PERGAMON

Journal of Quantitative Spectroscopy & Radiative Transfer 73 (2002) 23–27

Journal of Quantitative Spectroscopy & Radiative Transfer

www.elsevier.com/locate/jqsrt

Reciprocity principle for radiative transfer models that use periodic boundary conditions

Larry Di Girolamo *

Department of Atmospheric Sciences, University of Illinois at Urbana-Champaign, 105 S Gregory Street, Urbana, IL 61801-3070, USA

Received 6 February 2001; accepted 14 May 2001

Abstract

Many numerical models use periodic boundary conditions in solving the radiative transfer through heterogeneous media specified over a fixed domain. A reciprocity principle applicable to solutions from these models is derived for the common situation of a scattering and absorbing heterogeneous medium that is illuminated over the entire domain from a single direction. The derived reciprocity principle states that the domain-averaged bidirectional reflectance distribution function remains invariant when incoming and outgoing directions are interchanged, regardless of the heterogeneity of the medium and the size of the domain. This reciprocity principle provides a simple and useful benchmark test for radiative transfer models that use periodic boundary conditions. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Reciprocity; Three-dimensional radiative transfer

1. Introduction

Reciprocity principles in radiative transfer theory have been widely used in deriving analytical and numerical solutions of radiative transfer problems [1,2], in testing numerical models of radiative transfer [3], and in remote sensing applications [4,5]. In the most widely studied case, that of a horizontally homogeneous medium completely illuminated at the top boundary by a constant, unidirectional irradiance, the reciprocity principle states that the bidirectional reflectance distribution function (BRDF), R, at the top boundary of the medium is invariant under a change in the incident and outward directions [6]; that is, at the top boundary of the medium,

$$R(-\Omega_1;\Omega_2)=R(-\Omega_2;\Omega_1), \tag{1}$$

* Tel.: +1-217-333-3080; fax: +1-217-244-4393.
E-mail address: larry@atmos.uiuc.edu (L. Di Girolamo).

PII: S0022-4073(01)00166-2

where Ω represents the directional unit vector with the outward direction negative. For a unidirectional irradiance, F,

$$R(-\Omega_1;\Omega_2) = \frac{I(-\Omega_1;\Omega_2)}{\Omega_2 \cdot \boldsymbol{n} F(\Omega_2)}, \tag{2}$$

where I is the radiance and $\boldsymbol{n}$ is a unit vector that is outward normal at the top boundary (i.e., $\boldsymbol{n} \cdot \Omega < 0$ represents a direction incident at the top boundary of the medium).

Eq. (1) has been derived for the idealized case of a horizontally homogeneous medium completely illuminated at the top boundary by a constant irradiance [6]. For this case, the horizontal flux divergence is zero, and the solution of the radiative transfer is one-dimensional (1-D); that is, the radiance field only varies in the vertical dimension. Radiative transfer models for the solution of this idealized case require only the vertical distribution of the optical properties of the medium as input into the model.

In the case of a 3-D heterogeneous medium illuminated everywhere at the top boundary, the radiative transfer solution is much more complicated to handle than the 1-D case. One complicating factor is that the optical properties of the medium need to be specified in all three spatial dimensions. Often, only the optical properties of the medium over a finite horizontal domain are specified, even though the radiative transfer solution for the medium within the domain depends on the optical properties of the medium outside the domain. To handle this problem, periodic boundary conditions (PBC) are often employed. When PBC are employed, the optical properties of the medium within the model domain are assumed to indefinitely repeat themselves outside the model domain, as illustrated in Fig. 1. This type of boundary condition

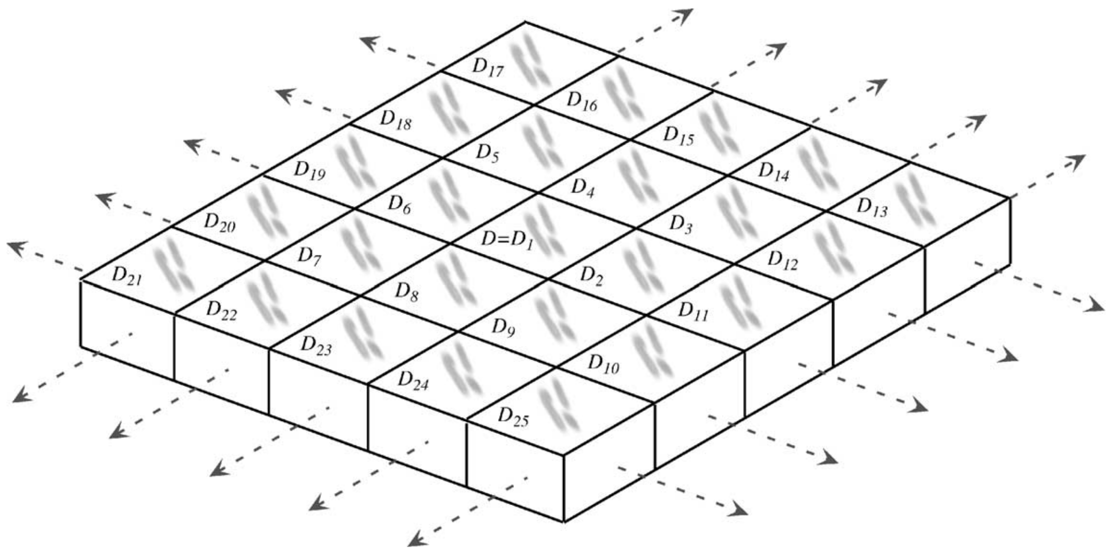

Fig. 1. An illustration of a medium modeled in a radiative transfer model that uses periodic boundary conditions (PBC). Only the optical properties of the medium (shaded in gray) within the model domain, which is bound at the top by the surface D, are specified in the model. By employing PBC, the radiative transfer model's solution to the radiance field across the surface D effectively solves for the case when the medium indefinitely repeats itself around D. The dashed arrows represent the medium extending out to infinity in all horizontal directions.

is easily handled in model calculations. For example, in Monte Carlo radiative transfer models that employ PBC, a photon leaving the model domain reappears on the opposite side of the domain, traveling in the same direction as when it left the domain.

In general, 3-D radiative transfer solutions obey reciprocity principles that have spatial and directional attributes [7,8]. Note that Eq. (1) is strictly a directional reciprocity principle. The purpose of this article is to provide a formal proof that the domain-averaged BRDF obeys a directional reciprocity principle, regardless of the heterogeneity of the medium and the size of the model domain, for cases when the top boundary of the model domain is illuminated by a constant unidirectional source and the model employs PBC.

2. Proof

For an external unidirectional illumination, Di Girolamo [8] derived the following reciprocity principle:

$$\Omega_2 \cdot \boldsymbol{n} \int_D F_2(\boldsymbol{r}, \Omega_2) I(\boldsymbol{r}, -\Omega_2; C, \Omega_1)\, \mathrm{d}\boldsymbol{r} = \Omega_1 \cdot \boldsymbol{n} \int_C F_1(\boldsymbol{r}, \Omega_1) I(\boldsymbol{r}, -\Omega_1; D, \Omega_2)\, \mathrm{d}\boldsymbol{r}, \tag{3}$$

where $I(\boldsymbol{r}, -\Omega_2; C, \Omega_1)$ is the radiance at position $\boldsymbol{r}$ in direction $-\Omega_2$ caused by illuminating the surface C with a unidirectional irradiance $F_1(\boldsymbol{r}, \Omega_1)$ from direction $\Omega_1, I(\boldsymbol{r}, -\Omega_1; D, \Omega_2)$ is the radiance at position $\boldsymbol{r}$ in direction $-\Omega_1$ caused by illuminating the surface D with a unidirectional irradiance $F_2(\boldsymbol{r}, \Omega_2)$ from direction Ω_2, and surface integration is taken over surfaces C and D. Eq. (3) is quite general and applies to any absorbing and scattering medium, regardless of its heterogeneity. The only assumption used in its derivation is that the scattering phase function of the scatterers that form the medium have time-reversal symmetry.

Surfaces C and D may represent any surface in space. However, for this proof, let C be the horizontal surface that extends out to infinity in all horizontal directions, located at the top boundary of the model domain, and illuminated everywhere with a constant unidirectional irradiance. In practice, the model domain is finite and is bound at the top by a horizontal surface, D. Thus, D is a subset of C. As shown in Fig. 1, the domain can be considered periodic out to infinity in all horizontal directions when PBC are used. Thus,

$$C = \sum_{i=1}^{N} D_i, \quad N \to \infty, \tag{4}$$

where D_i represents the top surface of an individual domain as shown in Fig. 1. Summing both sides of Eq. (3) over all D_i and dividing by N yields

$$\frac{\Omega_2 \cdot \boldsymbol{n}}{N} \sum_{i=1}^{N} \int_{D_i} F_2(\boldsymbol{r}, \Omega_2) I(\boldsymbol{r}, -\Omega_2; C, \Omega_1)\, \mathrm{d}\boldsymbol{r} = \frac{\Omega_1 \cdot \boldsymbol{n}}{N} \sum_{i=1}^{N} \int_C F_1(\boldsymbol{r}, \Omega_1) I(\boldsymbol{r}, -\Omega_1; D_i, \Omega_2)\, \mathrm{d}\boldsymbol{r}. \tag{5}$$

For the case of PBC, the integral on the left-hand side of Eq. (5) is the same for all D_i; that is

$$\int_{D_i} F_2(\boldsymbol{r}, \Omega_2) I(\boldsymbol{r}, -\Omega_2; C, \Omega_1)\, \mathrm{d}\boldsymbol{r} = \int_D F_2(\boldsymbol{r}, \Omega_2) I(\boldsymbol{r}, -\Omega_2; C, \Omega_1)\, \mathrm{d}\boldsymbol{r} \quad \forall\ D_i, \tag{6}$$

where D is any one realization of D_i, which is simply the surface of the top boundary of the model domain. With Eq. (6), Eq. (5) becomes

$$\Omega_2 \cdot \boldsymbol{n} \int_D F_2(\boldsymbol{r}, \Omega_2) I(\boldsymbol{r}, -\Omega_2; C, \Omega_1)\, \mathrm{d}\boldsymbol{r} = \frac{\Omega_1 \cdot \boldsymbol{n}}{N} \sum_{i=1}^{N} \int_C F_1(\boldsymbol{r}, \Omega_1) I(\boldsymbol{r}, -\Omega_1; D_i, \Omega_2)\, \mathrm{d}\boldsymbol{r}. \tag{7}$$

Note, with reference to Eq. (4), the integral on the right-hand side of Eq. (7) can be written as

$$\int_C F_1(\boldsymbol{r}, \Omega_1) I(\boldsymbol{r}, -\Omega_1; D_i, \Omega_2)\, \mathrm{d}\boldsymbol{r} = \sum_{j=1}^{N} \int_{D_j} F_1(\boldsymbol{r}, \Omega_1) I(\boldsymbol{r}, -\Omega_1; D_i, \Omega_2)\, \mathrm{d}\boldsymbol{r}. \tag{8}$$

Substituting Eq. (8) into Eq. (7) yields

$$\begin{aligned}
\Omega_2 \cdot \boldsymbol{n} \int_D F_2(\boldsymbol{r}, \Omega_2) I(\boldsymbol{r}, -\Omega_2; C, \Omega_1)\, \mathrm{d}\boldsymbol{r} &= \frac{\Omega_1 \cdot \boldsymbol{n}}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{D_j} F_1(\boldsymbol{r}, \Omega_1) I(\boldsymbol{r}, -\Omega_1; D_i, \Omega_2)\, \mathrm{d}\boldsymbol{r} \\
&= \frac{\Omega_1 \cdot \boldsymbol{n}}{N} \sum_{j=1}^{N} \sum_{i=1}^{N} \int_{D_j} F_1(\boldsymbol{r}, \Omega_1) I(\boldsymbol{r}, -\Omega_1; D_i, \Omega_2)\, \mathrm{d}\boldsymbol{r} \\
&= \frac{\Omega_1 \cdot \boldsymbol{n}}{N} \sum_{j=1}^{N} \int_{D_j} F_1(\boldsymbol{r}, \Omega_1) I(\boldsymbol{r}, -\Omega_1; C, \Omega_2)\, \mathrm{d}\boldsymbol{r}.
\end{aligned} \tag{9}$$

For the case of PBC, the integral on the right-hand side of Eq. (9) is the same for all D_j; that is

$$\int_{D_j} F_1(\boldsymbol{r}, \Omega_1) I(\boldsymbol{r}, -\Omega_1; C, \Omega_2)\, \mathrm{d}\boldsymbol{r} = \int_D F_1(\boldsymbol{r}, \Omega_1) I(\boldsymbol{r}, -\Omega_1; C, \Omega_2)\, \mathrm{d}\boldsymbol{r} \quad \forall\, D_j, \tag{10}$$

where D is any one realization of D_j, which is simply the surface of the top boundary of the model domain. Substituting Eq. (10) into Eq. (9) yields

$$\Omega_2 \cdot \boldsymbol{n} \int_D F_2(\boldsymbol{r}, \Omega_2) I(\boldsymbol{r}, -\Omega_2; C, \Omega_1)\, \mathrm{d}\boldsymbol{r} = \Omega_1 \cdot \boldsymbol{n} \int_D F_1(\boldsymbol{r}, \Omega_1) I(\boldsymbol{r}, -\Omega_1; C, \Omega_2)\, \mathrm{d}\boldsymbol{r}. \tag{11}$$

In the case when F_1 and F_2 are independent of $\boldsymbol{r}$ (i.e., constant illumination), Eq. (11) can be written as

$$\frac{\int_D I(\boldsymbol{r}, -\Omega_2; C, \Omega_1)\, \mathrm{d}\boldsymbol{r}}{\Omega_1 \cdot \boldsymbol{n} F_1(\Omega_1)} = \frac{\int_D I(\boldsymbol{r}, -\Omega_1; C, \Omega_2)\, \mathrm{d}\boldsymbol{r}}{\Omega_2 \cdot \boldsymbol{n} F_2(\Omega_2)}. \tag{12}$$

When divided through by the area of D, Eq. (12) states that the average BRDF over the domain D obeys directional reciprocity.

3. Discussion

It was shown in this article that the domain-averaged BRDF remains invariant when incoming and outgoing directions are interchanged, regardless of the heterogeneity of the medium and the size of the model domain, for cases when the top boundary of the model domain is illuminated by a constant unidirectional source and the model employs PBC. This statement is also true when vacuum boundary conditions (VBC) are employed; that is, photons exiting the domain never return. The proof is straightforward: for VBC, $C = D =$ top boundary of model domain in Eq. (3). Setting F_1 and F_2 independent of $\boldsymbol{r}$ leads directly to Eq. (12).

Eq. (12) provides a useful benchmark test for 3-D radiative transfer codes. These codes traditionally have had very few benchmark tests upon which to draw. Eq. (12) is currently being used as a benchmark test in the NASA/DOE Intercomparison of 3-D Radiation Codes project [9].

Acknowledgements

Partial support from the Jet Propulsion Laboratory of the California Institute of Technology under contract 1221756 is gratefully acknowledged.

References

[1] Stamnes K, Swanson RA. A new look at the discrete ordinate method for radiative transfer calculations in anisotropically scattering atmospheres. J Atmos Sci 1981;38:387–99.
[2] Van de Hulst HC. Reciprocity relations in radiative transfer by spherical clouds. JQSRT 2000;64:151–72.
[3] Herman BM, Asous W, Browning SR. A semi-analytic technique to integrate the radiative transfer equation over optical depth. J Atmos Sci 1980;37:1828–38.
[4] Suttles JT, Green RN, Minnis P, Smith GL, Staylor F, Wielicki BA, Walker IJ, Young DF, Taylor VR, Stowe LL. Angular radiation models for the Earth-atmosphere system, vol. I, shortwave radiation. NASA RP-1184, Washington, DC: NASA, 1988, 147pp.
[5] Stephens GL, Ellingson RG, Vitko J, Bolton W, Tooman TP, Valero FPJ, Minnis P, Pilewskie P, Phipps GS, Sekelsky S, Carswell JR, Miller SD, Benedetti A, McCoy RB, McCoy RF, Lederbuhr A, Bambha R. The Department of Energy's Atmospheric Radiation Measurement (ARM) Unmanned Aerospace Vehicle (UAV) program. Bull Amer Meteor Soc 2000;81:2915–37.
[6] Chandrasekhar S. Radiative transfer. Dover: New York, 1960.
[7] Case KM. Transfer problems and the reciprocity principle. Rev Mod Phys 1957;29:651–63.
[8] Di Girolamo L. Reciprocity principle applicable to reflected radiance measurements and the searchlight problem. Appl Opt 1999;38:3196–8.
[9] Cahalan RF. Personal communication; http://climate.gsfc.nasa.gov/I3RC/index.html.