{Et}{r}=2\sqrt{\frac{KEt}{12(1-\sigma^2)b}}$$

from which

$$K=\frac{Ebt}{r^2} \qquad (16)$$

Substitution of this value of K in equation (13) gives the following general expression for the critical compressive stress in a Robertson cylinder

$$S=\frac{\pi^2Et^2}{12(1-\sigma^2)\beta^2}\left[1+\frac{12(1-\sigma^2)\beta^4}{t^2r^2\pi^4}\right] \qquad (17)$$

Equation (17), while different in form, is in substance the same as equation (3). This fact will be shown by the following derivation: According to equation (3),

$$\frac{S}{E}=\frac{1}{\alpha^2}+A\alpha^2$$

and

$$S=E\alpha^2A\left[1+\frac{1}{\alpha^4A}\right] \qquad (18)$$

By definition

$$\alpha=\frac{\pi r}{\beta} \qquad (19)$$

Consequently, substitution of the values that define α and A in equation (18) gives equation (17).

BRIEF DISCUSSION OF FORCED FAILURE

From the form of equation (17), it may be concluded that Southwell's general equation (equation (1) of this report) contains the solutions for the buckling of both flat and curved plates subjected to edge compression. The second term in the brackets represents the effect of curvature on the critical stress. If $r=\infty$, this term becomes zero and equation (17) gives the critical compressive stress for a flat plate simply supported at the four edges. In a similar manner, if t and r have fixed values such as correspond to the dimensions of a particular cylinder and β is forced with stiffeners to be less than the value that causes S to be a minimum

$$\beta=\sqrt[4]{\frac{t^2r^2\pi^4}{12\,(1-\sigma^2)}} \qquad (20)$$

the second term in the brackets rapidly becomes a small fraction and the critical stress approaches that for a flat plate. If values of β less than half the value given by equation (20) are forced, the portion of the curved sheet under consideration may be regarded as flat with an error not greater than 6.3 percent.

Further consideration of the subject of forced failure is beyond the scope of the present report. However, in view of the correlation of the buckling of thin-walled cylinders with the buckling of plates, it would appear that perhaps many solutions obtained for problems in the buckling of plates can, with the proper factors, be applied to similar problems in the buckling of cylinders and curved sheets. It is therefore recommended that theoretical and experimental research be conducted to explore this field, particularly as regards the compressive strength of curved sheet and stiffener combinations.

REFERENCES

1. Lundquist, Eugene E.: Strength Tests on Thin-Walled Duralumin Cylinders in Torsion. T.N. No. 427, N.A.C.A., 1932.
2. Robertson, Andrew: The Strength of Tubular Struts. R. & M. No. 1185, British A.R.C., 1929.
3. Flügge, W.: Die Stabilität der Kreiszylinderschale. Ingenieur-Archiv, Band III, no. 5, pp. 463–506, December 1932.
4. Wilson, Wilbur M., and Newmark, Nathan M.: The Strength of Thin Cylindrical Shells as Columns. Bull. No. 255, Eng. Exp. Sta., Univ. Ill., Feb. 28, 1933.
5. Southwell, R. V.: On the General Theory of Elastic Stability. Phil. Trans. Roy. Soc. (London), vol. 213, series A, 1914, pp. 187–244.
6. Donnell, L. H.: Stability of Thin-Walled Tubes under Torsion. T.R. No. 479, N.A.C.A., 1933.
7. Prescott, John: Applied Elasticity. Longmans, Green and Co., London, 1924, chap. XVII.
8. Schuman, Louis, and Back, Goldie: Strength of Rectangular Flat Plates under Edge Compression. T.R. No. 356, N.A.C.A., 1930.
9. Timoshenko, S.: Problems Concerning Elastic Stability in Structures. Trans. Amer. Soc. Civ. Eng., vol. 94, 1930, pp. 1,000–1,035.
10. von Kármán, Theodor, Sechler, Ernest E., and Donnell, L. H.: The Strength of Thin Plates in Compression. Applied Mechanics Section of Trans. A.S.M.E., Jan. 30, 1932, p. 53.

BIBLIOGRAPHY

1905. Lilly, W. E.: The Strength of Columns. Proc. Inst. Mech. Eng., London, Parts 3 and 4, June 1905, p. 697.

1907. Lilly, W. E.: The Economic Design of Columns. Proc. Inst. Civ. Eng. of Ireland (Dublin), vol. 33, March 1907, p. 57.

1908. Lilly, W. E.: The Design of Struts. Engineering, Jan. 10, 1908.

1908. Mallock, A.: Note on the Instability of Tubes Subjected to End Pressure, and on the Folds in a Flexible Material. Proc. Roy. Soc. (London), series A, vol. 81, 1908, p. 388.

1908. Lorenz, Rudolph: Achsensymmetrische Verzerrungen in Dünnwandigen Hohlzylindern. Z.V.D.I., Oct. 24, 1908, vol. 52, pp. 1706–13.

1910. Timoshenko, S.: Einige Stabilitätsprobleme der Elastizitatstheorie. Z.f. Math. u. Physik, vol. 58, p. 337. (Abstract of a paper published in 1907 in the Bull. of the Polytechnical Institute at Kiew.)

1911. Lorenz, Rudolph: Die Nicht Achsensymmetrische Knickung Dunnwandiger Hohlzylinder. Physik. Zeits. Leipzig. Jahrg. 12, pp. 241–260.

1914. Timoshenko, S.: (Buckling of Cylinders—Title Unknown). Bull. Electrotechnical Inst., St. Petersburg, vol. XI.

1916. Timoshenko, S.: Theory of Elasticity. Vol. II, St. Petersburg.

1922. Westergaard, H. M.: Buckling of Elastic Structures. Trans. Amer. Soc. of Civ. Eng., pp. 576–676. Paper No. 1490. (Including discussions by George Paaswell, Eugene E. Halmos, and Edward Godfrey.)

1925. Dean, W. R.: On the Theory of Elastic Stability. Proc. Roy. Soc. (London) vol. 107, series A, pp. 734–59.

1928. Geckeler, J. W.: Plastiches Knicken der Wandung von Hohlzylindern und einige andere Faltungserscheinungen an Schalen und Blechen. Z.f.a.M.M., October 1928, vol. 8, pp. 341–52.

1928. Geckeler, J. W.: Elastostatics. Handbuch der Physik. H. Geiger und Karl Scheel, bearb. Band VI, Kapitel III, pp. 298–302. Julius Springer (Berlin) 1928.

1930. Timoshenko, S.: Stability and Strength of Thin-Walled Constructions. Proc. Third International Congress for Applied Mechanics, Stockholm, vol. III, 1930.

1931. Timoshenko, S.: Stabilitaetsprobleme der Elasticitaet. Handbuch der Physikalischen und Technischen Mechanik. F. Auerbach und W. Hort, bearb. Band IV, I. J. A. Barth, Leipzig, 1931, pp. 81–145.

1932. von Sanden, K. V., and Tölke, F.: Über Stabilitätsprobleme dünner, Kreiszylindrischer Schalen. Ingenieur-Archiv, vol. 3, no. 1, March 1932, pp. 24–66.

TABLE I.—PROPERTIES OF SHEET DURALUMIN USED IN STRENGTH TESTS ON THIN-WALLED CYLINDERS

[Longitudinal and transverse refer to specimens taken parallel and normal to the direction of rolling, respectively]

Material		Ultimate tensile strength (pounds per square inch)		Tensile yield point (pounds per square inch)		Elongation in 2 inches (percent)		Modulus of elasticity			
								Secant modulus at a stress of 5,000 pounds per square inch (pounds per square inch)		Secant modulus at a stress of 20,000 pounds per square inch (pounds per square inch)	
Lot	Specimen no.	Longitudinal	Transverse	Longitudinal	Transverse	Longitudinal	Transverse	Longitudinal	Transverse	Longitudinal	Transverse
1	17	58,500	55,500	41,400	36,500	18	16.5	10,620,000	10,720,000	10,510,000	10,520,000
1	34	59,400	56,600	42,500	37,100	15	16.5	10,720,000	10,770,000	10,580,000	10,570,000
2	48	----------	57,300	----------	36,800	----------	16.5	----------	10,570,000	----------	10,340,000
2	50	----------	56,800	----------	35,800	----------	13.0	----------	10,550,000	----------	10,380,000
2	55	----------	58,200	----------	36,500	----------	18.0	----------	10,470,000	----------	10,270,000
2	70	----------	57,800	----------	35,800	----------	16.0	----------	10,560,000	----------	10,440,000
3	2A	----------	62,850	----------	33,750	----------	19.3	----------	10,270,000	----------	10,130,000
3	14A	----------	62,800	----------	41,000	----------	14.0	----------	10,410,000	----------	10,190,000
3	46A	----------	61,550	----------	38,750	----------	19.0	----------	10,550,000	----------	10,350,000
3	63A	----------	61,400	----------	39,050	----------	18.5	----------	10,110,000	----------	10,060,000
3	68A	----------	61,400	----------	37,900	----------	18.5	----------	10,410,000	----------	10,070,000

TABLE II.—CALCULATED AND EXPERIMENTAL VALUES OF k FOR N.A.C.A. TESTS ON DURALUMIN CYLINDERS

$\frac{l}{r}$	Radius=7.5 inches												Radius=15.0 inches											
	$\frac{r}{t}$=333–362				$\frac{r}{t}$=455–460				$\frac{r}{t}$=625–714				$\frac{r}{t}$=670–757				$\frac{r}{t}$=909–920				$\frac{r}{t}$=1270–1415			
	k_{max}	$k\left(\frac{\lambda_x}{2}=\frac{l}{2}\right)$	$k\left(\frac{\lambda_x}{2}=l\right)$	k (exp.)	k_{max}	$k\left(\frac{\lambda_x}{2}=\frac{l}{2}\right)$	$k\left(\frac{\lambda_x}{2}=l\right)$	k (exp.)	k_{max}	$k\left(\frac{\lambda_x}{2}=\frac{l}{2}\right)$	$k\left(\frac{\lambda_x}{2}=l\right)$	k (exp.)	k_{max}	$k\left(\frac{\lambda_x}{2}=\frac{l}{2}\right)$	$k\left(\frac{\lambda_x}{2}=l\right)$	k (exp.)	k_{max}	$k\left(\frac{\lambda_x}{2}=\frac{l}{2}\right)$	$k\left(\frac{\lambda_x}{2}=l\right)$	k (exp.)	k_{max}	$k\left(\frac{\lambda_x}{2}=\frac{l}{2}\right)$	$k\left(\frac{\lambda_x}{2}=l\right)$	k (exp.)
0.25																					32.7	31.8	25.8	16
.50	17.2	16.6	13.3	12					23.2 23.3 23.9 22.8	21.0 21.1 21.0 20.4	16.3 16.3 16.1 15.7	13–17	24.4 25.2 25.0	21.3 21.8 21.7	16.3 16.6 16.6	12					32.7 34.5	25.8 27.0	10.3 20.2	13–14
.05									23.4 23.4 23.5	18.9 18.9 19.0	14.2 14.2 14.3	13–15									34.5	24.3	17.9	12–13
1.00	16.9 17.2 17.3 17.3	13.1 13.3 13.4 13.4	9.8 9.9 10.0 10.0	9–10	19.5 19.5 19.5 19.4	14.5 14.5 14.3 14.3	10.8 10.8 10.6 10.6	10	24.4 23.5 23.2 23.2	16.3 16.0 15.9 15.9	12.0 11.7 11.7 11.7	10–13	23.6 23.9	16.0 16.1	11.8 11.8	11	27.5 27.7	17.5 17.6	12.8 12.8	11–13	34.2 34.4	19.8 19.8	14.3 14.4	12–15
1.50									23.8 23.1 23.0 23.0	13.5 13.3 13.2 13.2	9.8 9.7 9.6 9.6	8–12	23.8 23.8	13.5 13.5	9.8 9.8	10								
2.00									23.4 23.6 22.8	11.7 11.8 11.5	8.4 8.5 8.3	8–10												
2.50									22.9 23.2	10.4 10.5	7.5 7.5	7–10												
3.00									23.0	9.6	6.8	8–10												

TABLE III.—CALCULATED AND EXPERIMENTAL VALUES OF k FOR TESTS ON STEEL CYLINDERS REPORTED BY ROBERTSON IN REFERENCE 2

$\frac{l}{r}$	r (in.)	$\frac{r}{t}$ (approx.)	k_{max}	k $\left(\frac{\lambda_x}{2}=\frac{l}{2}\right)$	k $\left(\frac{\lambda_x}{2}=l\right)$	k (exp.)
0.68	7.32	500	20.8	17.0	12.9	12
1.25	4.00	263	14.7	11.1	8.2	8
2.00	2.50	164	11.6	7.9	5.8	8
3.07	1.63	110	9.5	5.9	4.3	7

TABLE IV.—CALCULATED AND EXPERIMENTAL VALUES OF k FOR TESTS ON STEEL AND BRASS CYLINDERS BY DONNELL

Material	$\frac{l}{r}$	$\frac{r}{t}$	Radius=0.943				Radius=1.88				Radius=2.84				$\frac{S_e}{E}$
			k_{max}	$k\left(\frac{\lambda_e}{2}=\frac{l}{2}\right)$	$k\left(\frac{\lambda_e}{2}=l\right)$	k (exp.)	k_{max}	$k\left(\frac{\lambda_e}{2}=\frac{l}{2}\right)$	$k\left(\frac{\lambda_e}{2}=l\right)$	k (exp.)	k_{max}	$k\left(\frac{\lambda_e}{2}=\frac{l}{2}\right)$	$k\left(\frac{\lambda_e}{2}=l\right)$	k (exp.)	
Brass	2.12	483									20.0	10.5	7.5	10.0	0.000393
Do		923									27.6	12.4	8.9	10.0	.000138
Steel		971									28.4	12.6	9.0	11.4	.000146
Do		1,013									29.0	12.8	9.1	11.4	.000157
Do		1,284									32.6	13.6	9.7	12.0	.000062
Do		1,307									32.9	13.6	9.8	12.0	.000063
Brass		1,331									33.2	13.7	9.8	11.0	.000074
Do		1,383									33.8	13.8	9.9	10.5	.000073
Brass	3.20	311					16.0	7.7	5.5	8.8					.000435
Do		314					16.1	7.7	5.5	8.0					.000414
Steel		633					22.9	9.3	6.6	10.0					.000224
Do		660					23.4	9.4	6.7	9.6					.000292
Do		837					26.3	10.0	7.1	10.4					.000161
Do		864					26.7	10.1	7.2	10.0					.000198
Brass		864					26.7	10.1	7.2	9.7					.000120
Do		897					27.3	10.2	7.3	10.0					.000106
Brass	4.22	476									19.9	7.5	5.4	9.0	.000358
Do		490									20.1	7.6	5.4	9.0	.000508
Do		490									20.1	7.6	5.4	9.2	.000484
Steel		1,058									29.6	9.3	6.6	10.0	.000121
Do		1,383									33.8	9.9	7.0	10.0	.000071
Brass		1,440									34.5	10.0	7.1	10.0	.000091
Brass	6.38	160	11.5	4.7	3.3	8.0									.001200
Do		315	16.1	5.6	4.0	6.9									.000511
Do		315	16.1	5.6	4.0	7.4									.000572
Steel		328	16.5	5.6	4.0	6.7									.000530
Do		347	16.9	5.7	4.1	9.2									.000758
Do		451	19.3	6.1	4.3	6.7									.000348
Do		460	19.5	6.1	4.4	7.5									.000420
Brass	6.38	311					16.0	5.5	3.9	7.2					.000472
Steel		679					23.7	6.8	4.8	8.7					.000272
Brass		880					27.0	7.2	5.1	10.0					.000131
Steel		915					27.5	7.3	5.2	8.9					.000129
Brass	10.6	474									19.8	4.9	3.5	6.8	.000352
Brass	12.7	160	11.5	3.5	2.6	7.9									.001410
Steel		342	16.8	4.2	3.0	7.3									.000437
Brass	16.0	315					16.1	3.7	2.7	7.4					.000535
Steel	25.5	332	16.6	3.0	2.2	5.0									.000045
Brass	31.8	158	11.4	2.3	2—	8.3									.001472
Steel		469	19.7	2.9	2—	7.2									.000245

NOTE.—In calculating k for this table cylinders with $\frac{l}{r} > 6.38$ were considered as Southwell cylinders.

TABLE V.—CALCULATED AND EXPERIMENTAL VALUES OF k FOR TESTS ON RUBBER AND CELLULOID CYLINDERS REPORTED BY FLÜGGE IN REFERENCE 3

Material	$\frac{l}{r}$	r (in.)	$\frac{r}{t}$ (approx.)	k_{max}	$k\left(\frac{\lambda_e}{2}=\frac{l}{2}\right)$	$k\left(\frac{\lambda_e}{2}=l\right)$	k (exp.)	$\frac{S_e}{E}$
Rubber	1.76	1.77	90.0	8.6	7.0	5.3	5	0.00479
Celluloid	1.95	1.80	90.2	8.6	6.7	5.0	5	.00386
Rubber	3.53	1.77	90.0	8.6	5.3	3.8	2	.00374
Do	3.53	1.77	90.0	8.6	5.3	3.8	4	.00333
Celluloid	3.97	1.80	174.1	12.0	6.0	4.3	4	.00187
Do	4.01	1.80	90.2	8.6	5.0	3.6		.00317
Do	4.01	1.80	90.2	8.6	5.0	3.6		.00328
Do	4.01	1.80	90.2	8.6	5.0	3.6	4	.00317
Do	4.02	1.80	96.4	8.9	5.0	3.6	4	.00357
Do	4.02	1.80	90.2	8.6	5.0	3.6	4	.00394
Do	4.02	1.80	90.2	8.6	5.0	3.6	3	.00306
Do	4.02	1.80	90.2	8.6	5.0	3.6	4	.00317
Do	4.02	1.80	90.2	8.6	5.0	3.6	3	.00302
Do	4.02	1.80	90.2	8.6	5.0	3.6	4	.00291
Do	5.02	1.43	138.0	10.7	5.0	3.6	4	.00207
Do	5.05	1.43	76.6	8.0	4.3	3.1	4	.00281

TABLE VI.—CALCULATED AND EXPERIMENTAL VALUES OF k FOR TESTS ON STEEL CYLINDERS REPORTED BY WILSON AND NEWMARK IN REFERENCE 4

Types of construction	Test series	Specimen no.	$\frac{l}{r}$	r (in.)	$\frac{r}{t}$	k_{max}	$k\left(\frac{\lambda_s}{2}=\frac{l}{2}\right)$	$k\left(\frac{\lambda}{2}=l\right)$	k (exp.)	$\frac{S_c}{E}$ calculated from data given in reference 4
Machined cylinders with no seams	1	110	3.86	1.94	66.5	7.4	4.6	3.4	5	0.00186
		111	3.89	1.94	133.0	10.5	5.6	4.0	8	.00135
		112	3.89	1.94	67.5	7.5	4.7	3.4	5	.00176
		113	3.88	1.94	136.0	10.6	5.6	4.1	8	.00146
	6	610	1.39	6.82	478	19.9	12.6	9.2	10	.000525
		611	1.41	6.82	206	13.1	9.8	7.3	8	.00112
		612	1.39	6.82	330	16.5	11.4	8.3	10	.000822
		613	1.39	6.82	435	19.0	12.3	9.0	12	.000792
		614	1.40	6.82	763	25.1	14.3	10.4	12	.000286
		615	1.35	6.82	510	20.6	13.0	9.5	12	.000593
		616	1.39	6.82	340	16.8	11.5	8.4	10	.000777
		617	1.39	6.82	235	14.0	10.3	7.6	11	.00109
Fabricated cylinders with either riveted or welded seams—R denotes riveted seams.	E	816R	1.80	40.0	167	11.8	8.4	6.2	8	.000836
		817R	1.80	40.0	172	11.9	8.4	6.2	8	.000812
		114	1.80	40.0	165	11.7	8.4	6.1	8	.000900
		86	1.80	40.0	167	11.8	8.4	6.2	8	.000944
	7	710	6.00	5.0	166	11.7	4.8	3.5	8	.000346
		711	6.00	5.0	168	11.8	4.9	3.5	8–9	.000493
		721	3.00	10.0	329	16.5	8.0	5.8	11–12	.000302
		730	2.00	15.0	518	20.7	11.0	7.9	10–12	.000299
		731	2.00	15.0	450	19.3	10.5	7.6	10–12	.000339
		740	1.50	20.0	667	23.5	13.4	9.7	10	.000191
		741	1.50	20.0	654	23.3	13.3	9.6	10	.000192
		750	1.20	25.0	825	26.1	15.7	11.4	13	.000148
		751	1.20	25.0	848	26.5	15.8	11.5	10–13	.000143
		760	1.00	30.0	971	28.4	17.8	13.0	10	.000139
		761	1.00	30.0	990	28.6	18.0	13.0	10	.000129
	8	810	24.7	17.0	139	11.7	2.5	2—	(1)	.000666
		811	24.7	17.0	151	11.2	2.5	2—	7	.000741
		820	14.1	17.0	142	10.8	3.2	2.3	6–7	.000772
		821	14.1	17.0	142	10.8	3.2	2.3	7	.000778
		830	4.24	17.0	151	11.2	5.6	4.0	8	.000894
		831	4.24	17.0	145	11.0	5.5	4.0	7	.000941

[1] Irregular.

NOTE.—In calculating k for this table cylinders with $\frac{l}{r} > 6.00$ were considered as Southwell cylinders.

TABLE VII.—TABLE FOR DIFFERENTIATION BETWEEN SOUTHWELL AND ROBERTSON CYLINDERS

k	Term	Multiplying factor	$\epsilon=g/k=0.01$ S	C	R	$g/k=0.05$ S	C	R	$g/k=0.1$ S	C	R	$g/k=0.2$ S	C	R	$g/k=0.4$ S	C	R	$g/k=1.0$ S	C	R	$g/k=10.0$ S	[1] C	[1] R
1	1	10^{-3}	200	200	100	200	201	101	200	205	102	200	218	108	200	277	135	200	760	400	200	104	102
	2	10^{-2}	000	000	100	000	000	101	000	000	104	000	1	117	000	17	181	000	500	1,600	000	104	104
2	1	10^{-1}	200	200	160	200	201	161	200	204	163	200	217	173	200	273	216	200	784	640	200	164	163
	2	1	144	144	256	144	146	259	144	150	266	144	172	300	144	287	463	144	3,132	4,096	144	266	266
3	1	10^{-1}	900	900	810	900	905	814	900	918	826	900	973	874	900	1,220	1,094	900	3,560	3,240	900	828	826
	2	10	518	519	656	518	524	663	518	539	682	518	610	768	518	961	1,189	518	9,348	10,500	518	683	682
4	1	1	272	272	256	272	273	257	272	277	261	272	295	277	272	369	346	272	1,082	1,024	272	261	262
	2	10^{2}	576	576	656	576	582	662	576	599	682	576	676	767	576	1,056	1,186	576	9,832	10,490	576	682	682
5	1	1	650	650	625	650	653	628	650	664	638	650	703	675	650	879	844	650	2,590	2,500	650	638	638
	2	10^{3}	360	360	391	360	364	394	360	374	406	360	421	457	360	657	707	360	5,997	6,250	360	407	407
6	1	10	133	133	130	133	134	130	133	136	132	133	144	140	133	180	175	133	531	518	133	132	132
	2	10^{4}	159	159	168	159	160	170	159	165	175	159	186	197	159	289	304	159	2,611	2,687	159	176	176
7	1	10	245	245	240	245	246	241	245	250	245	245	265	259	245	331	324	245	978	960	245	245	245
	2	10^{5}	553	554	577	553	559	582	553	576	600	553	648	675	553	101	105	553	904	923	553	60	60
8	1	10	416	416	410	416	418	412	416	424	418	416	449	442	416	562	553	416	1,661	1,638	----	----	----
	1	10^{5}	163	163	168	163	164	169	163	169	175	163	190	196	163	296	304	163	2,642	2,685	----	----	----
9	1	10^{2}	664	664	656	664	668	659	664	678	689	664	717	708	664	898	886	664	265	262	----	----	----
	2	10^{5}	420	420	431	420	424	435	420	437	448	420	492	504	420	762	779	420	6,800	6,887	----	----	----
10	1	10^{2}	101	101	100	101	102	101	101	103	102	101	109	108	101	136	135	101	404	400	----	----	----
	2	10^{5}	980	980	1,000	980	990	1,010	980	1,019	1,040	980	1,147	1,170	980	1,778	1,810	980	15,840	16,000	----	----	----
15	1	10^{2}	509	509	506	509	511	509	509	519	516	509	549	547	509	687	684	----	----	----	----	----	----
	2	10^{7}	254	254	256	254	257	259	254	264	267	254	297	300	254	460	464	----	----	----	----	----	----
20	1	10^{3}	160	160	160	160	161	161	160	164	163	160	173	173	160	217	216	----	----	----	----	----	----
	2	10^{8}	255	255	256	255	257	259	255	265	266	255	298	300	255	461	463	----	----	----	----	----	----
25	1	10^{3}	391	391	391	391	393	393	391	399	398	391	423	422	391	528	528	----	----	----	----	----	----
	2	10^{9}	152	152	153	152	154	154	152	158	159	152	178	179	152	276	276	----	----	----	----	----	----

[1] Multiplying factor = 10^{4} × values given in third column.

TABLE VIII.—CALCULATED VALUES OF $\frac{\lambda_a}{2r}$ AND SMALLEST VALUES OF $\frac{l}{r}$

r (inches)	$\frac{r}{t}$ (average)	$\frac{\lambda_a}{2r}$ (equation 12)	$\frac{l}{r}$ smallest in tests	Remarks
1.63	110	0.33	3.07	Tests on steel cylinders by Robertson.
2.50	165	.27	2.00	
4.00	260	.21	1.25	
7.32	500	.15	.68	
7.5	350	.18	.50	Tests on duralumin cylinders by N.A.C.A.
7.5	460	.16	1.00	
7.5	670	.13	.50	
15.0	715	.13	.50	
15.0	915	.11	1.00	
15.0	1,340	.094	.25	