Preservation of Subband Symmetry in Multirate Signal Coding

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Abstract—This correspondence derives necessary and sufficient conditions for a multirate linear phase FIR filter bank, acting on a symmetrically extended finite-length input signal, to produce either symmetric or antisymmetric downsampled subbands. Such techniques have gained popularity recently for image coding applications, and we provide the solution to a key technical problem in terms of the input and filter symmetries and the downsampling ratio.

I. Introduction.

Figure 1 depicts the analysis half of a critically decimated M-channel multirate filter bank (MFB) [1]. Two-dimensional (product) MFB's have been used for image coding (see, e.g. [2], [3], [4], [5], [6]) because they furnish good space/frequency localization without the need for block coding as in DCT image coders [7], [8], thereby eliminating blocking artifacts inherent in block transforms. One problem with using filter banks for coding finite-duration data, like row or column vectors in a digital image, is handling boundary conditions at the ends of the input vector. It is necessary to define an extension, y, of a finite data vector, x, to compute transform coefficients when the filters extend beyond the given data. Periodic extension or zero-padding introduce jump discontinuities in the input signal; if x is fairly smooth, these discontinuities add variance to highpass subbands, reducing the filter bank's coding gain [9], [10]. Note that the DCT–II [7] is mathematically equivalent to a shift of the DFT of an even extension of x, and thus does not introduce a jump discontinuity in the input.

Another important fact is that the DCT conserves the number of samples in the signal by mapping N_0 points of real-valued input to N_0 real outputs. Borrowing terminology from [3], we say that the DCT is a *nonexpansive* transform. If an MFB's downsampling ratio, M, divides the length, N_0, of the input (denoted "$M \mid N_0$") and if the filters have lengths $\leq N_0$, then the MFB can be implemented nonexpansively by N_0-point circular convolution and circular M:1 downsampling, resulting in M subbands of period N_0/M. The same technique works with nonuniformly downsampled filter banks if all of the downsampling ratios divide N_0.

The jump discontinuity in circular implementations can be eliminated by transforming an even extension of the input signal. This doubles the length of the input, though, apparently doubling the number of output samples. The key to doing this nonexpansively is to use linear phase filters that produce *symmetric* subbands [2], [3], [11], [5], [6], [12]. By saving (roughly) one half of each symmetric subband's period, the overall size of this *symmetric extension transform* can be reduced to N_0 outputs, making the implementation nonexpansive.

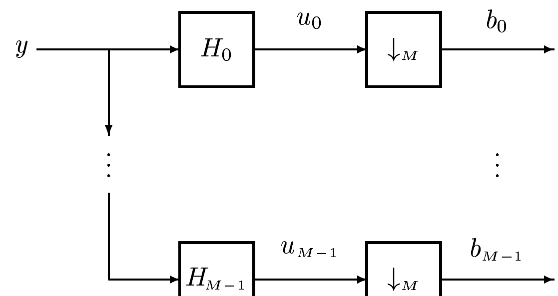

Fig. 1. An M-channel multirate analysis filter bank.

A key technical issue is determining the combinations of symmetric input extension and filter symmetry and phase that result in symmetric (or antisymmetric) subbands, a problem complicated by the lack of shift invariance in MFB's. By drawing many pictures (as was done in, e.g., [5], [6]), it is possible to convince oneself that one has exhausted all such combinations of extension symmetry, filter symmetry and phase, and downsampling ratio. The situation is complicated enough, however, that the author felt an algebraic derivation was also appropriate to confirm his pictorial intuition and enumerate the possibilities precisely and completely. Since the algebraic derivation turns out to be somewhat nontrivial and since a precise statement and correct derivation of necessary and sufficient conditions have not yet appeared in the literature, the author offers this correspondence, which furnishes technical results used in the complete classification [13] of all nonexpansive symmetric extension transforms.

A. Symmetric Extensions and Linear Phase Filters.

We follow [14], [15] and call a discrete-time signal *whole-sample symmetric* (WS) if it is symmetric about one of its samples and *half-sample symmetric* (HS) if it is symmetric about a point halfway between two samples. Similarly, there are corresponding properties of *whole-* and *half-sample antisymmetry* (WA/HA). Three different symmetric, periodic extensions, y, of an input, x, of length N_0 are shown in Figure 2. The open dots represent a second period of each extension; the period of y will be denoted N. The notation, $y = E_s^{(i,j)}x$ (i,j=1 or 2), means that y is a symmetric extension of x in which the first sample, $x(0)$,

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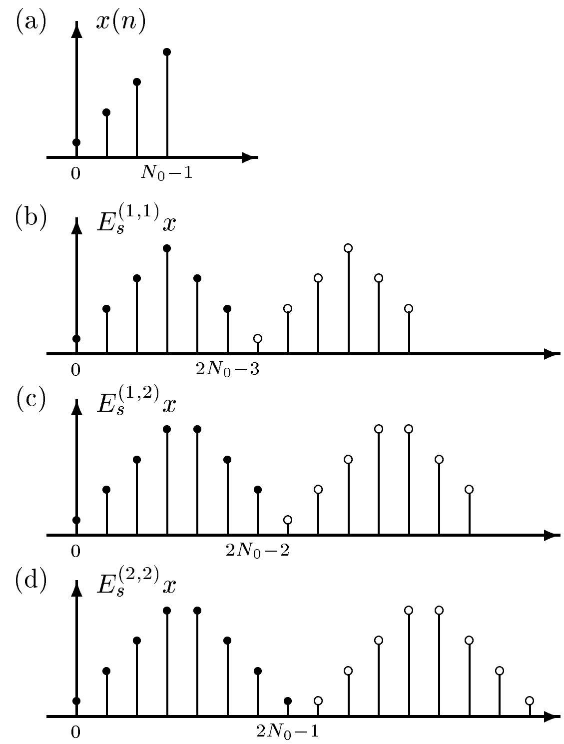

Fig. 2. Three symmetric extensions, $y(n)$, for signal analysis.

appears i times and the last sample, $x(N_0-1)$, appears j times. Extensions $E_s^{(1,1)}x$ (with period $N=2N_0-2$) and $E_s^{(1,2)}x$ ($N=2N_0-1$) are both WS about 0, while $E_s^{(2,2)}x$ ($N=2N_0$) is HS about $-1/2$. In [13] it is shown that these are the only symmetric extensions, up to trivial phase shifts, that produce nonexpansive transforms.

Real-coefficient linear phase FIR filters will be referred to in terms of their symmetry properties. If h is a linear phase filter, its center of symmetry, $n=\gamma$ (where γ is also h's group delay), is an integer for W-type (WS or WA) filters and an odd multiple of $1/2$ for H-type filters. Since filters are applied to y by N-point circular convolution, h is extended with zeros to compute its N-point DFT, $H(k)$. Symmetric filters, of either type, satisfy

$$H^*(k)=e^{j2\pi(2\gamma)k/N}H(k), \tag{1}$$

while antisymmetric filters satisfy

$$H^*(k)=-e^{j2\pi(2\gamma)k/N}H(k). \tag{2}$$

It is easy to deduce the symmetries that result from applying linear phase filters to symmetric extensions by circular convolution, $u=y*h$; e.g., if $y=E_s^{(1,1)}x$ then $Y^*(k)=Y(k)$, so if h is HS then (1) implies

$$U^*(k)=Y(k)H(k)e^{j2\pi(2\gamma)k/N}=e^{j2\pi(2\gamma)k/N}U(k).$$

Since 2γ is odd, u is HS with a center at γ. The other combinations of signal and filter symmetry are listed in Table I; these results also appeared, using different notation, in [3], [5], [6].

TABLE I
Symmetry and Center of Convolution Output, $u=y*h$, in Terms of Filter Group Delay, γ.

		Sym., center of y: WS, 0	Sym., center of y: HS, $-1/2$
Filter h:	WS	WS, γ	HS, $\gamma-1/2$
	WA	WA, γ	HA, $\gamma-1/2$
	HS	HS, γ	WS, $\gamma-1/2$
	HA	HA, γ	WA, $\gamma-1/2$

The more difficult problem addressed in detail below is determining precisely when $M:1$ decimation of a symmetric filter output, u, results in a symmetric subband, b. Besides giving a rigorous solution to this problem, we improve on previous presentations by tabulating subband symmetries as a function of signal and filter symmetries and downsampling ratio.

II. Symmetry of Downsampled Subbands.

We first observe the following intertwining relation (or "noble identity" [1]) for interchanging decimators and shifts: if $\delta_M(n)$ is a unit impulse at $n=M$ (an M-sample delay) and $b=\downarrow_M u$, then

$$\downarrow_M(\delta_M*u)=\delta_1*(\downarrow_M u)=\delta_1*b. \tag{3}$$

If u is N-periodic and $M|N$, (3) also holds for $M:1$ circular downsampling and circular shifts. We insist that the signal length, the analysis extension, and the decimation factor be constrained so that $M|N$ to ensure that N-point circular operations coincide with linear operations on N-periodic signals. The downsampling operation we use is:

$$(\downarrow_M u)(n)=u(Mn). \tag{4}$$

The paper [6] uses two different operators, termed "Decimation I," which is (4), and "Decimation II," which is (4) following a delay of u by $M/2$ samples (only for M even). Since such a shift can be incorporated into the phase of the analysis filter, it seems simpler to use the downsampler (4) and to classify symmetric transforms in terms of the group delays of the analysis filters, which have to be accounted for anyway.

Because of the complicated response of downsampling in the frequency domain, the author found that a time domain approach is more straightforward for analyzing the effects of downsampling on subband symmetry. The derivation that follows is presented in two parts: first for W-type filter outputs, then for H-type outputs.

A. Downsampling WS/WA Channels.

Consider the class of all WS signals, u, with center at 0; we shall begin by determining the (integer) values of c and n_c for which $b=\downarrow_M(\delta_c*u)$ is WS about n_c for all such u. The center, c, has been singled out to clarify the following arguments. Because of (3), we only need to consider $0 \leq c \leq M-1$. Note that we are determining the phases, c, for which $M:1$ decimation of *arbitrary* WS signals with center c will result in symmetric subbands, b. The subbands will be WS with center n_c if and only if $b(n_c - n) = b(n_c + n)$; i.e.,

$$u(M(n_c - n) - c) = u(M(n_c + n) - c) \quad \text{for all } n. \quad (5)$$

Since u is WS about 0, it is clear that (5) is satisfied if

$$M(n_c - n) - c = -(M(n_c + n) - c) \quad \text{for all } n,$$

which doesn't depend on n at all and is in fact equivalent to

$$Mn_c = c. \quad (6)$$

Thus, (6) is a sufficient condition on c and n_c to ensure that the class of all WS signals, u, produces WS subbands, $b = \downarrow_M (\delta_c * u)$, that are always symmetric about n_c.

Conversely, suppose that (5) holds for all WS signals, u, with center at 0. Define

$$m = M(n_c + 1) - c \quad \text{and} \quad u_m = \delta_m + \delta_{-m}. \quad (7)$$

By assumption, (5) holds for the special WS signal, $u_m(k)$, and evaluating u_m at $k = M(n_c - 1) - c$ gives

$$u_m(M(n_c - 1) - c) = u_m(M(n_c + 1) - c) = 1.$$

But $u_m(k) = 1$ if and only if $k = \pm m$, and since $M(n_c - 1) - c \neq m$, we conclude that

$$M(n_c - 1) - c = -(M(n_c + 1) - c),$$

which is equivalent to (6). Thus, condition (6) is also *necessary* for all downsampled subbands to be symmetric about n_c. Finally, for $0 \leq c \leq M-1$, the only solution to (6) for c and n_c is

$$c = 0, \quad n_c = 0.$$

The same result holds when u and b are WA.

It is also possible to get HS signals by decimating WS signals. We get HS signals centered at $n_c - 1/2$, $b(n_c - 1 - n) = b(n_c + n)$, if

$$u(M(n_c - 1 - n) - c) = u(M(n_c + n) - c) \quad \text{for all } n. \quad (8)$$

To have condition (8) hold for all WS signals, u, it is sufficient to have

$$M(n_c - 1 - n) - c = -(M(n_c + n) - c) \quad \text{for all } n,$$

which is equivalent to

$$2Mn_c - 2c = M. \quad (9)$$

To see that (9) is also *necessary* for (8) to hold, use the WS example (7) and evaluate at $k = M(n_c - 2) - c$. If condition (8) holds for all WS signals, u, then

$$u_m(M(n_c - 2) - c) = u_m(M(n_c + 1) - c) = 1,$$

and, as above, we conclude that $M(n_c - 2) - c = -(M(n_c + 1) - c)$, which is equivalent to (9). Thus, (9) is both necessary and sufficient for the class of all downsampled WS signals to be symmetric about $n_c - 1/2$. This constrains M to be even, and for $0 \leq c \leq M-1$, the only values of c and n_c satisfying (9) are

$$c = M/2, \quad n_c = 1. \quad (10)$$

Again, the same result holds when u is WA and b is HA.

Let's illustrate briefly how to handle signals with more general centers using (3). Suppose we have a WS signal centered at $c = M\nu + \mu$ (i.e., $u * \delta_{M\nu+\mu}$), $0 \leq \mu \leq M-1$; then

$$b = \downarrow_M (u * \delta_{M\nu+\mu}) = \delta_\nu * (\downarrow_M (\delta_\mu * u)).$$

If $\mu = M/2$, this says that b is a ν-sample delay of an HS subband centered, according to (10), at $n_c - 1/2 = 1/2$, so b is centered at $\nu + 1/2$. Thus, if $\delta_c * u$ is WS, we've shown that all we need for $b = \downarrow_M (\delta_c * u)$ to be HS is that M be even and that c be congruent to $M/2 \bmod M$ (denoted "$c \equiv M/2 \bmod M$"). This allows us to summarize the above results as follows.

Lemma 1: For all u in the class of WS (resp., WA) signals centered at 0, the subbands $b = \downarrow_M (\delta_c * u)$ are WS (resp., WA) with center at n_c if and only if c is a multiple of M (i.e., $c \equiv 0 \bmod M$), in which case $n_c = c/M$. Similarly, the subbands are HS (resp., HA) if and only if M is even and $c \equiv M/2 \bmod M$, in which case the subbands are centered at the half-integer point $n_c - 1/2 = c/M$.

B. Downsampling HS/HA Channels.

Now do the same analysis for the class of all HS signals centered at $-1/2$: $u(-1-n) = u(n)$. Can the subbands $b = \downarrow_M (\delta_c * u)$ be WS with center n_c: $b(n_c - n) = b(n_c + n)$? As above, this says

$$u(M(n_c - n) - c) = u(M(n_c + n) - c) \quad \text{for all } n. \quad (11)$$

Let $m = M(n_c + 1) - c$, and define the HS signal

$$u_m = \delta_m + \delta_{-1-m}. \quad (12)$$

If (11) holds for all HS signals centered at $-1/2$, then

$$u_m(M(n_c - 1) - c) = u_m(M(n_c + 1) - c) = 1.$$

Reasoning as before, $u_m(k) = 1$ if and only if $k = m$ or $-1-m$, and $M(n_c - 1) - c \neq m$, so we must have

$$M(n_c - 1) - c = -1 - (M(n_c + 1) - c). \quad (13)$$

But (13) simplifies to $2Mn_c - 2c = -1$, which has no solutions, so there are *no* phases, c, for which downsampling the class of HS signals produces WS subbands. The analogous result holds for HA signals.

There are phases, c, however, for which the subbands are HS with a center at $n_c - 1/2$: $b(n_c - 1 - n) = b(n_c + n)$. In terms of u,

$$u(M(n_c - 1 - n) - c) = u(M(n_c + n) - c) \quad \text{for all } n. \quad (14)$$

Using the same sample HS signal (12), one can prove that (14) holds for all HS signals if and *only* if

$$M(n_c - 1 - n) - c = -1 - (M(n_c + n) - c) \quad \text{for all } n,$$

or

$$2Mn_c - 2c = M - 1. \tag{15}$$

This means that M must be odd, and for $0 \le c \le M-1$ the only solution to (15) is

$$c = \frac{M+1}{2}, \quad n_c = 1. \tag{16}$$

Thus, the subbands are HS with center at $n_c - 1/2 = 1/2$; the same result holds for HA signals. Including the effect of shifts by multiples of M, we get the following.

Lemma 2: For all u in the class of HS (resp., HA) signals centered at $-1/2$, the subbands $b = \downarrow_M (\delta_c * u)$ are never WS or WA, and the subbands are HS (resp., HA) if and only if M is odd and $c \equiv (M+1)/2 \mod M$, in which case the subbands are centered at $n_c - 1/2 = (c - 1/2)/M$.

Now drop the restriction that "u" denotes a signal centered only at 0 or $-1/2$, so we can write $u = y * h$ where the group delay, γ, of h is arbitrary. From Table I we see that there are two ways in which u can arise as a WS signal. The γ's for which u will have a center $c \equiv 0 \mod M$ necessarily differ (by 1/2) depending on whether y and h are WS or HS. Similarly, there are two ways in which u can arise as a WS signal with $c \equiv M/2 \mod M$ when M is even. In Table II we combine the results of Lemmas 1 and 2 with Table I to enumerate all combinations of signal symmetry, filter symmetry and phase, and decimation factor that result in a symmetric subband, b.

III. Conclusion.

This correspondence has derived necessary and sufficient symmetry, phase, and downsampling conditions for a linear phase MFB to produce symmetric subbands when applied to a symmetric extension of a finite-duration input vector. The results in Lemmas 1 and 2 can be combined into the following intuitively appealing statement (note that the center, ϕ, is in general an integer multiple of 1/2):

Corollary 1: The class of all discrete-time signals symmetric (or antisymmetric) about ϕ remains symmetric (antisymmetric) after $M:1$ downsampling if and only if ϕ/M is an integer multiple of 1/2, in which case the downsampled subbands are symmetric (antisymmetric) about ϕ/M.

The completeness of the results in Table II forms part of a complete classification of nonexpansive symmetric extension transforms [13], and this work has had recent practical applications in subband image coding [4].

TABLE II
Symmetry of Linear Phase MFB Subbands.

Signal y: Sym., Center	Filter h: Sym.	Center γ	M	Subband Symmetry, Center: $u = y * h$	$b = \downarrow_M u$
WS, 0	WS/WA	$M\nu$	Any	WS/WA, $M\nu$	WS/WA, ν
		$M\nu + M/2$	Even	WS/WA, $M\nu + M/2$	HS/HA, $\nu + 1/2$
	HS/HA	$M\nu + M/2$	Odd	HS/HA, $M\nu + M/2$	HS/HA, $\nu + 1/2$
HS, $-1/2$	HS/HA	$M\nu + 1/2$	Any	WS/WA, $M\nu$	WS/WA, ν
		$M\nu + (M+1)/2$	Even	WS/WA, $M\nu + M/2$	HS/HA, $\nu + 1/2$
	WS/WA	$M\nu + (M+1)/2$	Odd	HS/HA, $M\nu + M/2$	HS/HA, $\nu + 1/2$

References

[1] P. P. Vaidyanathan, *Multirate Systems and Filter Banks.* Englewood Cliffs, NJ: Prentice Hall, 1993.

[2] M. J. T. Smith and S. L. Eddins, "Analysis/synthesis techniques for subband image coding," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 38, pp. 1446–1456, Aug. 1990.

[3] S. A. Martucci, "Signal extension and noncausal filtering for subband coding of images," in *Visual Commun.*, vol. 1605 of *Proc. SPIE*, (Boston), pp. 137–148, SPIE, Nov. 1991.

[4] Federal Bureau of Investigation, *WSQ Gray-Scale Fingerprint Image Compression Specification*, IAFIS-IC-0110v2 (rev. 2.0), Feb. 1993. Drafted by T. Hopper, C. Brislawn, and J. Bradley.

[5] R. H. Bamberger, S. L. Eddins, and V. Nuri, "Generalized symmetric extension for size-limited multirate filter banks," *IEEE Trans. Image Process.*, vol. 3, pp. 82–87, Jan. 1994.

[6] H. Kiya, K. Nishikawa, and M. Iwahashi, "A development of symmetric extension method for subband image coding," *IEEE Trans. Image Process.*, vol. 3, pp. 78–81, Jan. 1994.

[7] K. R. Rao and P. Yip, *Discrete Cosine Transform: Algorithms, Advantages, and Applications.* San Diego, CA: Academic Press, 1990.

[8] W. B. Pennebaker and J. L. Mitchell, *JPEG Still Image Data Compression Standard.* New York, NY: Van Nostrand Reinhold, 1992.

[9] N. S. Jayant and P. Noll, *Digital Coding of Waveforms.* Englewood Cliffs, NJ: Prentice Hall, 1984.

[10] V. Nuri and R. H. Bamberger, "A theory of size-limited filter banks," in *Proc. Int'l. Conf. Acoust., Speech, Signal Process.* [16].

[11] J. N. Bradley, C. M. Brislawn, and V. Faber, "Reflected boundary conditions for multirate filter banks," in *Proc. Int'l. Symp. Time-Freq. Time-Scale Analysis*, (Victoria, BC), pp. 307–310, IEEE Signal Process. Soc., Oct. 1992.

[12] L. Chen, T. Q. Nguyen, and K. P. Chan, "Symmetric extension methods for M-channel PR LP FIR analysis/synthesis systems," in *Proc. Int'l. Symp. Circuits Systems*, (London), IEEE Circuits Systems Soc., June 1994.

[13] C. M. Brislawn, "Classification of nonexpansive symmetric extension transforms for multirate filter banks," Tech. Rep. LA-UR-94-1747, Los Alamos Nat'l. Lab, May 1994. Submitted for publication.

[14] S. A. Martucci and R. M. Mersereau, "The symmetric convolution approach to the nonexpansive implementation of FIR filter banks for images," in *Proc. Int'l. Conf. Acoust., Speech, Signal Process.* [16], pp. V–65–68.

[15] S. A. Martucci, "Symmetric convolution and the discrete sine and cosine transforms," *IEEE Trans. Signal Process.*, vol. 42, pp. 1038–1051, May 1994.

[16] IEEE Signal Process. Soc., *Proc. Int'l. Conf. Acoust., Speech, Signal Process.*, (Minneapolis), Apr. 1993.