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RELATIVISTIC CHARGED FLUID FLOW
I: ENTROPY FORCE TENSOR

BY
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ABSTRACT

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The problem of the dynamical effects of ~~entropy generation, i.e.~~ heat absorption or rejection, on the flow of an electrically charged fluid is considered within the framework of Special Relativity. The source or sink for the heat transferred to or from the fluid may be imagined to be a heat reservoir whose velocity is in general not the same as the fluid velocity. The analysis of a relativistic Carnot cycle that operates between two heat reservoirs having relative motion with respect to each other provides a simple way to study the dynamical effects of heat transfer and to incorporate the moving heat reservoir model into the formalism of fluid dynamics. All the dynamical effects that are functions of the 4-gradient of the specific entropy of the fluid can be expressed in terms of an antisymmetric tensor, the entropy force tensor, whose role in the fluid equation of motion is analogous to that of the electromagnetic field tensor.

RELATIVISTIC CHARGED FLUID FLOW
I: ENTROPY FORCE TENSOR

I. INTRODUCTION

The familiar stress-energy tensor approach[1-3] to relativistic fluid dynamics, which is summarized in Section II, arrives at a fluid equation of motion in which the dynamical effects of heat flux in the fluid are given by the 4-divergence of a symmetric tensor Q^{jk} which is the contribution to the stress-energy tensor resulting from the presence of heat flux. Until more specific information concerning the form of Q^{jk} is provided, however, the analysis can be carried no further. This barrier can be circumvented by expressing the heat force term in the equation of motion directly in terms of a model that is both flexible enough to include most situations of physical interest, and simple enough to allow a direct insight into the processes producing the dynamical effects associated with heat transfer within the fluid. Such a model must also be related to the heat tensor Q^{jk} so that, given this tensor, the characteristic features of the model can be calculated. The intuitive thinking and calculation could, however, be carried out directly in terms of the model.

The <u>moving heat reservoir</u> model fulfills these requirements. The quantitative implementation of this model can best be carried out in terms of a relativistic Carnot cycle operating between two heat reservoirs having relative motion with respect to each other. This is done in Section IV. As an introduction to the relativistic Carnot cycle, a discussion is given in Section III of the Lorentz transformation of heat and temperature, which has been the subject of recent debate.

In Section V the moving heat reservoir model is incorporated into the fluid equation of motion, and the entropy-dependent contribution to the force is expressed in terms of the antisymmetric entropy force tensor which is analogous, in its dynamical effects, to the electromagnetic field tensor.

Notation

Since everything we do will be within the framework of Special Relativity, we shall assume throughout the flat-space form of the metric tensor g^{jk}:

$$
\begin{aligned}
g^{00} &= g_{00} = 1\,; \\
g^{jj} &= g_{jj} = -1, \qquad (j = 1,\ 2,\ 3); \qquad\qquad (1.1) \\
g^{jk} &= g_{jk} = 0, \qquad (j \neq k).
\end{aligned}
$$

The components of the 4-gradient operator ∂_j are as follows:

$$
(\partial_0,\ \partial_1,\ \partial_2,\ \partial_3) = (\partial^0,\ -\partial^1,\ -\partial^2,\ -\partial^3) = \left(\frac{1}{c}\,\frac{\partial}{\partial t}\,,\ \nabla\right) \qquad (1.2)
$$

where ∇ is the 3-gradient operator. (In the notation illustrated in (1.2) it will be understood throughout that the space-like components with indices 1, 2, and 3 represent the x, y, and z components respectively in a Cartesian coordinate system fixed in the observer's reference frame.)

With the exception of the gradient operator given above, it will always be the contravariant form of a 4-vector, rather than the covariant form, that is the relativistic generalization of the corresponding 3-vector. Thus the components of the contravariant 4-velocity v^j are as follows:

$$
(v^0,\ v^1,\ v^2,\ v^3) = \Gamma(c,\ \mathbf{v}) \qquad (1.3a)
$$

where

$$\Gamma = (1 - \beta^2)^{-1/2}, \quad \beta = v/c, \tag{1.3b}$$

and c is the speed of light. Note that the fact that

$$v^j \, v_j = c^2 \tag{1.4}$$

implies

$$v_j \, \partial_k \, v^j = \frac{1}{2} \partial_k \, (v_j \, v^j) = 0. \tag{1.5}$$

(Repeated indices, of course, indicate summation.)

Differentiation with respect to the local proper time τ of the fluid is defined as

$$\frac{d}{d\tau} = v^j \, \partial_j = \Gamma D \tag{1.6a}$$

where

$$D = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \tag{1.6b}$$

is the non-relativistic substantial time differentiation operator.

We shall also have need for the symmetric space-like projection operator σ^{jk} whose definition is

$$\sigma^{jk} = g^{jk} - v^j \, v^k / c^2. \tag{1.7}$$

This operator has the property that the contraction of either index with that of an arbitrary 4-vector b_k yields the projection of b_k that is orthogonal to the fluid velocity v_j:

$$v_j \, (\sigma^{jk} \, b_k) = v_j \, (b_k \, \sigma^{kj}) = 0. \tag{1.8}$$

Tensor indices will always be indicated by lower case Roman letters. When other identifying subscripts or superscripts are necessary, capital letters will be used. Whether a superscript or subscript is used to identify a 4-vector will depend on whether it is in its covariant or contravariant form respectively. Thus, for example, the contravariant and covariant forms of the external 4-force acting on the fluid will be designated by f_E^j and f_j^E respectively.

The invariant particle density (not mass density) will be designated by ρ and the particle rest mass by m. Thus if $\overset{\circ}{\mho}$ is the specific volume in the fluid rest-frame, we have

$$\overset{\circ}{\mho} = 1/\rho m. \tag{1.9}$$

The particle density and particle mass as seen in the observer's frame will be designated by ρ^* and m^* respectively. Thus

$$\rho^* = \rho\Gamma; \quad m^* = m\Gamma. \tag{1.10}$$

The specific volume $\mho$ in the observer's frame is given by

$$\mho = \overset{\circ}{\mho}/\Gamma = \sqrt{1 - \beta^2}\ \overset{\circ}{\mho} \tag{1.11}$$

Further notation will be introduced as needed.

II. STRESS-ENERGY TENSOR APPROACH

The stress-energy tensor T_I^{jk} for an ideal fluid[4] has the following well-known form:[5]

$$T_I^{jk} = \rho m(1 + h/c^2)\, v^j v^k - \wp\, g^{jk} \tag{2.1}$$

where h is the specific enthalpy, $\mathcal{P}$ is the pressure, and all the other quantities have already been defined. Using the thermodynamic relation

$$h = u + \mathcal{P}\overset{\circ}{U} = u + \mathcal{P}/\rho m, \tag{2.2}$$

where u is the specific internal energy, T_I^{jk} can also be written

$$T_I^{jk} = \rho m(1 + u/c^2)\, v^j v^k - \mathcal{P}\, \sigma^{jk} \tag{2.3}$$

where σ^{jk} is the space-like projection operator defined in (1.7).

Eckart[2] and Taub[6] have shown that the presence of heat flux and viscosity can be accounted for by the addition of two symmetric tensors, Q^{jk} and S^{jk}, to T_I^{jk}. Thus the total stress-energy tensor T^{jk} has the form

$$T^{jk} = T_I^{jk} + Q^{jk} + S^{jk}. \tag{2.4}$$

The tensors Q^{jk} and S^{jk} which account for the heat flux and the viscous stress respectively are, in the general case, restricted only by the following orthogonality requirements:

$$v_j\, S^{jk} = S^{kj}\, v_j = 0; \tag{2.5a}$$

$$v_j\, Q^{jk} = Q^{kj}\, v_j = q^k; \tag{2.5b}$$

$$v_k\, q^k = 0. \tag{2.5c}$$

We shall see below that, in the scalar form of the First Law of thermodynamics, the space-like 4-vector q^k defined in (2.5b) plays the role of heat flux density. In the 4-vector form of the First Law, however, the heat flux density is specified by the components Q^{0k}. This follows from the fact that the statement of conservation of energy and momentum in terms of T^{jk} is given by

$$\partial_k T^{jk} = f^j_E \tag{2.6}$$

where f^j_E is the external 4-force density (force per unit volume) acting on the fluid. (This will later be identified with the electromagnetic force.) The time-like component of this 4-vector equation is the statement of energy conservation and can be written as follows:

$$\partial T^{00}/\partial t + \nabla \cdot \mathfrak{U} = c\, f^0_E \tag{2.7}$$

where

$$T^{00} = \rho^* m^* (c^2 + h) - \mathbb{P} + Q^{00} + S^{00}$$

$$= \rho^* m^* (c^2 + u) + (\Gamma\beta)^2\, \mathbb{P} + Q^{00} + S^{00} \tag{2.8}$$

and

$$\mathfrak{U} = c(T^{01}, T^{02}, T^{03}) \tag{2.9a}$$

$$= \rho^* m^* (c^2 + h)\, \mathbf{v} + \mathfrak{Q} + \mathfrak{S}$$

where

$$\mathfrak{Q} = c(Q^{01}, Q^{02}, Q^{03}) \tag{2.9b}$$

and

$$\mathfrak{S} = c(S^{01}, S^{02}, S^{03}). \tag{2.9c}$$

T^{00} and $\mathfrak{U}$ are obviously the total energy density and energy flux density respectively in the fluid. $c\, f^0_E$ is the work per unit time per unit volume that is performed on the fluid by external forces. Q^{00} and S^{00} are small relativistic contributions to the energy density which, because of the conditions given in (2.5), vanish in the local rest-frame of the fluid.

It is evident that in the form of the First Law (i.e. conservation of energy) given in (2.7) the 3-vector $\mathcal{Q}$ plays the role of heat energy flux density. Referring to (2.5b), we see that $\mathbf{q} = (q^1, q^2, q^3)$ and $\mathcal{Q}$ differ only by the dot product of $\mathbf{v}$ with the heat stress dyadic, which is just the space-space part of Q^{jk}. This difference vanishes in the local fluid rest-frame, and is very small in the observer's frame, so $q \approx \mathcal{Q}$.

To arrive at the equation of motion for the fluid (Euler's equation) we must supplement (2.6) with the continuity equation for the fluid:

$$\partial_j (\rho v^j) = 0. \tag{2.10}$$

Using (2.4), (2.6), (2.10), and the form of T_I^{jk} given in (2.3), we arrive at the following form of the equation of motion:

$$\rho\, d\,[m(1 + u/c^2)\, v^j\,]/d\tau = \partial_k(\mathcal{P}\,\sigma^{jk}) - \partial_k(Q^{jk} + S^{jk}) + f_E^j. \tag{2.11}$$

As a preliminary to contracting this equation with v_j we note that using (1.7), (1.5), (2.10), and (1.9) we arrive at the following relation:

$$v_j\, \partial_k(\mathcal{P}\,\sigma^{jk}) = -\,\rho m \mathcal{P}\, d\overset{\circ}{\mathcal{U}}/d\tau. \tag{2.12}$$

The contraction of (2.11) with v_j thus becomes

$$du/d\tau = -\,\overset{\circ}{\mathcal{U}}\, d_k\, q^k - \mathcal{P}\, d\overset{\circ}{\mathcal{U}}/d\tau + \overset{\circ}{\mathcal{U}}\left[\tfrac{1}{2}\,(Q^{jk} + S^{jk})\,(\partial_j v_k + \partial_k v_j) + f_E^j\, v_j\right], \tag{2.13}$$

where we have made use of (2.5).

Comparing this with the familiar form of the First Law,

$$\Delta u = \Delta Q - \mathcal{P}\,\Delta\overset{\circ}{\mathcal{U}} + \Delta W, \tag{2.14}$$

where ΔQ and ΔW are the heat added to, and the work (excluding that done by the pressure) performed on unit mass of the fluid, we see that $-\partial_k q^k$ in (2.13) is the heat added to the fluid per unit volume per unit time, and the expression in the brackets represents the work per unit volume per unit time done by viscosity (and the heat stress) and the external forces.

Equation (2.13) is the invariant form of the First Law given by Eckart.[2] It is possible to arrive at the analog of the First Law in the form

$$\Delta h = \Delta Q + \overset{0}{\mathcal{V}} \Delta \wp + \Delta W \tag{2.15}$$

by repeating the derivation of (2.11) using (2.1) instead of (2.3). Doing this yields

$$\rho \, d[m(1 + h/c^2) \, v^j]/d\tau = \partial^j \wp - \partial_k (Q^{jk} + S^{jk}) + f^j_E \, . \tag{2.16}$$

The advantage of this form of Euler's equation over (2.11) is the absence of σ^{jk} in the pressure gradient term. For this reason (2.16) will be used henceforth in preference to (2.11). Contracting (2.16) with v_j yields

$$dh/d\tau = -\overset{0}{\mathcal{V}} \partial_k q^k + \overset{0}{\mathcal{V}} d\wp/d\tau + \overset{0}{\mathcal{V}} \left[\tfrac{1}{2} (Q^{jk} + S^{jk}) (\partial_j v_k + \partial_k v_j) + f^j_E v_j \right] , \tag{2.17}$$

which is the desired analog of (2.15).

The invariant forms of the First Law given in (2.13) and (2.17) refer to the local fluid rest-frame, which in general is different from point to point. Thus the invariant form of the First Law is more restrictive than the 4-vector form given in (2.16), which applies to any arbitrary fixed frame of reference. The time-like component of (2.16) can be put in the following form:

$$\begin{aligned} D(c^2 \Gamma + h^*) = &- \mathcal{V}(\partial Q^{00}/\partial t + \nabla . \mathcal{Q}) + \mathcal{V} \partial \wp/\partial t \\ &+ \mathcal{V} \, [c f^0_E - (\partial W^{00}/\partial t + \nabla . \mathcal{S})] , \end{aligned} \tag{2.18}$$

where $h^* = \Gamma h$, and $D(c^2\Gamma) \approx D(1/2\, v^2)$ is the substantial time rate of change of the kinetic energy per unit mass. This form of the First Law is equivalent to that given in (2.7). In both cases $\mathcal{Q}$ (rather than $\mathbf{q}$) plays the role of heat flux density, and comparing (2.15) with (2.18) it is evident that $-(\partial Q^{00}/\partial t + \nabla\cdot\mathcal{Q})$ is the heat added to the fluid per unit volume per unit time.

We conclude from this discussion that for relativistic fluid flow the most useful form of the First Law is the following 4-vector equation:

$$d[m(1 + h/c^2)\, v^j]/d\tau = (\partial^j P)/\rho + \pi^j + f_E^j/\rho - (\partial_k S^{jk})/\rho, \tag{2.19}$$

where

$$\pi^j = -(\partial_k Q^{jk})/\rho \tag{2.20}$$

represents the time rate at which heat energy and momentum are added per particle. In relativistic theory it is in general impossible to add heat energy to the fluid without also adding momentum as well, because the heat energy has a mass whose momentum is injected into the fluid along with the heat energy, and is dependent on the motion of the heat source. This point will be discussed at greater length in the next two sections.

III. LORENTZ TRANSFORMATION OF HEAT AND TEMPERATURE

The question of the correct relativistic formulation of thermodynamics, in particular the form of the Lorentz transformations of heat and temperature, has been the subject of recent discussion.[7-9] An outline of the history of the field and the points at issue will now be given as a preliminary to explaining the point of view taken in the present paper.

Relativistic thermodynamics was first formulated in 1907 by Planck.[10] (Hasenöhrl and Einstein derived similar results by different methods.[10]) This formulation (henceforth referred to as the "Planck formulation") yields the following velocity transformations for heat ΔQ, temperature T, and entropy S:

$$\text{(Planck)} \begin{cases} \Delta Q = \sqrt{1-\beta^2}\,\Delta\overset{\circ}{Q}; & (3.1a) \\ T = \sqrt{1-\beta^2}\,\overset{\circ}{T}; & (3.1b) \\ S = \overset{\circ}{S}; & (3.1c) \end{cases}$$

where $\Delta\overset{\circ}{Q}$, $\overset{\circ}{T}$, and $\overset{\circ}{S}$ are the values for $\beta = 0$.

In 1911 von Laue pointed out[11] that the formulation of relativistic thermodynamics is not unique in that it is possible to multiply ΔQ in the statements of the First and Second Laws by two different monotonic functions of β^2 that are arbitrary except for the requirement that they become unity for $\beta = 0$. Since these arbitrary functions will appear in the velocity transformations for ΔQ and T (S is invariant because of its statistical significance), these transformations cannot be uniquely formulated. That is, some subjective criterion must be invoked to choose one out of all the possible formulations. Having made this point, von Laue then went on to defend the choice that had been implicitly made in the Planck formulation. Subsequently, this formulation was absorbed into the standard texts[12] without, however, von Laue's observation concerning the lack of uniqueness.

Recently, however, this lack of uniqueness has been re-emphasized by Ott,[7] who then proceeded to give arguments in favor of a formulation that yields the following transformation laws:

$$
(\mathrm{Ott})\begin{cases} \Delta Q = \overset{o}{\Delta Q}/\sqrt{1-\beta^2}; & (3.2a) \\ T = \overset{o}{T}/\sqrt{1-\beta^2}; & (3.2b) \\ S = \overset{o}{S}. & (3.2c) \end{cases}
$$

This formulation (henceforth referred to as the "Ott formulation") has also received some support from arguments based on the consideration of special cases.[8,9]

Ott's formulation has the undeniable intuitive advantage that the heat, which is after all an energy, transforms as we would intuitively expect an energy to transform, namely as the time-like component of a 4-vector, whereas this is not true in the Planck formulation. It is obvious that the complete 4-vector ΔQ^j of which ΔQ is the time-like part has the form

$$
\Delta Q^j = (\overset{o}{\Delta Q}/c)\, v^j = \Delta Q\,(1,\ \mathbf{v}/c). \tag{3.3}
$$

Since in any heat transfer the donor and receiver may in general have different velocities, care must be taken in specifying which of the two velocities is intended in the (3.3). This matter will be discussed at greater length in the next section. For the present, however, we shall consider only the simple case in which the donor and receiver have the same velocity $\mathbf{v}$. The observer sees a heat energy transfer of amount ΔQ which, by the general energy-mass equivalence, implies a mass transfer of amount $\Delta Q/c^2$. Because this mass transfer occurs at velocity $\mathbf{v}$, there is necessarily a momentum transfer of magnitude $(\Delta Q/c^2)\,\mathbf{v} = c^{-1}(\Delta Q^1,\ \Delta Q^2,\ \Delta Q^3)$. Thus the space-like components of ΔQ^j give (except for the factor c) the relativistic momentum transfer that must accompany the heat energy transfer.

Ott has shown that the essential difference between his formulation and Planck's can be set forth in a manner that is directly relevant to the purposes of the present paper, namely that the two formulations effectively postulate two different forms for the equation of motion of a body of (variable) rest-mass M that is absorbing heat from a reservoir at the rate $-đQ_R/d\tau$ measured in rest-frame of M or at the rate $-\not{D}Q_R = -(đQ_R/d\tau)/\Gamma$ measured in the observer's frame. ($đ$ designates a small quantity that is not a perfect differential.) We take $đQ_R$ to be the quantity of heat absorbed by the reservoir, and so it is positive, according to the usual sign convention, if the reservoir absorbs heat, i.e., if the mass M loses heat. (We must not, however, jump tothe conclusion that $đQ_R = -đQ_M$ where $đQ_M$ is the heat absorbed by M. The difference between these two is the work associated with the momentum that must be transferred along with the heat. This will be discussed in the next section.) The form of the 4-vector equation of motion postulated by Ott is

$$\text{(Ott)} \quad d(Mv^j)/d\tau = F^j_E - đQ^j_R/c\,d\tau \tag{3.4a}$$

where

$$F^j_E = \Gamma[(\mathbf{v}\cdot\mathbf{F}_E/c), \mathbf{F}_E] \tag{3.4b}$$

is the external 4-force acting on M. (In writing (3.4b) we have assumed that $\mathbf{F}_E$, like the electromagnetic force, does not change the rest-mass M. Thus change in M is to be associated exclusively with heat transfer. This assumption is not vital, but it does simplify the discussion.) The 4-vector equation (3.4a) is equivalent to the following energy and momentum equations:

$$D(M^*c^2) = \mathbf{v}\cdot\mathbf{F}_E - \not{D}Q_R \tag{3.4c}$$

and, using (3.3),

$$D(M^* \mathbf{v}) = \mathbf{F}_E - \not{D}[(QR/c^2)\, \mathbf{v}_R] \tag{3.4d}$$

where $M^* = \Gamma M$ and $\mathbf{v}_R$ is the reservoir velocity which in general is different from the mass velocity $\mathbf{v}$.

Note that (3.4a) has just the well-known form of the rocket equation[13] in which the energy-momentum carried off by mass discharge has been replaced by that carried off (or injected) by heat transfer. It seems completely natural that the two problems should be handled in similar fashion.

As contrasted with (3.4a), the equation of motion that implicitly underlies the Planck formulation has the following form:

$$\text{(Planck)} \quad d(Mv^j)/d\tau = F_c^j \tag{3.5a}$$

where

$$F_c^j = \Gamma[c^{-1}(\mathbf{v}\cdot\mathbf{F}_c - \not{D}Q_R),\ \mathbf{F}_c] \tag{3.5b}$$

is the composite 4-force that includes both the effects of external fields, such as an electromagnetic field, and the effects of heat transfer. The energy and momentum equations corresponding to (3.5a) are

$$D(M^*c^2) = \mathbf{v}\cdot\mathbf{F}_c - \not{D}Q_R \tag{3.5c}$$

and

$$D(M^*\mathbf{v}) = \mathbf{F}_c. \tag{3.5d}$$

Note that the time-like component of F_c^j contains $(\Gamma/c)\not{D}Q_R$. It is well-known[14] that when heat transfer is incorporated into the 4-force in this manner, the transformation equation (3.1a) follows as a necessary consequence.

The differences between the two approaches are most simply illustrated by the case in which $\mathbf{F}_E = 0$ and $\mathbf{v} = \mathbf{v}_R =$ constant. In the common rest-frame of M and the reservoir it is obvious that

$$D(Mc^2) = -\,D\overset{o}{Q}_R \quad (\text{for } \mathbf{F}_E = 0,\ \mathbf{v} = \mathbf{v}_R = \text{const.}). \tag{3.6}$$

(In this case $D\overset{o}{Q}_R$ is obviously a perfect differential.) But from (3.4c) we find $-DQ_R = \Gamma DMc^2$. Substituting (3.6) into this, we have $DQ_R = \Gamma D\overset{o}{Q}_R$, which is just (3.2a). Thus (3.4a) implies the transformation law (3.2a).

Now let us consider the consequences of (3.5a). We note that if we use (3.6) in (3.5d) we obtain $\mathbf{F}_c = -\mathbf{v}\,\Gamma c^{-2} D\overset{o}{Q}_R$, where we have used the fact that $\mathbf{v}$ and Γ are constant. Thus $\mathbf{v}\cdot\mathbf{F}_c = -\beta^2\Gamma\, D\overset{o}{Q}_R$. Using this and (3.6) in (3.5c), we arrive at $DQ_R = \Gamma^{-1} D\overset{o}{Q}_R$, which is just (3.1a).

Thus we have demonstrated that the two different equations of motion do in fact lead to different transformation laws for the heat. This simple example can also be used to illustrate the remark made previously that the formulation of relativistic thermodynamics is indeterminate because of the possibility of multiplying the heat term in the statement of the First Law by $f(\beta^2)$ where f is an arbitrary monotonic function such that $f(0) = 1$. To do this we note that the First Law for the Ott and Planck formulations is given by (3.4c) and (3.5c) respectively, where M^*c^2 plays the role of the internal energy of the system, $D(-Q_R)$ is the rate at which heat is injected into the system, and $\mathbf{v}\cdot\mathbf{F}_E = 0$ or $\mathbf{v}\cdot\mathbf{F}_c = \beta^2\Gamma D(-\overset{o}{Q}_R) = \beta^2\Gamma^2 D(-Q_R)$ is the rate at which mechanical work is performed on the system in the Ott and Planck formulations respectively. Thus the two statements that are the analog of $\Delta U = \Delta Q + \Delta W$ (U, Q, and W being internal energy, heat, and work, respectively) are as follows:

$$(\text{Ott}) \quad D(M^* c^2) = D(-Q_R); \tag{3.7}$$

$$(\text{Planck}) \quad D(M^* c^2) = D(-Q_R) + [\beta^2 \Gamma^2 D(-Q_R)]. \tag{3.8}$$

But (3.8) may also be written

$$(\text{Planck}) \quad D(M^* c^2) = (1 - \beta^2)^{-1} D(-Q_R). \tag{3.9}$$

Thus we arrive at the Planck formulation by replacing $D(-Q_R)$ in (3.7) by $(1-\beta^2)^{-1} D(-Q_R)$. We see then that the difference between the two formulations can be described as a difference in the form of the heat term appearing in the First Law, the difference being the factor $f(\beta^2) = (1-\beta^2)^{-1}$.

An important intuitive difference between the two formulations is apparent from a comparison of (3.7) with (3.8). In (3.7) no mechanical work appears. In particular, no work must be done to provide the increase in kinetic energy of the moving mass that results because its rest-mass M increases when it absorbs heat. The reason for this is that in the Ott formulation the quantity of heat that is transferred increases with velocity by just the amount necessary to account for the increase in kinetic energy caused by the increase in M. Another way of saying this is that the work necessary to speed up the heat energy (which has mass) was already done when the heat reservoir was brought to the velocity v. This kinetic energy of the heat is then regarded as an integral part of the heat energy, and is carried along into the mass M^* when the heat transfer is made.

In contrast to this straight-forward explanation in the Ott formulation, the situation in the Planck formulation seems rather exotic. At first glance it might appear that the work $\beta^2 \Gamma^2 \; D(-Q_R)$ that appears in (3.8) is just what is necessary to account for the increase in the kinetic energy of the heat, which is $(\Gamma - 1) D(-\overset{0}{Q}_R) = \Gamma(\Gamma - 1) \; D(-Q_R)$. But to order β^2 we find that the work is exactly twice as large as the increase in the kinetic energy of the heat. The reason for this is that from (3.1a) it follows that the heat absorbed from the moving reservoir is <u>less</u> than that which would be absorbed from a stationary reservoir. (In both cases there is no relative motion between mass and reservoir.) Another way to say this is that the moving reservoir has less heat energy to give than the same reservoir at rest ($Q_R < \overset{0}{Q}_R$), i.e. the kinetic energy of heat is <u>negative</u>. This defies a simple intuitive explanation.

It was this exotic nature of the Planck formulation that led Ott to reject it. He also set forth more abstract arguments to show that only his formulation, out of the infinity of possible formulations of relativistic thermodynamics, meets the necessary criteria for a satisfactory theory. In the present paper the problem of choosing one formulation and of all possible ones will not be faced. We shall simply pose the much more modest and concrete question, "Given a choice between the Planck and the Ott formulations, which is more appropriate to the needs of fluid dynamics?" The answer is obvious. The Ott formulation is more appropriate. This is immediately evident when one compares (2.19) with (3.4a) and makes the following identifications:

$$M \to m(1 + h/c^2) \tag{3.10a}$$

$$F_E^j \to \rho^{-1} \; [\partial^j P - \partial_k \, S^{jk} + f_E^j] \; ; \tag{3.10b}$$

$$- \,đQ_R^j/d\tau \to \pi^j \; . \tag{3.10c}$$

The Ott formulation allows a clean separation of all aspects of the fluid flow that are dependent on the presence of heat flux from those aspects that are dependent on pressure, viscosity, or external fields. It makes possible, moreover, a conceptual model for the term π^j in (2.19), namely the moving heat reservoir. (In the Planck formulation, it is very awkward to carry out a covariant mathematical implementation of the reservoir concept because the effects of the reservoir appear explicitly only as a contribution to the time-like component of F^j_c. In the Ott formulation, by contrast, all effects associated with the reservoir appear unmixed with the effects of any external forces in a single covariant package, the 4-vector- $dQ^j_R/d\tau$, which in the fluid case is identified with the 4-vector π^j.) In keeping with the reservoir model, we imagine that the fluid is in thermal contact with a heat reservoir which might be a very concrete and obvious thing, as in the case of two fluids (e.g. electron and ion gases) that coexist in space, each serving as a thermal reservoir for the other; or it might be more abstract, as in the case of a single fluid. In this case the reservoir can be defined only in terms of the tensor Q^{jk} which in turn is defined either in terms of local averages over the statistical fluctuations in the components of the exact stress-energy tensor for the fluid or in terms of an empirically postulated tensor function of the fluid variables. Regardless of the way in which the heat reservoir is defined, it is a useful abstraction that allows a further analysis of the basic equations (2.19) that would otherwise be impossible in the absence of specific information about the functional form of the force π^j appearing in (2.19).

Temperature Four-Vector

The 4-vector $đQ^j$ is inconvenient from the analytical point of view because it is not a perfect differential, i.e. it depends on the nature of the process by which a body gains or loses heat, and not just on its initial and final states. If, however, the heat transfer is reversible, it is possible to express $đQ^j$ as the product of the entropy change dS which characterizes the process by which the heat is transferred but is nevertheless a perfect differential, and the temperature 4-vector T^j which characterizes the thermal state of the body. Thus

$$đQ^j = T^j \, dS \tag{3.11}$$

where

$$T^j = \overset{o}{T} v^j / c = T\,(1,\ v/c), \tag{3.12a}$$

where, as in (3.2b),

$$T = \Gamma \overset{o}{T}, \tag{3.12b}$$

where $\overset{o}{T}$ is the temperature of the body in its own rest-frame, and T is its temperature in the observer's frame. The legitimacy of (3.11) follows from the fact that in the rest-frame of the body it is true that, for a reversible heat transfer,

$$\overset{o}{đQ} = \overset{o}{T}\, dS. \tag{3.13}$$

If now we multiply this equation by v^j/c we have, referring to (3.3), equation (3.11).

IV. RELATIVISTIC CARNOT CYCLE

In order to exploit the heat reservoir model, it will be necessary first to describe in detail the energy-momentum exchange that takes place between two moving reservoirs in thermal contact. This can best be accomplished by studying a relativistic Carnot cycle operating between two heat reservoirs having relative motion with respect to each other.

Such a cycle is executed by an engine of some kind that interacts with each of the reservoirs only when its temperature and velocity are exactly equal to those of the reservoir in question. This requirement is necessary in order for the heat transfer to be reversible. During the time the engine is absorbing or rejecting heat, it does no work. Work is performed only during the adiabatic transitions between reservoirs. Because during these transitions the velocity of the engine must be changed from the velocity of one reservoir to that of the other, a momentum exchange, as well as an energy exchange (i.e. performance of work), is necessary.

The energy and momentum aspects of the cycle are illustrated in figures 1 and 2 respectively. Using the sign convention that energy or momentum into the Carnot engine is positive, we have the following expressions for the total engine energy E (in the observer's frame) at each of the four corners of the cycle:

$$E_B = E_A + \Delta_h Q; \tag{4.1a}$$

$$E_C = E_B - \Delta_1 W = E_A + \Delta_h Q - \Delta_1 W; \tag{4.1b}$$

$$E_D = E_C - \Delta_c Q = E_A + (\Delta_h Q - \Delta_c Q) - \Delta_1 W; \tag{4.1c}$$

$$E_A = E_D + \Delta_2 W = E_A + (\Delta_h Q - \Delta_c Q) - (\Delta_1 W - \Delta_2 W). \tag{4.1d}$$

From (4.1d) it follows that

$$\Delta W = \Delta_1 W - \Delta_2 W = \Delta_h Q - \Delta_c Q \tag{4.2}$$

where ΔW is the total work done by the Carnot engine during the cycle.

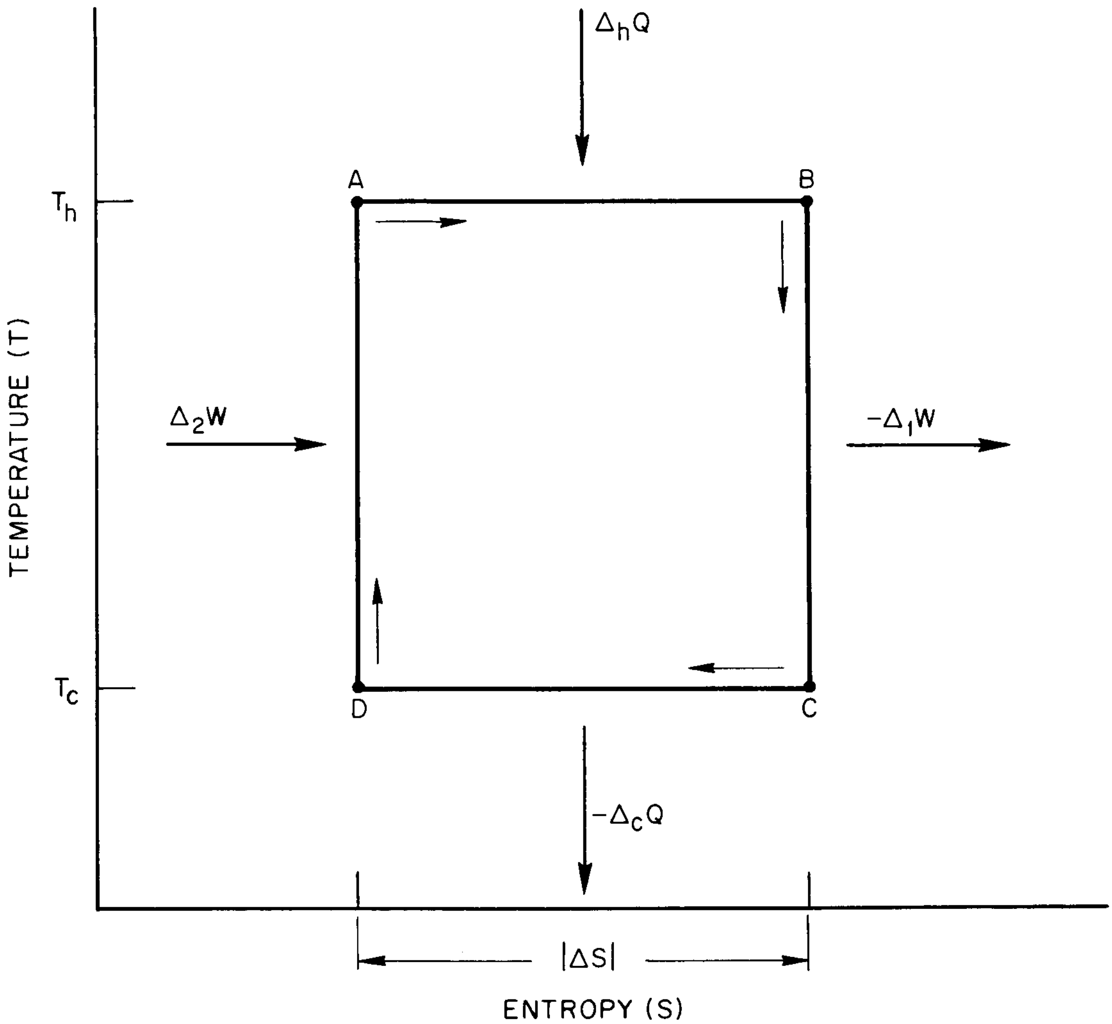

Figure 1–Energy Diagram for Carnot Cycle Between Moving Reservoirs

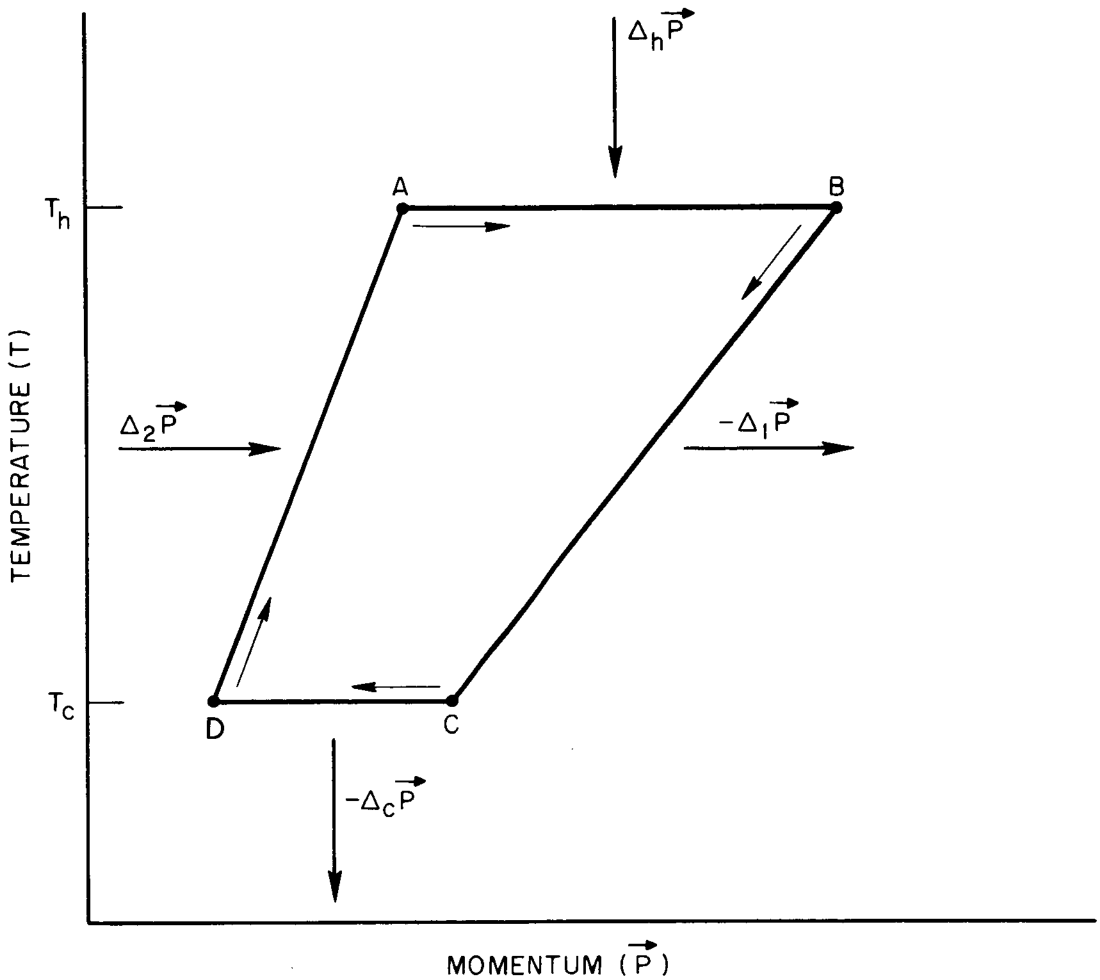

Figure 2–Momentum Diagram for Carnot Cycle Between Moving Reservoirs

Note that all of these relations are identical with the corresponding ones for the non-relativistic Carnot cycle between two stationary reservoirs. This would not have been the case if we had used the Planck, rather than the Ott, formulation, because then it would have been necessary to include extra terms in (4.1a) and (4.1c) corresponding to the term in brackets in (3.8).

Since the heat energy $\Delta_h Q$ that is transferred from the hot reservoir has a rest mass $\Delta_h \overset{o}{Q}/c^2$ and a velocity $\mathbf{v}_h$, we have for the transferred momentum $\Delta_h \mathbf{P}$ in the Ott formulation

$$\Delta_h \mathbf{P} = \Gamma_h (\Delta_h \overset{o}{Q}/c^2)\, \mathbf{v}_h = (\Delta_h Q/c^2)\, \mathbf{v}_h .$$

A similar relation holds for the momentum $-\Delta_c \mathbf{P}$ transferred to the cold reservoir. The momentum relations corresponding to the energy relations (4.1) are

$$\mathbf{P}_B = \mathbf{P}_A + (\Delta_h Q/c^2)\, \mathbf{v}_h; \tag{4.3a}$$

$$\mathbf{P}_C = \mathbf{P}_B - \Delta_1 \mathbf{P} = \mathbf{P}_A + (\Delta_h Q/c^2)\, \mathbf{v}_h - \Delta_1 \mathbf{P}; \tag{4.3b}$$

$$\mathbf{P}_D = \mathbf{P}_C - (\Delta_c Q/c^2)\, \mathbf{v}_c = \mathbf{P}_A + c^{-2}\, [\mathbf{v}_h \Delta_h Q - \mathbf{v}_c \Delta_c Q] - \Delta_1 \mathbf{P}; \tag{4.3c}$$

$$\mathbf{P}_A \,\square\, \mathbf{P}_D + \Delta_2 \mathbf{P} = \mathbf{P}_A + c^{-2}\, [\mathbf{v}_h \Delta_h Q - \mathbf{v}_c \Delta_c Q] - (\Delta_1 \mathbf{P} - \Delta_2 \mathbf{P}). \tag{4.3d}$$

Thus

$$\Delta \mathbf{P} = \Delta_1 \mathbf{P} - \Delta_2 \mathbf{P} = [\mathbf{v}_h \Delta_h Q - \mathbf{v}_c \Delta_c Q]/c^2 \tag{4.4}$$

where $\Delta \mathbf{P}$ is the total momentum transferred <u>away</u> from the Carnot engine during the cycle.

Now we note that if ΔQ_h is the heat gain of the hot reservoir, ΔS_h the corresponding entropy change, and T_h the reservoir temperature (all in the observer's frame), then using the time-like component of (3.11) we find

$$\Delta_h Q = -\Delta Q_h = -T_h \Delta S_h. \tag{4.5}$$

Similarly,

$$\Delta_c Q = \Delta Q_c = T_c \Delta S_c. \tag{4.6}$$

Because a Carnot cycle is reversible, the total entropy change of the two reservoirs must be zero. Thus

$$\Delta S_h = -\Delta S_c. \tag{4.7}$$

Using this in (4.5) we have

$$\Delta_h Q = T_h \Delta S_c. \tag{4.8}$$

Using (4.6) and (4.8) in (4.2) and (4.4) we find

$$\Delta W = (T_h - T_c)\, \Delta S_c \tag{4.9a}$$

and

$$\Delta \mathbf{P} = c^{-2}\, [T_h \mathbf{v}_h - T_c \mathbf{v}_c]\, \Delta S_c. \tag{4.9b}$$

Referring to (3.12a) we note that these two equations can be written as the following single 4-vector equation:

$$\Delta W^j = (\Delta W,\ c\Delta \mathbf{P}) = (T_h^j - T_c^j)\, \Delta S_c. \tag{4.10}$$

Thus the difference of the two temperature 4-vectors characterizing the hot and cold reservoirs multiplied by the entropy change that characterizes the Carnot cycle gives the energy-momentum 4-vector for the work and momentum that is extracted from the Carnot engine.

Application to Heat Reservoir Model

In order to relate the preceding analysis of the Carnot cycle to the problem of heat exchange between reservoir and fluid, it is necessary first to relate the idealizations of the former to the realities of the latter. The Carnot cycle is based on two idealizations: the ideal engine, and the ideal heat reservoir. An ideal engine can convert thermal energy and momentum into mechanical energy and momentum with no entropy generation. An ideal heat reservoir can, in its own rest-frame, reversibly absorb or reject heat energy, <u>but no momentum</u>. The point to be emphasized is that an <u>ideal</u> reservoir can in its own rest-frame absorb or reject <u>no momentum</u>. This is purely a matter of definition. A reservoir that violated this condition simply would not be "ideal."

To illustrate the point, consider a stationary probe of finite dimensions in a moving air stream. If the probe, which may be hotter or colder than the air stream and so exchanges heat with it, is so perfectly streamlined that the stream exerts no force on it, i.e. it absorbs no momentum from the stream, then it may be considered to constitute an ideal heat reservoir (if, of course the probe is connected to such a large heat source or sink that the heat exchange does not change its temperature). If, however, the streamlining of the probe is poor, so that a wake develops and momentum is absorbed from the stream, then the probe is not an ideal reservoir. Note that the absorption of momentum is associated with the generation of turbulence, and hence with an entropy increase. This illustrates that the condition that no momentum be absorbed is necessary in order for the heat exchange to be reversible.

We are now in a position to relate the Carnot cycle to the heat exchange between a heat reservoir (always regarded as "ideal") and a fluid. To do this we first identify the hot and cold reservoirs of the Carnot cycle with the ideal heat reservoir and the fluid. The fluid, however, is not an ideal reservoir. In particular, it is capable of absorbing (or rejecting) momentum in its own rest-frame. Thus the fluid is available to absorb the energy and momentum delivered by the Carnot engine to the external environment. It is not necessary that this energy and momentum be absorbed by the fluid. If suitable mechanisms are present, it could for example be absorbed by an electromagnetic field. For simplicity, however, we shall assume that the fluid absorbs the energy and momentum made available in the Carnot cycle. Thus the fluid is to be regarded as a combination of Carnot engine, heat reservoir, and sink (or source) for the energy and momentum delivered by (or to) the Carnot engine. In general these different aspects of the fluid will be so intimately intermingled that it would be difficult, if not impossible, to separate them. For example, consider again the probe in the air stream. If the probe is hotter than the stream, we may imagine the probe to emit radiation, i.e. photons, that are absorbed by the steam. The momentum as well as the energy of each photon is transferred to the fluid at the instant of absorption. It is obviously impossible to construct a detailed correspondence between the steps of the Carnot cycle and the emission and absorption of the photons.

The reverse case of a cold probe in a hot stream is a bit more complicated. In this case the boundary layer around the probe corresponds to the Carnot engine, i.e. the thermal intermediary between fluid and reservoir. We may imagine the boundary layer, which is at rest with respect to the probe, to consist of small blobs of gas, which for simplicity we may regard as being all of the same size. These are constantly being knocked out of the layer and replaced by similar blobs in the hot stream. Each time this occurs the newcomer gives all its momentum to the blob that it knocks out and, once settled in the boundary layer, gives up heat to the probe before being itself knocked out of the layer. Note that, because they are colder, the blobs that are knocked out of the boundary layer are slightly less massive than the blobs that replace them. They must, however, have the same momentum. Thus the blobs leaving the boundary layer have a slightly higher speed than those entering it. Since the speed of the stream after its encounter with the probe is slightly greater than before, whereas its momentum is the same, the kinetic energy of the stream is greater after passing the probe. We may interpret this increase in kinetic energy as the result of the work done by a "thermal recoil force" associated with the loss of heat to the probe. In the reverse case considered previously, when the fluid absorbed heat from the probe, there was a decrease in its kinetic energy which could be interpreted as work done by the fluid against the "thermal drag force" accompanying the heat absorption.

By describing the heat exchange between reservoir and fluid in terms of a Carnot cycle, it is possible to relate the magnitude of the thermal-recoil or drag force and the work done by it to the quantity of heat exchanged, and this is the justification for introducing the Carnot cycle even though it is usually very difficult to find a detailed correspondence between the steps of the cycle and the steps by which the heat exchange takes place in a real fluid.

We start by rewriting (4.9) as follows:

$$\Delta W = \Delta_R Q - \Delta_F Q; \tag{4.11a}$$

$$\Delta \mathbf{P} = (\Delta_R Q/c^2)\ \mathbf{v}_R - (\Delta_F Q/c^2)\ \mathbf{v}_F; \tag{4.11b}$$

where the subscripts R and F refer to "reservoir" and "fluid" respectively. The quantities $\Delta_R Q$ and $\Delta_F Q$ are both positive if heat is transferred from reservoir to fluid, and are negative if the transfer is in the opposite direction.

Since we are assuming that all the work ΔW performed during the cycle goes into (or comes out of) the kinetic energy of the fluid, and that the momentum $\Delta \mathbf{P}$ is absorbed by the fluid, we impose the following requirement:

$$\Delta W = \mathbf{v}_F \cdot \Delta \mathbf{P}. \tag{4.12}$$

(4.11a) then becomes

$$\Delta_R Q = \Delta_F Q + \mathbf{v}_F \cdot \Delta \mathbf{P}. \tag{4.13}$$

The interpretation of this equation is very simple. For the case in which heat flows from reservoir to fluid (positive $\Delta_R Q$ and $\Delta_F Q$) (4.13) says that if $\mathbf{v}_F \cdot \Delta \mathbf{P} > 0$ then, of the heat energy $\Delta_R Q$ that is surrendered by the reservoir, the quantity $\Delta_F Q$ is absorbed by the fluid and the amount $\mathbf{v}_F \cdot \Delta \mathbf{P}$ is converted into an increase in kinetic energy of the fluid. If $\mathbf{v}_F \cdot \Delta \mathbf{P} < 0$, then the heat $\Delta_F Q$ that is absorbed by the fluid is greater than the heat $\Delta_R Q$ given up by the reservoir. The difference is made up by a conversion of fluid kinetic energy into heat. Similar interpretations apply when $\Delta_R Q$ and $\Delta_F Q$ are negative.

The equation (4.11b) gives (when divided by Δt) the relativistic thermal force acting on the fluid. When heat flows from reservoir to fluid, it is a drag force that tends to accelerate the fluid in the direction in which the reservoir is moving. When heat flows from fluid to reservoir, it is a recoil force.

Note that, because of the requirement (4.12), no work is delivered by the Carnot cycle in the rest-frame of the fluid. If work were delivered in the fluid rest-frame, the only way to dispose of it would be to convert it into heat and inject it into the fluid. But this would produce an entropy increase, with the consequence that the cycle as a whole, including injection of ΔW and $\Delta \mathbf{P}$ into the fluid, would have a positive entropy increase, and so would be irreversible. Thus the condition (4.12) is equivalent to the requirement that the heat transfer between reservoir and fluid be reversible. When (4.12) holds, then the 4-vector ΔW^j defined in (4.10) has the form

$$\Delta W^j = [(\mathbf{v}_F \cdot \Delta \mathbf{P}),\ c\Delta \mathbf{P}] \tag{4.14}$$

with the result that

$$v_F^j \, \Delta W_j = 0 . \tag{4.15}$$

We may rewrite (4.10) as follows:

$$T_R^j = T_F^j + \Delta W^j / \Delta S_F . \tag{4.16}$$

Using the fact that $T^j_R = T_R(1, v_R/c)$ and $T^j_F = T_F(1, v_F/c)$ where T_R and T_F are respectively the reservoir and fluid temperatures in the observer's frame, and v_R and v_F are the respective 3-velocities, and substituting (4.14) into (4.16), we arrive at the following relations:

$$T_R/T_F = 1 + (v_F \cdot \Delta P)/T_F \Delta S_F; \tag{4.17a}$$

$$v_R = (T_R/T_F)^{-1} \; [v_F + \Delta P/(T_F \Delta S_F/c^2)] . \tag{4.17b}$$

These relations give the reservoir temperature and velocity (in the case of reversible heat transfer) in terms of the fluid temperature and velocity and the momentum and heat increments of the fluid, given by ΔP and $T_F \, \Delta S_F$ respectively.

In order to cast these relations into a form suitable for use in the fluid equation of motion, we must specify that ΔS_F and ΔW^j refer to increments occurring in unit rest volume of fluid per unit proper time. (Referring to (1.6a) and (1.11), we note that $(1/\overset{\circ}{U}) \; d/d\tau = (1/U) D$. Therefore it makes no difference whether ΔS_F and ΔW^j refer to unit volume and time in the fluid rest frame or in the observer's frame.) Thus $\Delta S_F / \Delta\tau$ is the entropy change per unit <u>volume</u> per unit time. Since $\overset{\circ}{U}$ is the volume occupied by unit <u>mass</u> of the fluid, $\overset{\circ}{U}(\Delta S_F/\Delta\tau) = (\rho m)^{-1} \; (\Delta S_F/\Delta\tau)$ is the entropy change per unit <u>mass</u> per unit time. But entropy per unit mass is just the specific entropy s. Thus we arrive at the relation

$$\Delta S_F = \rho m \dot{s} \, \Delta\tau, \tag{4.18}$$

where $\dot{s} = ds/d\tau$. Note that $m\dot{s}$ may be regarded as the entropy rate of change <u>per particle.</u>

To arrive at the corresponding relation for ΔW^j we note that the space-like part of $\Delta W^j/c$ is a momentum transfer, so $\Delta W^j/c\Delta\tau$ is the 4-force acting on unit rest volume of the fluid and $\overset{o}{U}\,(\Delta W^j/c\Delta\tau)$ is the 4-force acting on unit mass of the fluid. Thus $m\overset{o}{U}\,(\Delta W^j/c\Delta\tau) = (\rho c)^{-1}\ \Delta W^j/\Delta\tau$ is the 4-force <u>per particle</u>. Designating this thermal 4-force per particle by φ^j we have

$$\Delta W^j = \rho c \varphi^j \ \Delta\tau . \tag{4.19}$$

From (4-15) it is evident that

$$v_j \, \varphi^j = 0, \tag{4.20}$$

which means that φ^j has the form

$$\varphi^j = \Gamma\left[(\mathbf{v}\cdot\boldsymbol{\varphi}/c),\ \boldsymbol{\varphi}\right] \tag{4.21}$$

where $\boldsymbol{\varphi}$ is the thermal 3-force per particle.

We shall henceforth drop the subscript F since it will be understood that all quantities without a subscript refer to the fluid.

Dividing (4.19) by (4.18) and using (4.16) and (3.12a), we arrive at the following 4-vector equation:

$$m\dot{s}\, T_R^j = m\dot{s}\, \overset{o}{T}\, v^j/c + c\varphi^j \tag{4.22a}$$

which is equivalent to

$$mT_R\, Ds = mT\, Ds + \mathbf{v}\cdot\boldsymbol{\varphi} \tag{4.22b}$$

and

$$(mT_R Ds/c^2)\, \mathbf{v}_R = (mT\, Ds/c^2)\, \mathbf{v} + \boldsymbol{\varphi}. \tag{4.22c}$$

It will prove convenient to introduce the 3-velocity $\mathbf{w}$ defined by the following relation:

$$\mathbf{w} = (mT\,Ds/c^2)^{-1}\, \boldsymbol{\varphi} = (m\overset{\circ}{T}\, \dot{s}/c^2)^{-1}\, \boldsymbol{\varphi}. \tag{4.23a}$$

The corresponding 4-vector equation can be written

$$\varphi^j = (m\overset{\circ}{T}\, \dot{s}/c^2)\, w^j . \tag{4.23b}$$

Referring to (4.20) and (4.21) we see that

$$v_j\, w^j = 0 \tag{4.24}$$

and

$$w^j = \Gamma[(\mathbf{v}\cdot\mathbf{w}/c),\ \mathbf{w}] . \tag{4.25}$$

In terms of w^j and $\mathbf{w}$, equations (4.22) become

$$T^j_R = (\overset{\circ}{T}/c)\, (v^j + w^j)\, ; \tag{4.26a}$$

$$T^0_R = T_R = T\,(1 + \mathbf{v}\cdot\mathbf{w}/c^2); \tag{4.26b}$$

$$c(T^1_R,\ T^2_R,\ T^3_R) = T_R\, \mathbf{v}_R = T(\mathbf{v} + \mathbf{w}). \tag{4.26c}$$

It follows from (4.26a), (4.24), and (1.4) that

$$v_j\, (m\, T^j_R\, \dot{s}/c) = m\overset{\circ}{T}\dot{s}. \tag{4.27a}$$

This result is a consequence of the condition (4.12) which, as previously explained, is equivalent to the requirement that the heat transfer between reservoir and fluid be reversible. From (4.22a), (4.26a), and the definition of σ_{jk} given in (1.7), we have

$$\sigma_{jk}\,(m\,T_R^k\,\dot{s}/c) = \varphi_j = (m\overset{\circ}{T}\,\dot{s}/c^2)\;w_j\,. \tag{4.27b}$$

The two relations given in (4.27) will be of interest in the next section where we shall identify the 4-vector $m\,T_R^j\,\dot{s}/c$ with the vector π^j appearing in the fluid equation of motion (2.19).

V. ENTROPY FORCE TENSOR

We must now incorporate the formalism of the heat reservoir model into the fluid equation of motion (2.19). The first step is to eliminate from this equation the term $(\partial^j \mathrm{P})/\rho$ in favor of terms involving the 4-gradients of the specific enthalpy h and the specific entropy s, which will then be regarded as the two independent thermodynamic variables of the fluid. This can be accomplished by means of the identity

$$\begin{aligned}\partial^j h(\mathrm{P},\ s) &= (\partial h/\partial \mathrm{P})_s\ \partial^j \mathrm{P} + (\partial h/\partial s)_{\mathrm{P}}\ \partial^j s\\ &= \overset{\circ}{\mathrm{U}}\ \partial^j \mathrm{P} + \overset{\circ}{T}\ \partial^j s\end{aligned} \tag{5.1}$$

where for the purpose of arriving at the desired relation we have regarded h to be a function of P and s. In the second line of (5.1) we have made use of two well-known thermodynamic identities. Since $\overset{\circ}{\mathrm{U}} = 1/\rho m$, (5.1) can be written

$$(\partial^j \mathrm{P})/\rho = m\,\partial^j h - m\overset{\circ}{T}\ \partial^j s \tag{5.2}$$

which is the identity we need in order to eliminate the pressure gradient from the equation of motion (2.19), which now becomes

$$d[m(1 + h/c^2)\, v^j]/d\tau = m\partial^j h - m\overset{0}{T}\, \partial^j s + \pi^j - (\partial_k S^{jk})/\rho + f^j_E/\rho, \qquad (5.3a)$$

where

$$\pi^j = -(\partial_k Q^{jk})/\rho. \qquad (5.3b)$$

Note that only the <u>sum</u> of the tensors Q^{jk} and S^{jk} appears in the equation of motion in the form $\rho^{-1}\, \partial_k\, (Q^{jk} + S^{jk})$, and the distinction between the two tensors is not uniquely determined by the orthogonality conditions (2.5). We could, for example, replace Q^{jk} and S^{jk} with the tensors $\hat{Q}^{jk} = Q^{jk} + \tilde{S}^{jk}$ and $\hat{S}^{jk} = S^{jk} - \tilde{S}^{jk}$ where $\tilde{S}^{jk}$ is an arbitrary symmetric tensor for which $\tilde{S}^{jk} u_k = 0$. The tensors $\hat{Q}^{jk}$ and $\hat{S}^{jk}$ would satisfy the same orthogonality conditions (2.5) as Q^{jk} and S^{jk}, and since $\hat{Q}^{jk} + \hat{S}^{jk} = Q^{jk} + S^{jk}$ the equation of motion would be unchanged by the substitution of $\hat{Q}^{jk}$ and $\hat{S}^{jk}$ for Q^{jk} and S^{jk}. This indeterminacy can be partially removed by imposing the condition

$$v_j\, \partial_k S^{jk} = 0. \qquad (5.4)$$

The interpretation of this condition is that the viscosity stress tensor S^{jk} is not to be associated with heat injection into the fluid. In actual fact, of course, viscosity is accompanied by heat generation, but we regard this as being described by the heat tensor Q^{jk}, along with any other sources of heat injection. In other words, condition (5.4) simply says that Q^{jk} describes <u>all</u> the heat transferred to and from the fluid. As an extension of this requirement we assume that the external force f^j_E also satisfies the condition

$$v_j\, f^j_E = 0. \qquad (5.5)$$

Contracting (5.3a) with v_i and using (1.4), (1.5), (5.4), and (5.5), we arrive at the following result:

$$v_j\,(\pi^j - m\overset{\circ}{T}\,\partial^j s) = 0. \tag{5.6}$$

We shall see that the heat reservoir formalism automatically fulfills this requirement.

Because π^j is just the 4-vector that describes the time rate per particle at which heat energy and momentum are injected into the fluid, we make the following identification:

$$\pi^j = mT_R^j\,\dot{s}/c = (m/c)\,T_R^j\,(ds/d\tau). \tag{5.7}$$

Since from (3.12a) $T_R^j = T_R\;(1,\; \mathbf{v}_R/c)$, we have

$$\pi^j = [(mT_R\dot{s})/c,\; (mT_R\dot{s}/c^2)\,\mathbf{v}_R]\,. \tag{5.8}$$

Since $mT_R\,\dot{s}$ is the time rate (in the fluid rest-frame) at which heat energy is injected per particle, it is obvious that $c\pi^0$ gives the rate of energy injection, and the space-like part of π^j gives the momentum that is injected by virtue of the fact that the injected heat energy, which necessarily has mass, is thrown off from a reservoir moving with the velocity $\mathbf{v}_R$. Note that from (4.27a) and (5.7) it follows that $v_j\,\pi^j = m\overset{\circ}{T}\,\dot{s}$. Thus the expression for π^j given in (5.7) is consistent with the condition (5.6).

Using (4.27a) in (5.7), we can write π^j in the following form:

$$\pi^j = (m\overset{\circ}{T}\dot{s}/c^2)\,v^j + \varphi^j \tag{5.9}$$

where $(m\overset{o}{T}\dot{s}/c^2)\, v^j$ is just the momentum (and energy) per particle that the fluid absorbs along with the heat that it absorbs, and φ^j is the thermal 4-force per particle that acts on the fluid as a result of the fact that the reservoir velocity is not identical with the fluid velocity. Using the expression (5.9) for π^j, the fluid equation of motion (5.3a) can be cast into the following form:

$$d(\tilde{m}\, v^j)/d\tau - (m\overset{o}{T}\,\dot{s}/c^2)\, v^j = \partial^j(\tilde{m}\,c^2) - m\overset{o}{T}\,\partial^j s + \varphi^j - (\partial_k S^{jk})/\rho + f^j_E/\rho \tag{5.10}$$

where

$$\tilde{m} = m(1 + h/c^2)\,. \tag{5.11}$$

The variable mass $\tilde{m}$ includes the relativistic mass per particle that results from the heat energy of the fluid as represented by the enthalpy h, which plays the role of a thermal potential. This variable mass plays the role of the particle mass except in those terms involving the specific entropy which, by definition, is the entropy of a given sample of fluid divided by the rest mass of this sample for $\overset{o}{T} = 0$, i.e. the mass Nm where N is the number of particles in the sample. The mass $\tilde{m}$ plays a double role in the sense that the inertial properties of the particle are specified by $\tilde{m}\, v^j$, while the energy $\tilde{m}\, c^2$ acts as a potential function whose gradient is one of the applied forces acting on the particle.

The left side of (5.10) is just that part of the particle momentum change per unit time that must be accounted for by the application of external forces. As was previously explained, the momentum change rate $(m\overset{o}{T}\dot{s}/c^2)\, v^j$ does not have to be produced by an external applied force, because it is delivered along with the heat absorbed by the fluid, and so it must be subtracted from the total particle momentum change $d(\tilde{m}\, v^j)/d\tau$ if only applied external forces are to appear on the right side of the equation.

Although (5.10) provides a clear insight into the dynamical effects of heat absorption, a more advantageous form of the equation from the formal point of view is provided by introducing an antisymmetric tensor Θ^{jk} defined as follows:

$$\Theta^{jk} = T^j_R\, \partial^k s - T^k_R\, \partial^j s. \tag{5.12}$$

Using (4.27a) and (5.7) we find that

$$(m/c)\, \Theta^{jk}\, v_k = (m/c)\, T^j_R\, \dot{s} - m\overset{\circ}{T}\, \partial^j s = \pi^j - m\overset{\circ}{T}\, \partial^j s. \tag{5.13}$$

(Note that the condition (5.6) is automatically satisfied because of the antisymmetry of Θ^{jk}.) Using (5.13) in (5.3a), the equation of motion becomes

$$d(\tilde{m}\, v^j)/d\tau = \partial^j(\tilde{m}\, c^2) + (m/c)\, \Theta^{jk}\, v_k - (\partial_j\, S^{jk})/\rho + f^j_E/\rho. \tag{5.14}$$

As previously noted, we regard h and s as the two independent thermodynamical variables of the fluid. (Very often the relation $h = c_p \overset{\circ}{T}$ is a valid approximation where the constant-pressure specific heat c_p is constant. In such cases, the thermodynamical variables could be regarded as $\overset{\circ}{T}$ and s.) Because $\tilde{m}$ depends only on h, all of the dynamical effects of the entropy are described by the tensor Θ^{jk}, which we shall call the <u>entropy force tensor</u>. This tensor vanishes if s is everywhere constant, and (5.14) becomes the equation for relativistic isentropic flow, which has (for $S^{jk} = f^j_E = 0$) already been given by Khalatnikov.[15]

The introduction of the entropy force tensor Θ^{jk} allows the entropy-dependent force to be handled in a way that is completely analogous to the electromagnetic force. To make this more apparent, let us assume that we are dealing with a fluid of charged particles, such as an electron or ion gas, and that f_E^j is just the electromagnetic 4-force per unit volume. Then f_E^j/ρ is the electromagnetic force per particle which is

$$f_E^j/\rho = (q/c)\, F^{jk}\, v_k \tag{5.15}$$

where q is the particle charge and F^{jk} is the electromagnetic field tensor which is antisymmetric and, in terms of the electric and magnetic field intensities **E** and **B**, has the form

$$(F^{10},\ F^{20},\ F^{30}) = \mathbf{E}; \tag{5.16a}$$

$$(F^{23},\ F^{31},\ F^{12}) = -\mathbf{B}. \tag{5.16b}$$

Using (5.15), the equation of motion (5.14) becomes

$$d(\tilde{m}\, v^j)/d\tau = \partial^j(\tilde{m}\, c^2) + (m/c)\, \Theta^{jk}\, v_k - (\partial_k\, S^{jk})/\rho + (q/c)\, F^{jk}\, v_k. \tag{5.17}$$

It is evident then that the entropy-dependent force term has the form of an electromagnetic force that is coupled to the fluid through the particle mass m rather than the particle charge q, and for which Θ^{jk} plays the role of the electromagnetic field tensor.

Because $T_R^j = T_R\,(1,\ v_R/c)$, (5.12) can be written in more explicit form as follows:

$$(\Theta^{10},\ \Theta^{20},\ \Theta^{30}) = T_R\, \nabla s + \mathbf{v}_R\,(T_R\, \partial s/c^2\, \partial t); \tag{5.18a}$$

$$(\Theta^{23},\ \Theta^{31},\ \Theta^{12}) = (T_R\, \nabla s) \times \mathbf{v}_R/c. \tag{5.18b}$$

Summarizing the analogy, we have

$$\Theta^{jk} \to F^{jk}, \tag{5.19a}$$

$$m \to q, \tag{5.19b}$$

$$T_R \nabla s + v_R (T_R \partial s/c^2 \partial t) \to \mathbf{E}, \tag{5.19c}$$

$$(T_R \nabla s) \times v_R/c \to -\mathbf{B}, \tag{5.19d}$$

where the arrow is to be read "corresponds to."

Before this formalism can be put to use in any specific problem, some way must be found to specify the 4-vector T^j_R. There are two approaches to this problem: (1) We could take the point of view that the form of the heat tensor Q^{jk} is known, and that the time rate of entropy generation is given as a known function of other variables of the problem. Then from (5.3b) and (5.7) we have

$$T^j_R = -(\partial_k Q^{jk})/(\rho m \dot{s}/c), \tag{5.20}$$

which we would regard as defining T^j_R. (2) The physical conditions of the problem may be such that it is very easy to specify T^j_R directly. For example, if we are dealing with a fluid that is either losing heat through radiation, or gaining it because of nuclear or chemical reactions taking place within the fluid itself, we are dealing with the simple case of a <u>co-moving heat reservoir</u>. In this case $T^j_R = T^j = \overset{o}{T} v^j/c$. A slightly more complicated example is provided by a plasma consisting of an ion and an electron gas. If the ion gas is gaining heat through nuclear reactions taking place within itself, and then passing some of this heat on to the electron gas in such a way that we could assume that the temperatures of the two gases as seen in the electron rest-frame are equal, then $T^j_R = T^j_{ion}$. Thus the reservoir is co-moving with the ion gas, but has a relative velocity with respect to the electron gas. In all of these cases it would be necessary to derive an independent expression for $\dot{s}$ from the physics of the situation before the formalism could be applied to the solution of the problem.

VI. CONCLUSIONS

We have seen that the Ott formulation of relativistic thermodynamics, which treats temperature and heat as the time-like components of two 4-vectors, allows the dynamical effects of heat flux in the fluid to be treated in terms of a moving heat reservoir model. This in turn makes it possible to describe the dynamical effects of the temporal and spatial variation of the specific entropy of the fluid in terms of an antisymmetric entropy force tensor, which is analogous to the electromagnetic field tensor. In the following paper this analogy will be further developed in that a generalized Larmor theorem will be derived in which the role of the entropy force tensor will be seen to be completely analogous to that of the electromagnetic field tensor.

REFERENCES

1. Surveys of relativistic fluid dynamics can be found in most Relativity textbooks. Of the available brief surveys, probably the most relevant to the present work is the one given in Chap. XV of Fluid Mechanics by L. D. Landau and E. M. Lifshitz (Addison-Wesley, Reading, Mass., 1959). More detailed discussions are given by A. Lichnerowicz, Théories Relativistes de la Gravitation et de l'Électromagnetisme (Masson, Paris, 1955), Chaps. IV-VI, and by F. Halbwachs, Théorie Relativiste des Fluides à Spin (Gauthier-Villars, Paris, 1960), Chap IV, § 2, and Appendix B.

   The relativistic equations of fluid dynamics were first derived independently by G. Herglotz, Ann. d. Physik (4) 36, 493 (1911) and E. Lamla, Ann. d. Physik (4) 37, 772 (1912). Lamla's equations were less general in that they were limited to the case of isentropic flow, whereas this restriction was not imposed in Herglotz's derivation. Subsequent research has been limited almost exclusively either to isentropic flow [e.g. I. M. Khalatnikov, J. Exptl. Theoret. Phys. (U.S.S.R.) 5, 529 (1954), a brief review of which is given in Landau and Lifshitz cited above, and A. H. Taub, Phys. Rev. 103, 454 (1956)] or to the somewhat more general case of a barotropic fluid, i.e. one for which the pressure may be regarded as a function of the density alone (e.g. J. L. Synge, Proc. London Math. Soc. (2) 43, 376 (1937), and the work of Lichnerowicz, which is summarized in his book cited above). The papers cited in references 2 and 3 are not, however, restricted to either of these two special cases.

2. C. Eckart, Phys. Rev. 58, 919 (1940). This paper concentrates on the thermodynamical aspects of relativistic fluid dynamics. Of all the fluid dynamical

references cited, it and Khalatnikov's paper are the most relevant to the present work. Both of these papers, like the present work, are within the framework of Special, rather than General, Relativity.

3. G. Pichon, Ann. Inst. Henri Poincaré A2, 21 (1965). This work, like that of Synge, Lichnerowicz, and Taub cited in reference 1, is within the framework of General Relativity. Its relevance to the present work lies in the fact that it presents alternatives to the treatment of viscosity and heat conduction in a charged fluid that differ from the approach taken in the present work.

4. In the relativistic literature it is customary to call a fluid in which heat flux and viscosity are absent a "perfect fluid." This can, however, cause confusion with the term "perfect gas" which is almost universally used to designate a fluid that obeys the Perfect Gas Law, and indicates nothing about the absence or presence of viscosity or heat flux. (Kinetic theory tells us in fact that both can be present in a perfect gas.) To avoid any confusion, I shall follow the nomenclature that is common in non-relativistic fluid dynamics and use the term "ideal fluid" to indicate the absence of viscosity and heat flux, and the isotropy of pressure.

5. See, for example, L. Landau and E. Lipshitz, (cited in reference 1), Chap. XV.

6. A. H. Taub, Phys. Rev. 103, 454 (1956). Eckart (ref. 2) derived his formulation of the stress-energy tensor by means of macroscopic thermodynamic arguments. Taub arrived at the same form for the tensor through a kinetic theory argument.

7. H. Ott, Z. Physik 175, 70 (1963)

8. H. Arzeliès, Nuovo cimento 35, 792 (1965).

9. A. Gamba, Nuovo cimento 37, 1792 (1965).

10. M. Planck, Ann. d. Physik (4) 76, 1 (1908). Further references are given by W. Pauli, Theory of Relativity (Pergamon Press, London, 1958), p. 134.

11. M. von Laue, Die Relativitätstheorie I (Friedr. Vieweg and Sohn, Braunschweig, 1961), 7th ed. (First edition, 1911). See footnotes on p. 138 and p. 177–178.

12. See, for example, W. Pauli (cited in reference 10), p. 134; R. Tolman, Relativity, Thermodynamics, and Cosmology (Oxford, 1934), Chap. V.; and C. Møller, The Theory of Relativity (Oxford, 1952), § 78.

13. See, for example, A. Sommerfeld, Mechanics (Academic Press, New York, 1952), p. 29, eq. 4.

14. See, for example, C. Møller (cited in reference 12), p. 106.

15. Cited in reference 1. Khalatnikov's equation amounts to a rediscovery of the equations of Herglotz and Lamla (cited in reference 1), except that the role of the specific enthalpy as the thermal potential is more explicit in Khalatnikov's work than in that of either Herglotz or Lamla.

The relativistic equation of motion for an ideal charged barotropic fluid is given in Lichnerowicz's book, which was cited in reference 1 (chap. VI, eq. 55-7). This equation is equivalent to (5.14) above (for $\partial_j S^{jk} = 0$) for the isentropic case ($\Theta^{jk} = 0$) except that the thermal potential is not the specific

enthalpy, but rather a generalized barotropic thermal potential called the index-function, which was first introduced by Synge (cited in reference 1) and further developed by Lichnerowicz. (A similar function had already been introduced by L. P. Eisenhart, Trans. Am. Math. Soc. 26, 205 (1924)).