FAQs about CP Violation in B Physics

Informal Notes

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These informal notes are not to be cited. They are entirely pedagogical in character and do not have appropriate references. A number of excellent reviews exist, including

- Y. Nir and H. Quinn, CP Violation in B Physics, Ann. Rev. Nucl. Part. Sci., **42**, 2111 (1992)
- Y. Nir, CP Violation, in Twentieth Annual Summer Institute on Particle Physics, SLAC 1992.
- J. Richman, Heavy-Quark Physics and CP Violation, Les Houches lectures, 1998.
- CP Violation Primer, Chapter 1 of BABAR Physics Workshop, SLAC, 1998
- CP Violation, in *Review of Particle Properties, 1998*, by L. Wolfenstein, p. 107.
- $B-\overline{B}$ Mixing, in *Review of Particle Properties, 1998*, by H. Quinn, p. 555.

1. *What is direct CP violation?*
Direct CP violation is the inequality in decay rates between two CP-conjugate processes that occurs in the absence of mixing.

2. *Do we expect to see the CP violation $\Gamma(K^+ \to \pi^+\pi^0) \neq \Gamma(K^- \to \pi^-\pi^0)$?*
Let us write the decay amplitudes for the two processes as

$$\begin{aligned}\mathcal{M}(K^+ \to \pi^+\pi^0) &= \sum_i e^{i\phi_i} e^{i\delta_i} A_i \\ \mathcal{M}(K^- \to \pi^+\pi^0) &= \sum_i e^{-i\phi_i} e^{i\delta_i} A_i \end{aligned} \tag{1}$$

The sum is over various contributions to the amplitudes. The A_i's are real. The phase ϕ is the weak phase. It changes sign because the part of the weak Hamiltonian that gives rise the K^+ decay must be the adjoint (conjugate) of the part that gives rise to the K^- decay. This follows from unitarity: the Hamiltonian must be Hermitian. The phase δ arises from the strong interaction. It does not change sign because the strong interactions are invariant under CP. As we demonstrate below, in the case of K decay we can state precisely what these phases are.

The final state here is purely $I = 2$. Combining two πs can give $I = 0, 1,$ or 2, but $I = 0$ is impossible since we have $I_z = \pm 1$. Also, $I = 1$ is impossible by bose statistics: the even orbital angular momentum state must go with an even isospin state. The phase δ is precisely the $I = 2$ $\pi\pi$ phase shift.

We now see that the inequality is impossible since we can factor out the $e^{i\delta}$ and

$$|\mathcal{M}(K^+ \to \pi^+\pi^0)| = |\sum_i e^{i\phi_i} A_i| = |\mathcal{M}(K^- \to \pi^-\pi^0)| = |\sum_i e^{-i\phi_i} A_i| \tag{2}$$

3. *What determines the final-state interaction phase shifts?*

These follow from Watson's theorem, which we prove here. The S-matrix for channels with a single angular momentum (like a decay) can be taken to be symmetric. Consider the simple case $K \to \pi\pi$ in which we consider only a single final isospin, say $I = 2$. Now the S-matrix here contains two channels, the K and the $i = 2$ $\pi\pi$ state:

$$S = \begin{pmatrix} a & x \\ x & be^{2i\delta} \end{pmatrix} \tag{3}$$

We want to determine the possible values of $a, b,$ and x. We have already inserted the factor $e^{2i\delta}$ representing the $\pi\pi$ S-matrix. The off diagonal elements represent the weak decay $K \to \pi\pi$ and are thus small numbers. The lower right hand entry is essentially the S-matrix for $\pi\pi \to \pi\pi$.

The S-matrix is unitary, so

$$\begin{aligned} |a|^2+|x|^2 &= \; = 1 \\ ax^*+xbe^{-2i\delta} &= 0 \end{aligned} \tag{4}$$

We see that both $|a|^2$ and $|b|^2$ are nearly unity, while

$$x = i|x|e^{i\delta} \tag{5}$$

This is Watson's theorem: the weak decay picks up half the phase of the S-matrix. The factor i is the familiar one relating the S-matrix to the scattering amplitude.

4.*How does this generalize to a process with many channels?*

First we must break up the problem into sectors that are not mixed by the strong interactions. For $K \to \pi\pi$ we separate the $I=0$ and $I=2$ sectors. In this instance there remains only a single channel in each sector and we can apply Watson's theorem directly.

Consider, however, $B \to \pi\pi$. Once again, only $I=0$ and $I=2$ occur. There are many states with $I=0$ and many with $I=2$. We can imagine diagonalizing the $I=0$ states. Call the eigenstates $1,2,3,\ldots$. They satisfy

$$S|i\rangle = e^{i\delta_i}|i\rangle \tag{6}$$

though we don't know the actual content of the states $|i\rangle$ or the eigenphase shifts, δ_i Now express the $I=0$ $\pi\pi$ state with an invariant mass m_B as

$$|\pi\pi\rangle = \sum_i c_i|i\rangle \tag{7}$$

Then if the decay amplitude to one of the eigenchannels is

$$\mathcal{M}(B\to i) = A_i e^{i\phi_i}e^{i\delta_i} \tag{8}$$

we have

$$\mathcal{M}(B\to\pi\pi) = \sum_i c_i^* A_i e^{i\phi_i}e^{i\delta_i} \tag{9}$$

5. *Do we expect to see the CP violation* $\Gamma(B^+\to\pi^+\pi^0) \neq \Gamma(B^-\to\pi^-\pi^0)$*?*

Now there are contributions only from $I = 2$. But because there are many hadronic channels, many strong phases contribute. This is in contrast to the situation for $K \to \pi\pi$. However, if the weak phase ϕ_i is the same for every contribution, again we can factor out its corresponding phase. We do, in fact, expect only one weak phase, the phase of the tree amplitude (see below). The reason for this is that the other potential contribution, the penguin amplitude, has $\Delta I = 1/2$ and thus cannot actually participate in making the $I = 2$ final state.

Consider, on the other hand, $\Gamma(B^+ \to \pi^+\eta^0)$ vs. $\Gamma(B^- \to \pi^-\eta^0)$. Now the final state is $I = 1$ and there are contributions from both trees and penguins and we may see an inequality of the decay rates.

6. *What is the general description of mixing?*

For pseudoscalars P^0 and $\overline{P}^0$ the time dependent mixing of the state

$$|\Psi(t)\rangle = a(t)|P^0\rangle + b(t)|\overline{P}^0\rangle \tag{10}$$

is determined by

$$i\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = H\begin{pmatrix} a \\ b \end{pmatrix} \tag{11}$$

We refer to P^0 as 1 and to $\overline{P}^0$ as 2. Now CPT tells us that

$$\langle 1|\mathcal{H}|1\rangle = H_{11} = \langle 2|\mathcal{H}|2\rangle = H_{22} \tag{12}$$

The off diagonal elements, e.g. that changing K^0 to $\overline{K}^0$, can come from a single "superweak" interaction or by the action twice by an ordinary weak interaction that changes strangeness or bottomness by one unit. Let us ignore the superweak possibility. Then

$$\langle 1|H|2\rangle = \sum_n \frac{\langle 1|\mathcal{H}|n\rangle\langle n|\mathcal{H}|2\rangle}{m_P - E_n + i\epsilon} \tag{13}$$

We separate two pieces

$$\begin{aligned} M_{12} &= P\sum_n \frac{\langle 1|\mathcal{H}|n\rangle\langle n|\mathcal{H}|2\rangle}{m_P - E_n} \\ \Gamma_{12} &= 2\pi\sum_n \langle 1|\mathcal{H}|n\rangle\langle n|\mathcal{H}|2\rangle\delta(m_B - E_n) \end{aligned} \tag{14}$$

so

$$H_{12} = M_{12} - i\Gamma_{12}/2 \tag{15}$$

It should be noted that M_{12} and Γ_{12} are not real. In fact,they depend on the phases we assign to $|1\rangle$ and $|2\rangle$. What we can conclude, however, is that

$$H_{21} = M_{12}^* - i\Gamma_{12}^*/2 \tag{16}$$

If CP is conserved, then $CP\mathcal{H}CP = \mathcal{H}$. Suppose we have the general convention

$$\begin{aligned} CP|P^0\rangle &= e^{i\zeta}|\overline{P}^0\rangle \\ CP|\overline{P}^0\rangle &= e^{-i\zeta}|P^0\rangle \end{aligned} \tag{17}$$

Then we find, if CP is conserved,

$$\begin{aligned} M_{12} = e^{-2i\zeta}M_{21} = e^{-2i\zeta}M_{12}^* \\ \Gamma_{12} = e^{-2i\zeta}\Gamma_{21} = e^{-2i\zeta}\Gamma_{12}^* \end{aligned} \tag{18}$$

That is, M_{12} and Γ_{12} have the same phase, namely $e^{-i\zeta}$.

Returning again to CP violation, we write

$$H_{11} = H_{22} = M - i\Gamma/2 \tag{19}$$

we have to diagonalize

$$\begin{pmatrix} M - i\Gamma/2 & M_{12} - i\Gamma_{12}/2 \\ M_{12}^* - i\Gamma_{12}^*/2 & M - i\Gamma/2 \end{pmatrix} \tag{20}$$

The eigenvalues are

$$M_\pm - i\Gamma_\pm/2 = M - i\Gamma/2 \pm \sqrt{|M_{12}|^2 - |\Gamma_{12}|^2/4 - i\mathrm{Re}M_{12}\Gamma_{12}^*} \tag{21}$$

It is worth noting that this expression is independent of our choice of phases of the states $|1\rangle$ and $|2\rangle$. We call the heavy solution P_H and the light one P_L. We then find that

$$\begin{aligned} |P_L\rangle &= p|P^0\rangle + q|\overline{P}^0\rangle \\ |P_H\rangle &= p|P^0\rangle - q|\overline{P}^0\rangle \end{aligned} \tag{22}$$

where

$$\frac{q}{p} = -\sqrt{\frac{M_{12}^* - i\Gamma_{12}^*/2}{M_{12} - i\Gamma_{12}/2}} \approx -\frac{|M_{12}|}{M_{12}}\left(1 - \frac{1}{2}\mathrm{Im}\frac{\Gamma_{12}}{M_{12}}\right) \tag{23}$$

and where we take

$$|p|^2 + |q|^2 = 1 \tag{24}$$

7. *How does this apply to K decays?*
Traditionally we write, with $CP|K^0\rangle = -|\overline{K}^0\rangle$

$$\begin{aligned} |K_S\rangle &= \frac{1}{\sqrt{2}}[(1+\epsilon)|K^0\rangle - (1-\epsilon)|\overline{K}^0\rangle \\ |K_L\rangle &= \frac{1}{\sqrt{2}}[(1+\epsilon)|K^0\rangle + (1-\epsilon)|\overline{K}^0\rangle \end{aligned} \tag{25}$$

It follows that

$$\frac{q}{p} = -\frac{1-\epsilon}{1+\epsilon} \approx -(1-2\epsilon) \tag{26}$$

and

$$1 - 4\epsilon = \frac{M_{12}^* - i\Gamma_{12}^*/2}{M_{12} - i\Gamma_{12}/2} \tag{27}$$

The Wu-Yang convention specifies fixing the phase of $|K^0\rangle$ so that the $K^0 \to (\pi\pi)_{I=0}$ decay amplitude, without the final state interaction, is real. Now the real transitions govern Γ and are dominated by the $I = 0$ final state (remember the lifetime difference between K_S, which can decay to the $I = 0$ state and $K^\pm$, which cannot.). Thus, with the Wu-Yang convention, Γ_{12} is nearly real. In this case,

$$\begin{aligned} M_\pm - i\Gamma_\pm/2 &= M - i\Gamma/2 \pm \sqrt{(\mathrm{Re}M_{12} - i\Gamma_{12}/2)^2 + (\mathrm{Im}M_{12})^2} \\ &\approx M - i\Gamma/2 \pm (\mathrm{Re}M_{12} - i\Gamma_{12}/2) \end{aligned} \tag{28}$$

Thus, assuming $\mathrm{Re}M_{12} > 0$,

$$\begin{aligned} \Delta M &= M_+ - M_- = 2\mathrm{Re}M_{12} \\ \Delta\Gamma &= \Gamma_+ - \Gamma_- = -4\Gamma_{12} \end{aligned} \tag{29}$$

In fact, the measured values are (here $M_L = M_H$, $M_S = M_L$!!. K_L is heavy, K_S is light.)

$$\begin{aligned}
M_L - M_S &= 0.5311 \pm 0.001 \times 10^{10}\mathrm{s}^{-1} \\
1/\Gamma_L &= 5.15 \pm 0.04 \times 10^{-8} s \\
1/\Gamma_S &= 0.8940 \pm 0.0009 \times 10^{-10} s
\end{aligned} \tag{30}$$

We can write

$$1 - 4\epsilon = \frac{\mathrm{Re}M_{12} - i\mathrm{Im}M_{12} - i\Gamma_{12}/2}{\mathrm{Re}M_{12} + i\mathrm{Im}M_{12} - i\Gamma_{12}/2} = 1 - \frac{2i\mathrm{Im}M_{12}}{M_{12} - i\Gamma_{12}} \tag{31}$$

Thus

$$\epsilon \approx \frac{i\mathrm{Im}M_{12}}{M_L - M_S + i\Gamma_S/2} \tag{32}$$

From $(M_L - M_S)/\Gamma_S = 0.474$ we find that the phase of ϵ is $43.5°$.

8. *What is the time dependence of the* $|K^0\rangle - |\overline{K}^0\rangle$ *oscillations?* From

$$\begin{aligned}
|K_L\rangle &= p|K^0\rangle - q|\overline{K}^0\rangle \\
|K_S\rangle &= p|K^0\rangle + q|\overline{K}^0\rangle
\end{aligned} \tag{33}$$

we have

$$\begin{aligned}
|K^0\rangle &= \frac{1}{2p}(|K_S\rangle + |K_L\rangle) \\
|\overline{K}^0\rangle &= \frac{1}{2q}(|K_S\rangle - |K_L\rangle)
\end{aligned} \tag{34}$$

The states $|K_L\rangle$ and $|K_S\rangle$ remain pure. A state that begins a pure $|K^0\rangle$ evolves as

$$|K^0_{phys}(t)\rangle = \frac{1}{2p}(e^{-im_S t - \Gamma_S t/2}|K_S\rangle + e^{-im_L t - \Gamma_L t/2}|K_L\rangle) \tag{35}$$

The amplitude to be a $|K^0\rangle$ is

$$\langle K^0|K^0_{phys}(t)\rangle = \frac{1}{2}(e^{-im_S t - \Gamma_S t/2} + e^{-im_L t - \Gamma_L t/2}) \tag{36}$$

so

$$|\langle K^0|K^0_{phys}(t)\rangle|^2 = \frac{1}{4}(e^{-\Gamma_L t} + 2e^{-(\Gamma_L+\Gamma_s)t/2}\cos\Delta m t + e^{-\Gamma_s t}) \qquad (37)$$

while

$$|\langle\overline{K}^0|K^0_{phys}(t)\rangle|^2 = \frac{1}{4}|\frac{q}{p}|^2(e^{-\Gamma_L t} - 2e^{-(\Gamma_L+\Gamma_s)t/2}\cos\Delta m t + e^{-\Gamma_s t}) \qquad (38)$$

Note that for $|K^0\rangle - |\overline{K}^0\rangle$

$$|\frac{q}{p}|^2 = |1-2\epsilon|^2 \approx 1 - 4\mathrm{Re}\epsilon \qquad (39)$$

We can calculate the integrated rates using

$$\int_0^\infty dt e^{-\Gamma t}\cos\Delta m t = \frac{\Gamma}{\Gamma^2+\Delta m^2} \qquad (40)$$

In this way we find

$$\frac{\int_0^\infty dt|\langle\overline{K}|K(t)\rangle|^2}{\int_0^\infty dt|\langle K|K(t)\rangle|^2} = |\frac{q}{p}|^2\frac{x^2+y^2}{2+x^2-y^2} \qquad (41)$$

where $x = \Delta m/\Gamma$, $y = \Delta\Gamma/2\Gamma$, $\Gamma = (\Gamma_L+\Gamma_s)/2$. Similarly,

$$\frac{\int_0^\infty dt|\langle K|\overline{K}(t)\rangle|^2}{\int_0^\infty dt|\langle K|\overline{K}(t)\rangle|^2} = |\frac{p}{q}|^2\frac{x^2+y^2}{2+x^2-y^2} \qquad (42)$$

Even more simply, consider a pure K_L beam. Compare its decays to $\mu^+\pi^-\nu$ and to $\mu^-\pi^+\overline{\nu}$. The amplitudes are proportional to

$$\begin{aligned}\langle K^0||K_L\rangle &= pe^{-im_L t-\Gamma_L t/2}\\ \langle\overline{K}^0|K_L\rangle &= -qe^{-im_L t-\Gamma_L t/2}\end{aligned} \qquad (43)$$

so

$$\frac{\Gamma(K_L\to\pi^-\mu^+\nu)-\Gamma(K_L\to\pi^+\mu^-\overline{\nu})}{\Gamma(K_L\to\pi^-\mu^+\nu)+\Gamma(K_L\to\pi^+\mu^-\overline{\nu})} = \frac{|p|^2-|q|^2}{|p|^2+|q|^2} = \frac{(1+\epsilon)|^2-|(1-\epsilon)|^2}{|(1+\epsilon)|^2+|(1-\epsilon)|^2} = 2\mathrm{Re}\epsilon \qquad (44)$$

9. *What is ϵ' and how does it fit in $K \to \pi\pi$?*

Perhaps the longest lasting program in high-energy physics is the measurement of $K \to \pi\pi$. We write the amplitudes, inserting the final-state-interaction phases.

$$\begin{aligned}
\langle (2\pi) I = 0 \; | H_{\rm wk} | K^0 \rangle &= e^{i\delta_0} A_0 \\
\langle (2\pi) I = 0 \; | H_{\rm wk} | \overline{K}^0 \rangle &= -e^{i\delta_0} A_0^* \\
\langle (2\pi) I = 2 \; | H_{\rm wk} | K^0 \rangle &= e^{i\delta_2} A_2 \\
\langle (2\pi) I = 2 \; | H_{\rm wk} | \overline{K}^0 \rangle &= -e^{i\delta_2} A_2^*
\end{aligned}$$

When these results are assembled, the various $K^0 \to \pi\pi$ amplitudes turn out to be

$$\langle \pi^+\pi^- | H_{\rm wk} | K_L^0 \rangle = \sqrt{2/3} e^{i\delta_2} (\epsilon {\rm Re}\, A_2 + i {\rm Im}\, A_2) + 2\sqrt{1/3} e^{i\delta_0} (\epsilon {\rm Re}\, A_0 + i {\rm Im}\, A_0)$$

$$\langle \pi^0\pi^0 | H_{\rm wk} | K_L^0 \rangle = 2\sqrt{1/3} e^{i\delta_2} (\epsilon {\rm Re}\, A_2 + i {\rm Im}\, A_2) - \sqrt{2/3} e^{i\delta_0} (\epsilon {\rm Re}\, A_0 + i {\rm Im}\, A_0)$$

$$\langle \pi^+\pi^- | H_{\rm wk} | K_S^0 \rangle = \sqrt{2/3} (e^{i\delta_2} {\rm Re}\, A_2 + \sqrt{2} e^{i\delta_0} {\rm Re}\, A_0)$$

$$\langle \pi^0\pi^0 | H_{\rm wk} | K_S^0 \rangle = \sqrt{2/3} (\sqrt{2} e^{i\delta_2} {\rm Re}\, A_2 - e^{i\delta_0} {\rm Re}\, A_0)$$

These results can be simplified by observing the following. First, the much faster decay of the K_S compared to $K^+ \to \pi^+\pi^0$ shows that $|A_0| >> |A_2|$. This is known as the $\Delta I = 1/2$ rule since the $\Delta I = 3/2$ interaction responsible for $K^+ \to \pi^+\pi^0$ is weaker than the $\Delta I = 1/2$ operator responsible for $K_S^0 \to \pi^+\pi^-$. Secondly, the phase of the K^0 state is still a matter of convention. It can be chosen so that A_0 is real. This is the convention of T. T. Wu and C. N. Yang. Thus we can drop the ${\rm Im}\, A_0$ terms, and terms of order $\epsilon A_2/A_0$. To compare the CP violating $K_L \to 2\pi$ decay amplitudes to the CP-nonviolating $K_S \to 2\pi$ amplitudes we define the ratios

$$\eta_{+-} = \frac{\langle \pi^+\pi^-|H_{\rm wk}|K_L^0\rangle}{\langle \pi^+\pi^-|H_{\rm wk}|K_S^0\rangle} = \epsilon + \epsilon'$$

$$\eta_{00} = \frac{\langle \pi^0\pi^0|H_{\rm wk}|K_L^0\rangle}{\langle \pi^0\pi^0|H_{\rm wk}|K_S^0\rangle} = \epsilon - 2\epsilon'$$

where

$$\epsilon' = \frac{1}{\sqrt{2}}\frac{\mathrm{Im}\,A_2}{A_0}\exp(i\pi/2 - i\delta_0 + i\delta_2)$$

In particular,

$$\left|\frac{\eta_{00}}{\eta_{+-}}\right|^2 \approx 1 - 6\mathrm{Re}\ \epsilon/\epsilon' \tag{45}$$

Data on the $\pi\pi$ phase shifts indicate that the phase of ϵ' is about 48°, similar to the phase of ϵ (RPP 1998, p. 107), so $\mathrm{Re}\epsilon'/\epsilon \approx \epsilon'/\epsilon$.

10.*How do oscillations appear in the $B^0 - \overline{B}^0$ system?*

Here we can neglect Γ_{12} because we expect the lifetime difference between the two mass eigenstates to be small. This follows from the large number of channels available. A state that at $t=0$ is purely B^0 evolves as

$$|{B^0}_{phys}(t)\rangle = e^{-\Gamma t/2}(\cos(\Delta m t/2)|B^0\rangle + i\frac{q}{p}\sin(\Delta m t/2)|\overline{B}^0\rangle) \tag{46}$$

where we have dropped an inessential phase. Similarly

$$|\overline{B}^0_{phys}(t)\rangle = e^{-\Gamma t/2}(\cos(\Delta m t/2)|\overline{B}^0\rangle + i\frac{p}{q}\sin(\Delta m t/2)|B^0\rangle) \tag{47}$$

Now for any final state f, we can compute the decay rates from states that are initially B^0 or $\overline{B}^0$:

$$\begin{aligned} \langle f|{B^0}_{phys}(t)\rangle &= e^{-\Gamma t/2}(\cos(\Delta m t/2)A_f + i\frac{q}{p}\sin(\Delta m t/2)\overline{A}_f) \\ \langle f|\overline{B}^0_{phys}(t)\rangle &= e^{-\Gamma t/2}(\cos(\Delta m t/2)\overline{A}_f + i\frac{p}{q}\sin(\Delta m t/2)A_f) \end{aligned} \tag{48}$$

where $A_f = \langle f|B^0\rangle$, $\overline{A}_f = \langle f|\overline{B}^0\rangle$. Introducing

$$\lambda = \frac{q}{p}\frac{\overline{A}_f}{A_f} \tag{49}$$

we have

$$
\begin{aligned}
|\langle f|B^0{}_{phys}(t)\rangle|^2 &= e^{-\Gamma t}|A_f|^2\left[\tfrac{1}{2}(1+|\lambda|^2)+\tfrac{1}{2}(1-|\lambda|^2)\cos\Delta mt-\mathrm{Im}\lambda\sin\Delta mt\right] \\
|\langle f|\overline{B}^0_{phys}(t)\rangle|^2 &= e^{-\Gamma t}|A_f|^2\left[\tfrac{1}{2}(1+|\lambda|^2)-\tfrac{1}{2}(1-|\lambda|^2)\cos\Delta mt+\mathrm{Im}\lambda\sin\Delta mt\right]
\end{aligned}
\tag{50}
$$

If the time dependence is not observed, we have

$$
\begin{aligned}
\Gamma\int_0^\infty dt|\langle f|B^0{}_{phys}(t)\rangle|^2 &= \frac{1+x^2/2}{1+x^2}+\frac{x^2/2}{1+x^2}|\lambda|^2-\frac{x}{1+x^2}\mathrm{Im}\lambda \\
\Gamma\int_0^\infty dt|\langle f|\overline{B}^0_{phys}(t)\rangle|^2 &= \frac{1+x^2/2}{1+x^2}|\lambda|^2-\frac{x^2/2}{1+x^2}+\frac{x}{1+x^2}\mathrm{Im}\lambda
\end{aligned}
\tag{51}
$$

11. *How are the phases of the weak amplitudes determined in the Standard Model?*

The weak interactions of the quarks is given by

$$
\mathcal{H}=\frac{G_F}{\sqrt{2}}J_\mu^\dagger J^\mu \tag{52}
$$

while the currents are determined by

$$
J_\mu=(u_L\ c_L\ t_L)V\gamma_\mu\begin{pmatrix} d_L \\ s_L \\ b_L\end{pmatrix} \tag{53}
$$

The rule of thumb is that a diagram with an outgoing u quark and an incoming d quark gets a factor V_{ud}. If the u is incoming and the d is outgoing, the factor is V_{ud}^*. Replacing the quarks with antiquarks results in complex conjugation.

The Wolfenstein form for the KM matrix

$$
\begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb}\end{bmatrix} \tag{54}
$$

is

$$\begin{bmatrix} 1-\lambda^2/2 & \lambda & \lambda^3 A(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & \lambda^2 A \\ \lambda^3 A(1-\rho-i\eta) & -\lambda^2 A & 1 \end{bmatrix}. \tag{55}$$

12. *How is the unitarity triangle related to the KM matrix?*

From $V^\dagger V = 1$ we have, in particular

$$\begin{aligned} V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* &= 0, \\ \frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} + 1 + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} &= 0. \end{aligned} \tag{56}$$

We can regard the three terms of the last equation as three complex numbers whose sum is zero. Equivalently, the three complex numbers form a triangle as shown.

Inspecting the geometry reveals that

$$-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}$$

has the phase $+\gamma$. Similarly,

$$-\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}$$

has the phase $-\beta$. Thinking about it a bit more we see that

$$-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}$$

has the phase α. We can write all this conveniently as

$$\begin{aligned} e^{2i\alpha} &= \frac{V_{td}V_{tb}^*V_{ud}^*V_{ub}}{V_{td}^*V_{tb}V_{ud}V_{ub}^*}, \\ e^{2i\beta} &= \frac{V_{td}^*V_{tb}V_{cd}^*V_{cb}}{V_{td}V_{tb}^*V_{cd}V_{cb}^*}, \\ e^{2i\gamma} &= \frac{V_{ud}V_{ub}^*V_{cd}^*V_{cb}}{V_{ud}^*V_{ub}V_{cd}V_{cb}^*}. \end{aligned} \tag{57}$$

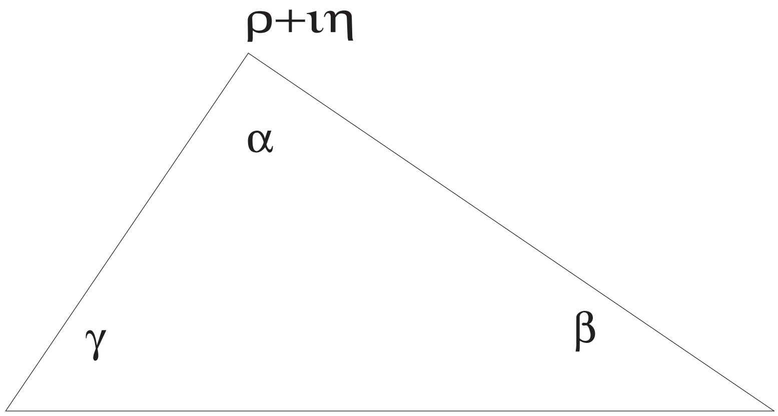

Figure 1:

In the Wolfenstein parameterization we see that the single vertex in the complex plane is at

$$-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}*} \approx \frac{1 \times A\lambda^3(\rho + i\eta)}{(-\lambda) \times (A\lambda^2)} \approx \rho + i\eta,$$

the well known result.

13. *What is the GIM mechanism?*

The Glashow-Iliopoulos-Maiani mechanism cancels potential neutral flavor-changing currents. Consider, for example $K_L \to \mu^+\mu^-$. This doesn't happen very often (BR=7×10^{-9}). It could, in principle, occur, for example, in a two-step process: $K_L \to W^+W^-$, $W^+W_- \to \mu^+\mu^-$. In the first step the annihilation of the $\overline{d}s$ or $\overline{s}d$ occurs through intermediate states u, c, or t. These have amplitudes proportional to $V_{ud}V_{us}^*$, $V_{cd}V_{cs}^*$, and $V_{td}V_{ts}^*$. If we ignore (!) the masses of the u, c, and t quarks, the amplitude is proportional to

$$V_{ud}V_{us}^* + V_{cd}V_{cs}^* + V_{td}V_{ts}^* = 0 \tag{58}$$

where the result follows from the unitarity of the CKM matrix. Of course the masses cannot be ignored. However, in doing the (one-loop) calculation of our two-step process, the integral, which appears to be divergent, actually converges because at high momenta we can ignore the masses and the integrand vanishes. This is an essential feature of the Standard Model.

14. *What phases are associated with tree and penguin diagrams?*

To a good approximation, the only transitions that introduce weak phases are those between the first and third generations. The phase of V_{ub} is $-\gamma$. The phase of V_{td} is $-\beta$.

The dominant tree processes are $b \to c, (d\overline{u}), (s\overline{c})$. Neither has a phase. Even the Cabibbo suppressed variants $b \to c, (d\overline{c}), (s\overline{u})$ have no phase. None has a first-to-third transition.

The suppressed tree transitions $b \to u$ all introduce V_{ub}, whose phase is $-\gamma$

Penguin transitions involve the virtual emission and absorption of u, c, and t. The t dominates because of its large mass (remember that such loops vanish if all quark masses are degenerate). Thus the $b \to s$ penguin has a factor $V_{tb}V_{ts}^*$. This has no first-to-third transition and thus has no phase. By contrast, the $b \to d$ penguin comes with $V_{tb}V_{td}^*$, whose phase is $+\beta$.

From this we can conclude the following:

$\overline{B}^0 \to J/\psi K_S$: From tree $b \to c(\overline{c}, s)$ no phase. From penguin $b \to s(c\overline{c})$, no phase.

$\overline{B}^0 \to D, D_s$. Identical to $\overline{B}^0 \to J/\psi K_S$.

$\overline{B}^0 \to \pi\pi$. From tree $b \to u(\overline{u}, d)$, phase is $-\gamma$. From penguin, $b \to d(\overline{u}, u), (\overline{d}, d)$, phase is $+\beta$.

15. *What is the phase of q/p?*

We established that

$$\frac{q}{p} = -|M_{12}|/M_{12} \tag{59}$$

The matrix element M_{12} can be calculated by computing a box diagram, inserting as usual, u, c, and t quarks. The result is dominated by the t quark it turns out that (with $CP|B^0\rangle = e^{2i\xi}|\overline{B}^0\rangle$)

$$M_{12} \propto -(V_{tb}V_{td}^*)^2 e^{-2i\xi} \tag{60}$$

so

$$\frac{q}{p} = e^{-2i\beta}e^{2i\xi} \tag{61}$$

This result depends on the Standard Model giving the correct description not just for physical decays, but for the virtual transitions responsible for

$B^0 - \overline{B}^0$ mixing. It is therefore the most likely place to find a breakdown in the canonical predictions.

16. *What can we learn from lepton asymmetries in B decays?*

If $B\overline{B}$ pair is produced, because of oscillations, sometimes both will decay semileptonically as if they were both B or both $\overline{B}$. From FAQ 8, we see that

$$\frac{N(BB) - N(\overline{BB})}{N(BB) - N(\overline{BB})} = -\frac{|\frac{q}{p}|^4 - 1}{|\frac{q}{p}|^4 + 1} \approx \mathrm{Im}\frac{\Gamma_{12}}{M_{12}} \tag{62}$$

In the Standard Model, $\mathrm{Im}\frac{\Gamma_{12}}{M_{12}}$ is small. This is not hard to see. M_{12} is dominated by th t quark intermediate state and carries $(V_{tb}V_{td}^*)^2$. Now consider Γ_{12}. This comes from transitions with physical intermediate states:$u\overline{u}$, $c\overline{c}$, $u\overline{c}$, $c\overline{u}$. These transitions are produced with the CKM factors

$$\begin{aligned}&(V_{ub}V_{ud}^*)^2 + (V_{cb}V_{cd}^*)^2 + 2(V_{cb}V_{ud}^*)(V_{ub}V_{cd}^*) + \\ &\quad = [(V_{ub}V_{ud}^*) + (V_{cb}V_{cd}^*)]^2 = (V_{tb}V_{td}^*)^2\end{aligned} \tag{63}$$

where we have used unitarity. Thus Γ_{12} and M_{12} are expected to have about the same phase.

17. *What are the Standard Model predictions for M_{12} and Γ_{12}?*

The transition $\overline{B}^0 \to B^0$ occurs, at the quark level, through box diagrams where the intermediate states are $t\overline{t}, t\overline{c}, t\overline{u}, c\overline{c}\ldots$ etc. The sum of all the diagrams would vanish if the quark masses were zero (or just all identical). The result then is dominated by the t quark contribution (since there is not too much suppression by the CKM matrix):

$$M_{12}^{SM} = \frac{G_F^2}{12\pi^2}(B_B f_B^2) m_B m_t^2 \eta (V_{tb}V_{td}^*)^2 f(x_t) \tag{64}$$

Here we have

- $G_F = 1.166 \times 10^{-5}\mathrm{GeV}^{-2}$
- $B_B \approx 1.3$ is the bag parameter, relating the matrix element of a quark operator between physical states to the value obtained naively. This is obtained in lattice calculations.

- $f_B \approx 175 \pm 25$ MeV is the decay constant for the B meson. This is also obtained from lattice calculations.
- $\eta = 0.55$ is a QCD correction.
- $$f(x_t) = \frac{4 - 11x_t + x_t^2}{4(1-x_t)^2} - \frac{3x_t^2 \ln x_t}{2(1-x_t)^3}$$
  is a kinematical factor with $x_t = m_t^2/m_W^2$. We find $f(x_t) = 0.54$.
- $V_{tb} \approx 1, V_{td} \approx (0.004 - 0.013)e^{-i\beta}$ (PDG 1998).

If we take $|V_{td}| = 0.009$ we get good agreement with $\Delta M = x_d/\tau_B = 3.2 \times 10^{13}\text{GeV} = 2|M_{12}|$. However, there is still a very large uncertainty here from the value of V_{td}. We can use this method to determine $|V_{td}|$ only if we assume that there are no other contributions to M_{12} except those given by the CKM model.

The prediction for Γ_{12} is closely related and

$$\frac{\Gamma_{12}}{M_{12}} = -\frac{3\pi}{2}\frac{1}{\eta f(x_t)}\frac{m_b^2}{m_t^2}\left(1 + \frac{8}{3}\eta_5 \frac{m_c^2}{m_b^2}\frac{V_{cb}V_{cd}^*}{V_{tb}V_{td}^*}\right) \tag{65}$$

The first term is real and approximately -1.1×10^{-2}. The quantities $\eta \approx 0.55$ and $\eta_5 \approx 0.88$ are QCD corrections. The second term is about $-0.24 \times e^{-i\beta}$ times the first.

17. *How is the oscillation behavior simplified when the observed final state is a CP eigenstate?*

Think of the weak Hamiltonian as having several pieces, each corresponding to a different mechanism (tree, penguin, ...). Each piece comes with a weak phase. Because the Hamiltonian must be Hermitian, the adjoint (conjugate) of each piece must occur, too.

$$\mathcal{H} = \sum_j e^{i\alpha_j}\mathcal{H}_j + \sum_j e^{-i\alpha_j}\mathcal{H}_j^\dagger \tag{66}$$

Now under CP

$$CP\mathcal{H}_j CP = \mathcal{H}_J^\dagger \tag{67}$$

so

$$CP\mathcal{H}CP = \sum_j e^{i\alpha_j}\mathcal{H}_j^\dagger + \sum_j e^{-i\alpha_j}\mathcal{H}_j \tag{68}$$

Now suppose the final state has CP eigenvalue $\eta_{CP} = \pm 1$ and suppose the phase convention on B^0 and $\overline{B}^0$ is

$$CP|B^0\rangle = e^{2i\xi}|\overline{B}^0\rangle \tag{69}$$

Then

$$\begin{aligned}\langle f|\mathcal{H}|\overline{B}^0\rangle &= \langle f|\mathcal{H}CP|B^0\rangle e^{-2i\xi} \\ &= \eta_{CP}e^{-2i\xi}\langle f|CP\mathcal{H}CP|B^0\rangle\end{aligned} \tag{70}$$

Thus $\langle f|\mathcal{H}|\overline{B}^0\rangle$ is obtained by reversing the weak phases in $\langle f|\mathcal{H}|B^0\rangle$ and multiplying by $\eta_{CP}e^{-2i\xi}$.

If there is a single weak phase, say β in the transition $B^0 \to f$, then

$$\frac{\langle f|\overline{B}^0\rangle}{\langle f|B^0\rangle} = e^{-2i\beta}e^{-2i\xi}\eta_{CP} \tag{71}$$

18. *What are the predictions for the asymmetry in $B^0 \to J/\psi K_S$?*

Since there are no weak phases in $B^0 \to J/\psi K_S$,

$$\langle J/\psi K_S|B^0\rangle = -\langle J/\psi K_S|\overline{B}^0\rangle e^{2i\xi} \tag{72}$$

The minus sign comes from the CP of $J/\psi K_S$ ($CP(J/\psi) = +1, CP(K_S) = +1$ and -1 from the $L = 1$ orbital angular momentum). Altogether, we see that

$$\lambda = \frac{q}{p}\frac{\overline{A}_f}{A_f} = -e^{-2i\beta} \tag{73}$$

and

$$\begin{aligned}\mathrm{Im}\lambda &= \sin 2\beta \\ |\lambda| &= 1\end{aligned} \tag{74}$$

Now we have

$$\begin{aligned}|\langle f|B^0{}_{phys}(t)\rangle|^2 &\propto e^{-\Gamma t}\left[1 - \sin 2\beta \sin \Delta m t\right] \\ |\langle f|\overline{B}^0_{phys}(t)\rangle|^2 &\propto e^{-\Gamma t}\left[1 + \sin 2\beta \sin \Delta m t\right]\end{aligned} \tag{75}$$

19. *What about $B^0 \to \pi^+\pi^-$?*

From the discussion above, indicating the tree amplitude without the weak phase by T and the penguin by P

$$\begin{aligned}\langle \pi^+\pi^-|\overline{B}^0\rangle &= e^{-i\gamma}T + e^{+i\beta}P \\ \langle \pi^+\pi^-|B^0\rangle &= [e^{i\gamma}T + e^{-i\beta}P]e^{-2i\xi}\end{aligned} \tag{76}$$

We can put these in the equations for the time-dependent amplitudes:

$$\begin{aligned}\langle f|B^0{}_{phys}(t)\rangle &= e^{-\Gamma t/2}(\cos(\Delta mt/2)(e^{i\gamma}T + e^{-i\beta}P) \\ &\qquad +ie^{-2i\beta}\sin(\Delta mt/2)(e^{-i\gamma}T + e^{+i\beta}P) \\ \langle f|\overline{B}^0_{phys}(t)\rangle &= e^{-\Gamma t/2}(\cos(\Delta mt/2)(e^{-i\gamma}T + e^{+i\beta}P) \\ &\qquad +ie^{2i\beta}\sin(\Delta mt/2)(e^{+\gamma}T + e^{-i\beta}P)\end{aligned} \tag{77}$$

Removing a factor of $e^{-i\beta}$ from one and $e^{-i\beta}$ from the other,

$$\begin{aligned}\langle f|B^0{}_{phys}(t)\rangle &= e^{-\Gamma t/2}(\cos(\Delta mt/2)(e^{i\gamma+i\beta}T + P) \\ &\qquad +i\sin(\Delta mt/2)(e^{-i\gamma-i\beta}T + P) \\ \langle f|\overline{B}^0_{phys}(t)\rangle &= e^{-\Gamma t/2}(\cos(\Delta mt/2)(e^{-i\gamma-i\beta}T + P) \\ &\qquad +i\sin(\Delta mt/2)(e^{+i\gamma+i\beta}T + P)\end{aligned} \tag{78}$$

we that it is only the combinations $-e^{\pm i\alpha}T+P$, where we have assumed that $\alpha+\beta+\gamma$.

The effective λ here is

$$\lambda = \frac{-e^{i\alpha}T + P}{-e^{-i\alpha}T + P} \tag{79}$$

If we ignore the penguins, then $\mathrm{Im}\lambda = \sin 2\alpha$. In all likelihood, this is not a good approximation. Note that the extent of direct CP violation is given by

$$|\lambda| = \left|\frac{1 - e^{-i\alpha}\frac{P}{T}}{1 - e^{i\alpha}\frac{P}{T}}\right| \tag{80}$$

If we expand in P/T,

$$|\lambda| = 1 - 2\sin 2\alpha\ \mathrm{Im}\frac{P}{T} + \ldots \tag{81}$$

Naturally, there is direct CP violation only if we have two weak phases (here β and γ) and a relative strong phase between two amplitudes (here T and P).

20. *What determines the time dependence of a process in which one B from an $\Upsilon(4S)$ is tagged and the other is observed in a CP eigenstate, $|f\rangle$?*

The $B^0 - \overline{B}^0$ state produced by the $\Upsilon(4S)$ can be expressed as

$$\int d^3q\phi(q)a_q^\dagger\overline{a}_{-q}^\dagger|0\rangle \tag{82}$$

where the $a_q^\dagger$ creates a B^0 and the $\overline{a}_{-q}^\dagger$ creates a $\overline{B}^0$. Because the Υ is a $J=1$ state, the $B^0\overline{B}^0$ state is p-wave, so $\phi(-p)=-\phi(p)$.

Suppose at $t=0$ we observe a B^0 with momentum p. Then the remaining state is

$$\phi(p)\overline{a}_{-q}^\dagger|0\rangle = \phi(p)|\overline{B}^0(-p)\rangle \tag{83}$$

The amplitude to find a state $|f\rangle$ at a later time, t, is *(N.B.$B^0(t=0)$ means B^0 is observed (tagged) at $t=0$, not that the remaining meson at $t=0$ is a B^0!!)*

$$\mathcal{M}(B^0 \ at\ t=0, f\ at\ t>0) = \phi(p)e^{-i(M-i\Gamma/2)t}$$
$$\times\left(\cos\tfrac{1}{2}\Delta mt\ \langle f|\mathcal{H}|\overline{B}^0\rangle + i\frac{p}{q}\sin\tfrac{1}{2}\Delta mt\ \langle f|\mathcal{H}|B^0\rangle\right)$$

Suppose, on the other hand, we observe $|f\rangle$ with momentum p at $t=0$. Then the remaining state is

$$\phi(p)|\overline{B}^0(-p)\rangle\langle f|\mathcal{H}|B^0(p)\rangle + \phi(-p)|B^0(-p)\rangle\langle f|\mathcal{H}|\overline{B}^0(p)\rangle \tag{84}$$

The amplitude to observe a B^0 at a time t later is thus, dropping the common exponential factor,

$$\phi(p)\langle f|\mathcal{H}|B^0(p)\rangle i\frac{p}{q}\sin\tfrac{1}{2}\Delta mt + \phi(-p)\langle f|\mathcal{H}|\overline{B}^0(p)\rangle\cos\tfrac{1}{2}\Delta mt \tag{85}$$

Since ϕ is odd, we have

$$\phi(p)\left[\langle f|\mathcal{H}|B^0(p)\rangle i\frac{p}{q}\sin\tfrac{1}{2}\Delta mt - \langle f|\mathcal{H}|\overline{B}^0(p)\rangle\cos\tfrac{1}{2}\Delta mt\right] \tag{86}$$

Note the change relative sign of the two terms, a reflection of the p-wave state.

Now let us indicate

$$\begin{aligned} \mathcal{A} &= \langle f|\mathcal{H}|B^0\rangle, \\ \overline{\mathcal{A}} &\equiv \langle \overline{f}|\mathcal{H}|\overline{B}^0\rangle = \langle f|\mathcal{H}|\overline{B}^0\rangle \end{aligned} \tag{87}$$

where in the last equation we assume that $|f\rangle$ has positive CP. Now the decay rate is

$$\Gamma(B^0 \ at \ t=0, f \ at \ t>0) = e^{-\Gamma t}\left|\cos\frac{\Delta m}{2}t\overline{\mathcal{A}} + i\frac{p}{q}\sin\frac{\Delta m}{2}t\mathcal{A}\right|^2$$

$$= e^{-\Gamma t}\left[\tfrac{1}{2}(|\overline{\mathcal{A}}|^2 + |\mathcal{A}|^2) + \tfrac{1}{2}\cos\Delta mt(|\overline{\mathcal{A}}|^2 - |\mathcal{A}|^2) - \mathrm{Im}\frac{p}{q}\sin\Delta mt\mathcal{A}\overline{\mathcal{A}}^*\right] \tag{88}$$

Assuming $|p/q| = 1$, and writing

$$\lambda = \frac{q}{p}\frac{\overline{\mathcal{A}}}{\mathcal{A}} = \frac{p^*}{q^*}\frac{\overline{\mathcal{A}}}{\mathcal{A}} \tag{89}$$

we have

$$\begin{aligned} &\Gamma(B^0(tag \ at \ t=0); f(t>0)) = \\ &\quad e^{-\Gamma t}|A|^2\left[\tfrac{1}{2}(1+|\lambda|^2) - \tfrac{1}{2}\cos\Delta mt(1-|\lambda|^2) + \mathrm{Im}\lambda\sin\Delta mt\right] \end{aligned} \tag{90}$$

and, *mutatis mutandis*,

$$\begin{aligned} &\Gamma(\overline{B}^0(tag \ at \ t=0); f(t>0)) = \\ &\quad e^{-\Gamma t}|A|^2\left[\tfrac{1}{2}(1+|\lambda|^2) + \tfrac{1}{2}\cos\Delta mt(1-|\lambda|^2) - \mathrm{Im}\lambda\sin\Delta mt\right] \end{aligned} \tag{91}$$

On the other hand, if f is observed at $t=0$ and B^0 at $t>0$, we find

$$\begin{aligned} &\Gamma(f(t=0); B^0(tag \ at \ t>0)) \\ &\quad = e^{-\Gamma t}|A|^2\left[\tfrac{1}{2}(1+|\lambda|^2) - \tfrac{1}{2}\cos\Delta mt(1-|\lambda|^2) - \mathrm{Im}\lambda\sin\Delta mt\right] \end{aligned} \tag{92}$$

and analogously

$$\Gamma(f(t=0);\overline{B}^0(tag\ at\ t>0))$$
$$= e^{-\Gamma t}|A|^2\left[\tfrac{1}{2}(1+|\lambda|^2)+\tfrac{1}{2}\cos\Delta mt(1-|\lambda|^2)+\mathrm{Im}\lambda\sin\Delta mt\right] \quad (93)$$

If no timing information is available, the best we can do is observe correlations between tagged Bs and a final state f:

$$\int dt\Gamma(f\ observed; B^0\ tagged) = \frac{|A|^2}{\Gamma}\left[(1+|\lambda|^2)-(1-|\lambda|^2)\frac{\Gamma^2}{\Gamma^2+\Delta m^2}\right] \quad (94)$$
$$\int dt\Gamma(f\ observed; \overline{B}^0\ tagged) = \frac{|A|^2}{\Gamma}\left[(1+|\lambda|^2)+(1-|\lambda|^2)\frac{\Gamma^2}{\Gamma^2+\Delta m^2}\right] \quad (95)$$

Note that only direct CP violation can be seen in this way. This is the drawback of B-factories, especially those that are symmetric.

In the event that there is a single weak phase, $|\overline{\mathcal{A}}/\mathcal{A}| = 1$ and $|\lambda|^2 = 1$ and the time dependence becomes much simpler:

$$\Gamma(B^0(t=0); f(t>0)) = e^{-\Gamma t}|A|^2\left[1+\mathrm{Im}\lambda\sin\Delta mt\right] \quad (96)$$
$$\Gamma(\overline{B}^0(t=0); f(t>0)) = e^{-\Gamma t}|A|^2\left[1-\mathrm{Im}\lambda\sin\Delta mt\right] \quad (97)$$
$$\Gamma(f(t=0); B^0(t>0)) = e^{-\Gamma t}|A|^2\left[1-\mathrm{Im}\lambda\sin\Delta mt\right] \quad (98)$$
$$\Gamma(f(t=0); \overline{B}^0(t>0)) = e^{-\Gamma t}|A|^2\left[1+\mathrm{Im}\lambda\sin\Delta mt\right] \quad (99)$$

21. *How can non-CP eigenstates be used?*

If we consider $B^0 \to \rho^+\pi^-$ and $\overline{B}^0 \to \rho^+\pi^-$ we see that there is no reason to expect them to have similar rates. For example, the $\overline{b}$ may emit a W^+ in decaying to $\overline{c}$ and the W^+ may become a ρ^+, but in the $\overline{B}^0$ decay it is a W^- that is emitted, which would become a ρ^-, not a ρ^+. On the other hand, $B^0 \to \rho^+\pi^-$ and $\overline{B}^0 \to \rho^-\pi^+$ are related by CP and their amplitudes must have a close connection.

Let us introduce four amplitudes:

$$\begin{aligned}
\mathcal{A} &= \langle f|\mathcal{H}|B^0\rangle && (100)\\
\mathcal{A}' &= \langle \overline{f}|\mathcal{H}|B^0\rangle && (101)\\
\overline{\mathcal{A}}' &= \langle f|\mathcal{H}|\overline{B}^0\rangle && (102)\\
\overline{\mathcal{A}} &= \langle \overline{f}|\mathcal{H}|\overline{B}^0\rangle && (103)
\end{aligned}$$

Now we would find

$$\begin{aligned}
\Gamma(B^0\ at\ 0, f\ at\ t>0) &= e^{-\Gamma t}\left|\cos\frac{\Delta m}{2}t\overline{\mathcal{A}}' + i\frac{p}{q}\sin\frac{\Delta m}{2}t\mathcal{A}\right|^2 && (104)\\
\Gamma(f\ at\ 0, B^0\ at\ t>0) &= e^{-\Gamma t}\left|-\cos\frac{\Delta m}{2}t\overline{\mathcal{A}}' + i\frac{p}{q}\sin\frac{\Delta m}{2}t\mathcal{A}\right|^2 && (105)\\
\Gamma(\overline{B}^0\ at\ 0, f\ at\ t>0) &= e^{-\Gamma t}\left|\cos\frac{\Delta m}{2}t\mathcal{A} + i\frac{q}{p}\sin\frac{\Delta m}{2}t\overline{\mathcal{A}}'\right|^2 && (106)\\
\Gamma(f\ at\ 0, \overline{B}^0 at\ t>0) &= e^{-\Gamma t}\left|-\cos\frac{\Delta m}{2}t\mathcal{A} + i\frac{q}{p}\sin\frac{\Delta m}{2}t\overline{\mathcal{A}}'\right|^2 && (107)\\
\Gamma(B^0\ at\ 0, \overline{f}\ at\ t>0) &= e^{-\Gamma t}\left|\cos\frac{\Delta m}{2}t\overline{\mathcal{A}} + i\frac{p}{q}\sin\frac{\Delta m}{2}t\mathcal{A}'\right|^2 && (108)\\
\Gamma(\overline{f}\ at\ 0, B^0\ at\ t>0) &= e^{-\Gamma t}\left|-\cos\frac{\Delta m}{2}t\overline{\mathcal{A}} + i\frac{p}{q}\sin\frac{\Delta m}{2}t\mathcal{A}'\right|^2 && (109)\\
\Gamma(\overline{B}^0\ at\ 0, \overline{f}\ at\ t>0) &= e^{-\Gamma t}\left|\cos\frac{\Delta m}{2}t\mathcal{A}' + i\frac{q}{p}\sin\frac{\Delta m}{2}t\overline{\mathcal{A}}\right|^2 && (110)\\
\Gamma(\overline{f}\ at\ 0, \overline{B}^0 at\ t>0) &= e^{-\Gamma t}\left|-\cos\frac{\Delta m}{2}t\mathcal{A}' + i\frac{q}{p}\sin\frac{\Delta m}{2}t\overline{\mathcal{A}}\right|^2 && (111)
\end{aligned}$$

We now define

$$\begin{aligned}
\lambda &= \frac{q}{p}\frac{\overline{\mathcal{A}}'}{\mathcal{A}} = \frac{p^*}{q^*}\frac{\overline{\mathcal{A}}'}{\mathcal{A}} && (112)\\
\lambda' &= \frac{q}{p}\frac{\overline{\mathcal{A}}}{\mathcal{A}'} = \frac{p^*}{q^*}\frac{\overline{\mathcal{A}}}{\mathcal{A}'} && (113)\\
& && (114)
\end{aligned}$$

In terms of these, we have

$$\begin{aligned}
\Gamma(B^0 \ at \ 0, f \ at \ t>0) &= e^{-\Gamma t}|A|^2\left[\tfrac{1}{2}(1+|\lambda|^2)-\tfrac{1}{2}(1-|\lambda|^2)\cos\Delta mt+\mathrm{Im}\lambda\sin\Delta mt\right]\\
\Gamma(f \ at \ 0, B^0 \ at \ t>0) &= e^{-\Gamma t}|A|^2\left[\tfrac{1}{2}(1+|\lambda|^2)-\tfrac{1}{2}(1-|\lambda|^2)\cos\Delta mt-\mathrm{Im}\lambda\sin\Delta mt\right]\\
\Gamma(\overline{B}^0 \ at \ 0, f \ at \ t>0) &= e^{-\Gamma t}|A|^2\left[\tfrac{1}{2}(1+|\lambda|^2)+\tfrac{1}{2}(1-|\lambda|^2)\cos\Delta mt-\mathrm{Im}\lambda\sin\Delta mt\right]\\
\Gamma(f \ at \ 0, \overline{B}^0 at \ t>0) &= e^{-\Gamma t}|A|^2\left[\tfrac{1}{2}(1+|\lambda|^2)+\tfrac{1}{2}(1-|\lambda|^2)\cos\Delta mt+\mathrm{Im}\lambda\sin\Delta mt\right]\\
\Gamma(B^0 \ at \ 0, \overline{f} \ at \ t>0) &= e^{-\Gamma t}|A'|^2\left[\tfrac{1}{2}(1+|\lambda'|^2)-\tfrac{1}{2}(1-|\lambda'|^2)\cos\Delta mt+\mathrm{Im}\lambda'\sin\Delta mt\right]\\
\Gamma(\overline{f} \ at \ 0, B^0 \ at \ t>0) &= e^{-\Gamma t}|A'|^2\left[\tfrac{1}{2}(1+|\lambda'|^2)-\tfrac{1}{2}(1-|\lambda'|^2)\cos\Delta mt-\mathrm{Im}\lambda'\sin\Delta mt\right]\\
\Gamma(\overline{B}^0 \ at \ 0, \overline{f} \ at \ t>0) &= e^{-\Gamma t}|A'|^2\left[\tfrac{1}{2}(1+|\lambda'|^2)+\tfrac{1}{2}(1-|\lambda'|^2)\cos\Delta mt-\mathrm{Im}\lambda'\sin\Delta mt\right]\\
\Gamma(\overline{f} \ at \ 0, \overline{B}^0 at \ t>0) &= e^{-\Gamma t}|A'|^2\left[\tfrac{1}{2}(1+|\lambda'|^2)+\tfrac{1}{2}(1-|\lambda'|^2)\cos\Delta mt+\mathrm{Im}\lambda'\sin\Delta mt\right]
\end{aligned} \tag{115}$$

If f is an even CP eigenstate, then all the primes can be removed and we have the situation discussed earlier. If f is an odd CP eigenstate, the primes can be removed in favor of a minus sign.

If instead f is not a CP eigenstate, there is still the possibility of some simplification. In particular, suppose a single weak decay mechanism is responsible for the decays. Then we have

$$\frac{\mathcal{A}}{\overline{\mathcal{A}}} = e^{2i\phi_{\mathcal{A}}} \tag{116}$$

$$\frac{\mathcal{A}'}{\overline{\mathcal{A}}'} = e^{2i\phi_{\mathcal{A}}} \tag{117}$$

This would be the case, for instance, in $B\to\rho\pi$ if we ignore penguins. We see that with this assumption, $\mathcal{A}$ and $\overline{\mathcal{A}}$ have the same strong phase, and $\mathcal{A}'$ and $\overline{\mathcal{A}}'$ have the same strong phase. Thus

$$\mathcal{A} = |\mathcal{A}|e^{i\phi_{\mathcal{A}}}e^{i\delta_{\mathcal{A}}} \tag{118}$$

$$\mathcal{A}' = |\mathcal{A}'|e^{i\phi_{\mathcal{A}}}e^{i\delta'_{\mathcal{A}}} \tag{119}$$

$$\overline{\mathcal{A}}' = |\mathcal{A}'|e^{-i\phi_{\mathcal{A}}}e^{i\delta'_{\mathcal{A}}} \tag{120}$$

$$\overline{\mathcal{A}} = |\mathcal{A}|e^{-i\phi_{\mathcal{A}}}e^{i\delta_{\mathcal{A}}} \tag{121}$$

In this instance, we can express

$$\lambda = \frac{q}{p}\frac{\overline{\mathcal{A}}'}{\mathcal{A}} = \frac{q}{p}e^{-2i\phi_{\mathcal{A}}}e^{i(\delta'_{\mathcal{A}}-\delta_{\mathcal{A}})}\left|\frac{\mathcal{A}'}{\mathcal{A}}\right| \tag{122}$$

$$\lambda' = \frac{q}{p}\frac{\overline{\mathcal{A}}}{\mathcal{A}'} = \frac{q}{p}e^{-2i\phi_{\mathcal{A}}}e^{i(\delta_{\mathcal{A}}-\delta'_{\mathcal{A}})}\left|\frac{\mathcal{A}}{\mathcal{A}'}\right| \tag{123}$$

and, in particular, for $B \to \rho\pi$

$$\mathrm{Im}\lambda = -\sin(2\alpha + \Delta\delta)\left|\frac{\mathcal{A}'}{\mathcal{A}}\right| \tag{124}$$

$$\mathrm{Im}\lambda' = -\sin(2\alpha - \Delta\delta)\left|\frac{\mathcal{A}}{\mathcal{A}'}\right| \tag{125}$$

$$\tag{126}$$

so that by measuring both the $\rho^{\pm}\pi^{\mp}$ final states, we can determine both α and the difference of the phase shifts. Unfortunately, penguins will subvert this result to some extent.