2. Conservation of momentum, heat and energy

2.1 Equation of motion

If the only external force is gravity, then the equation of motion of an inviscid fluid in an inertial reference frame is

$$\rho d\boldsymbol{V}/dt = -\nabla p - \rho\nabla\Phi_g . \tag{2.1}$$

Here $\rho(\boldsymbol{x}, t)$ is the density, $p(\boldsymbol{x}, t)$ the pressure, and $\Phi_g(\boldsymbol{x}, t)$ is the gravitational potential. We ignore the time-dependent, tide-generating part of Φ_g resulting from the gravitational fields of the sun and moon. Furthermore, we consider Φ_g as given, the mass of the atmosphere being negligible compared to that of the earth.

Much of the complexity of fluid dynamics results from the nonlinearity hidden in the acceleration $d\boldsymbol{V}/dt = (\partial/\partial t + \boldsymbol{V}\cdot\nabla)\boldsymbol{V}$: the velocity advects itself. The vector notation $(\boldsymbol{V}\cdot\nabla)\boldsymbol{V}$ is an awkward one, and it is sometimes clearer to work explicitly in Cartesian coordinates, so that $(\boldsymbol{V}\cdot\nabla)V_i = V_j\partial V_i/\partial x_j$. One can then write 2.1 in flux form,

$$\partial(\rho V_i)/\partial t + \partial(\rho V_i V_j)/\partial x_j = -\partial p/\partial x_i - \rho\partial\Phi_g/\partial x_i . \tag{2.2}$$

An alternative form of the equation of motion is obtained by using the identity

$$(\boldsymbol{V}\cdot\nabla)\boldsymbol{V} = (\nabla\times\boldsymbol{V})\times\boldsymbol{V} + \nabla(|\boldsymbol{V}|^2/2) \ , \tag{2.3}$$

with the result

$$\boldsymbol{V}_t = -(\omega\times\boldsymbol{V}) - \rho^{-1}\nabla p - \nabla(\Phi_g + |\boldsymbol{V}|^2/2) , \tag{2.4}$$

where $\omega \equiv \nabla\times\boldsymbol{V}$ is the *vorticity* of the flow.

There is no need to memorize numerous vector identities such as 2.3; nearly all of them can be obtained from the single result

$$\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} . \tag{2.5}$$

Here δ_{ij} is the Kronecker delta (equal to unity if $i = j$ and zero if $i \neq j$) while ε_{ijk} equals zero if any of the three indices are equal, +1 if the indices are an even permutation of (x, y, z) and -1 if they are an odd permutation. In this notation, $(\nabla\times\boldsymbol{V})_i = \varepsilon_{ijk}\partial V_k/\partial x_j$. The validity of 2.5 can be checked by direct calculation. It can then be used to show that

$$(\boldsymbol{V}\times\omega)_i = \varepsilon_{ijk}\varepsilon_{klm}V_j\partial V_m/\partial x_l = V_j\partial V_j/\partial x_i - V_j\partial V_i/\partial x_j , \tag{2.6}$$

which yields 2.4.

2.2 Rotation

We are accustomed to looking at the atmosphere from a a coordinate system rotating with the angular velocity Ω of the solid earth. We can assume that Ω is constant, with magnitude $7.2\times10^{-5}\ \mathrm{s}^{-1}$.

The sidereal day is $2\pi/\Omega$. The solar day, 86400 s, is $2\pi/(\Omega+\Omega_s)$, where $\Omega_s = 2\pi/(1\ \mathrm{yr})$ is the angular velocity of the earth in its orbit around the sun. Fluctuations in the length of the day, with an amplitude of a fraction of a millisecond, are observed to occur on time scales of days to years. Since these fluctuations result primarily from exchange of angular momentum between the atmosphere and the solid earth, they are an important diagnostic of the atmospheric state, but they are obviously much too small to have any effect on its dynamics.

The earth's rotation rate is slowly decreasing due to tidal friction that causes some of the angular momentum of the rotation to be transferred to the earth-moon system. One billion years ago, the length of the day is estimated to have been ?? hours.

If the velocity at the point $\boldsymbol{r}$ in an inertial frame is $\boldsymbol{V}$ (where $\boldsymbol{r} = 0$ is on the axis of rotation) then the velocity in the rotating frame is

$$\boldsymbol{V}^* = \boldsymbol{V} - \Omega\times\boldsymbol{r}. \tag{2.7}$$

More generally, the material derivative of any vector $\boldsymbol{A}$ in rotating and inertial frames are related by

$$\frac{d^*\boldsymbol{A}}{dt} = \frac{d\boldsymbol{A}}{dt} - \Omega\times\boldsymbol{A}. \tag{2.8}$$

Using both 2.7 and 2.8, we find for the acceleration of a fluid particle in the inertial frame

$$\frac{d\boldsymbol{V}}{dt} = \frac{d\boldsymbol{V}^*}{dt} + \Omega\times\boldsymbol{V} = \frac{d^*\boldsymbol{V}^*}{dt} + 2\Omega\times\boldsymbol{V}^* + \Omega\times(\Omega\times\boldsymbol{r}). \tag{2.9}$$

The last term, the centripetal acceleration, is a function of space only and can therefore be incorporated into the gravitational potential. The second term, the Coriolis acceleration, gives rotating flows their distinctive, and often counterintuitive, character.

The equation of motion in the rotating frame now reads (dropping the asterisks)

$$\rho\left(\frac{d\boldsymbol{V}}{dt} + 2\Omega\times\boldsymbol{V}\right) = -\nabla p - \rho\nabla\Phi, \tag{2.10}$$

where $\Phi \equiv \Phi_g - \Omega^2R^2/2$ is the geopotential, and R is the distance to the axis of rotation ($R = r\cos\theta$ in spherical coordinates). The alternative form 2.4 becomes

$$\boldsymbol{V}_t = -(2\Omega+\zeta)\times\boldsymbol{V} - \rho^{-1}\nabla p - \nabla(\Phi + |\boldsymbol{V}|^2/2)\ ; \qquad \zeta = \nabla\times\boldsymbol{V}. \tag{2.11}$$

The quantity $\omega = 2\Omega+\zeta$ is the total vorticity of the flow, the first term, 2Ω, being the vorticity of the solid-body rotation. The total vorticity is often referred to as the *absolute vorticity*, and the vorticity as measured in a rotating frame, ζ, as the *relative vorticity.*

A fluid at rest in an inertial frame of reference must be in *hydrostatic balance*, with the pressure gradient balancing the gravitational force: $\nabla p = -\rho \nabla \Phi_g$. In such a state of rest, surfaces of constant pressure are also surfaces of constant gravitational potential. If the fluid is instead at rest in a rotating system, we have

$$\nabla p = -\rho \nabla \Phi = -\rho \nabla (\Phi_g - \Omega^2 R^2 / 2). \qquad (2.12)$$

Surfaces of constant pressure are now surfaces of constant geopotential Φ. This balance is again referred to as hydrostatic.

It is important to think of the local vertical as being determined by the geopotential and not the gravitational potential (see problem 4). For our purposes it is sufficient to assume that the surface of the earth is a surface of constant Φ (the "geoid") plus perturbations (mountains, etc.) and that this surface is spherical. Since the atmosphere is contained in a thin shell, we can simply set $\Phi = gz$.

The equation of motion in spherical coordinates is

$$\begin{aligned} \frac{du}{dt} &= 2\Omega(\sin\theta)v - 2\Omega(\cos\theta)w + uv(\tan\theta)/r - uw/r - (\rho r \cos\theta)^{-1}\frac{\partial p}{\partial \lambda} \\ \frac{dv}{dt} &= -2\Omega(\sin\theta)u - u^2(\tan\theta)/r - vw/r - (\rho r)^{-1}\frac{\partial p}{\partial \theta} \\ \frac{dw}{dt} &= 2\Omega(\cos\theta)u + (u^2+v^2)/r - g - \rho^{-1}\frac{\partial p}{\partial r}\ , \end{aligned} \qquad (2.13)$$

where $d/dt = \partial/\partial t + (r\cos\theta)^{-1} u \partial/\partial\lambda + r^{-1} v \partial/\partial\theta + w\partial/\partial r$.

2.3 Angular momentum conservation

In the special case that all variables are independent of longitude (a zonally symmetric flow), the zonal equation of motion becomes

$$\frac{\partial u}{\partial t} + r^{-1} v \frac{\partial u}{\partial \theta} + w \frac{\partial u}{\partial r} = 2\Omega(\sin\theta)v + uv(\tan\theta)/r - 2\Omega(\cos\theta)w - uw/r. \qquad (2.14)$$

If we define

$$M = (\Omega r \cos\theta + u) r \cos\theta, \qquad (2.15)$$

then 2.14 can be rewritten in the simpler form

$$\frac{\partial M}{\partial t} + r^{-1} v \frac{\partial M}{\partial \theta} + w \frac{\partial M}{\partial r} = 0 \qquad (2.16)$$

or, equivalently, $dM/dt = 0$. M is the component of the angular momentum of the fluid, per unit mass, parallel to the axis of rotation. Note how the metric terms in 2.14 arise from the difference between the derivatives of u and of $ur\cos\theta$, while the Coriolis terms arise from the advection of the angular momentum of the solid body rotation, $\Omega r^2 \cos^2\theta$.

If a ring of air moves poleward in a zonally symmetric flow, in the absence of any frictional torques, it must conserve its value of M. If it starts at the equator with no relative motion ($u = 0$), when it reaches the latitude θ, the zonal flow will be

$$u = u_m = \Omega a(\sin^2\theta)/\cos\theta \tag{2.17}$$

(here we have set $r = a$, the radius of the earth). The value of Ωa for the earth is $465\ \mathrm{m/s}$. At a latitude of 30 deg, westerlies of strength $135\ \mathrm{m/s}$ with respect to the surface will have been created. If the ring starts with no relative motion at a point off the equator, easterlies will be generated if the ring moves equatorward. One often speaks equivalently of the flow in these rings of air as being turned eastward or westward by the Coriolis force.

2.4. Dry thermodynamics for an ideal gas

If we ignore forcing and dissipation, we can complete our set of equations by assuming that the specific entropy s (the entropy per unit mass) is conserved following a fluid particle: $ds/dt = 0$. This assumes that the evolution of the state of each small parcel of fluid is sufficiently slow, from the perspective of the molecular dynamics that enforce local thermodynamic equilibrium, so as to be reversible. In the presence of a heating rate per unit mass Q, we have instead

$$ds/dt = Q/T, \tag{2.18}$$

where T is the temperature. In the atmosphere, Q includes radiative heating, the release of latent heat associated with the phase changes of water, the effects of molecular diffusion of heat (important adjacent to the surface), and, if we are striving for completeness, the heating due to frictional dissipation of kinetic energy.

We consider here the thermodynamics of dry air. The entropy is related to p and ρ by the *equation of state* of the fluid:

$$s = S(p, \rho). \tag{2.19}$$

Equations 2.18 and 2.19 combine with the statement of conservation of mass and the three components of the equation of motion to form a closed set of 6 equations for the 6 unknown fields $\boldsymbol{V}$, ρ, p and s.

The atmosphere is an ideal gas to an excellent approximation. We write the ideal gas law in the form $p\alpha = RT$, where R is the gas constant, T the absolute temperature, and $\alpha = \rho^{-1}$. (With this notation, $R = nk$, where k is Boltzmann's constant, and n is the number of molecules per unit mass.) For dry air, $R = 287\ \mathrm{J/(kg\ K)}$.

If $z = r - a$ is the radial coordinate, so that $\Phi = gz$ near the surface, then for an atmosphere at rest,

$$\partial p/\partial z = -\rho g = -p/H. \tag{2.20}$$

The second equality holds for an ideal gas, with $H = RT/g$. A typical value for the local-scale height *H* of the lower atmosphere is $6\ \mathrm{km}$. If the atmosphere is isothermal, its pressure and density are exponential functions of height, with the e-folding distance *H*.

The first and second laws of thermodynamics imply that

$$\begin{aligned} T\frac{ds}{dt} = Q &= \frac{de}{dt} + p\frac{d\alpha}{dt} \\ &= \frac{dh}{dt} - \alpha\frac{dp}{dt}, \end{aligned} \tag{2.21}$$

where *e* is the *internal energy* per unit mass. Under adiabatic conditions, $Q = 0$, expansion of a fluid parcel requires work to be performed against the pressure force, and the internal energy of a fluid parcel changes to compensate. The alternative form in 2.21 follows from the definition of the *enthalpy*, $h = e + p\alpha$. For an ideal gas, $h = e + RT$.

The specific heats at constant volume and pressure determine how much the temperature changes for a given input of heat. We can write

$$T\frac{ds}{dt} = \left.\frac{\partial e}{\partial T}\right|_{\alpha}\frac{dT}{dt} + \left(p + \left.\frac{\partial e}{\partial \alpha}\right|_{T}\right)\frac{d\alpha}{dt} \tag{2.22}$$

or

$$T\frac{ds}{dt} = \left.\frac{\partial h}{\partial T}\right|_{p}\frac{dT}{dt} + \left(\left.\frac{\partial h}{\partial p}\right|_{T} - \alpha\right)\frac{dp}{dt}. \tag{2.23}$$

Therefore, the heat capacity at constant volume c_v (equivalently, constant α) and the heat capacity at constant pressure c_p are defined as

$$c_v = \left.\frac{\partial e}{\partial T}\right|_{\alpha} \quad \text{and} \quad c_p = \left.\frac{\partial h}{\partial T}\right|_{p}. \tag{2.24}$$

From the kinetic theory of gases, the internal energy *e* of an ideal gas is dependent only on *T*. In fact, the internal energy is equal to $kT/2$ per molecule, or $RT/2$ per unit mass, for each degree of freedom excited. For point particles with no internal degrees of freedom, only the translational degrees of freedom are available, and $e = 3RT/2$. Our atmosphere is composed primarily of diatomic molecules (N_2 and O_2) whose rotational internal degrees of freedom are fully excited at the temperatures of interest, but whose vibrational degrees of freedom are not excited appreciably. Since these diatomic molecules consist of identical atoms, quantum theory tells us that there are only two rotational degrees of freedom per molecule, so that $e = 5RT/2$. (This approximation turns out to be very accurate for the atmosphere.) It follows that $c_v = 5R/2$ and $c_p = 7R/2 = 10^3\ \mathrm{J/(kg\ K)}$. Also, the internal energy, *e*, is simply c_vT, while the enthalpy, h, is $c_p T$. It is convenient to define

$$\kappa = R/c_p \quad \text{and} \quad \gamma = c_p/c_v = 7/5. \tag{2.25}$$

For an ideal gas under adiabatic conditions, 2.21 reduces to

$$\frac{dT}{dt} = (\kappa T/p)\frac{dp}{dt}. \tag{2.26}$$

The cooling of air as it rises into regions of decreasing pressure, coupled to the well-known fact that colder air can hold less water vapor, is, needless to say, centrally important to our weather.

It is customary to define the potential temperature Θ such that

$$s = c_p \ln(\Theta) \tag{2.27}$$

and to use Θ as a variable instead of s. Under adiabatic conditions, 2.21 can be replaced by $d\Theta/dt = 0$. From 2.23, $c_p d(\ln\Theta)/dt = c_p d(\ln T)/dt - Rd(\ln p)/dt$, an equation satisfied by

$$\Theta = (p_*/p)^{\kappa} T = (p_*/p)^{\kappa} p/(R\rho). \tag{2.28}$$

Here p_* is a reference pressure. The usual choice for p_* is $10^5 \text{ Pa} = 1 \text{ bar}$, a value close to the mean atmospheric pressure at sea level. The notation $\Pi = (p/p_*)^{\kappa}$ is customary, so that $T = \Pi\Theta$. It follows that Θ can be thought of as the temperature that a parcel of air would have if its pressure were adiabatically changed to the value p_*. In the presence of heating Q per unit mass, we can choose between the alternative *thermodynamic equations*:

$$c_p \frac{d\Theta}{dt} = Q/\Pi \tag{2.29}$$

$$c_v \frac{dT}{dt} = -RT\nabla \cdot \boldsymbol{V} + Q \tag{2.30}$$

and

$$\frac{dp}{dt} = -\gamma p \nabla \cdot \boldsymbol{V} + \rho Q(R/c_v) \tag{2.31}$$

2.5. Ocean thermodynamics

The equation of state of sea water is quite complex -- it is most often written as an expression for the density as a function of temperature, salinity (S), and pressure. While it is very often adequate for idealized theories to assume that

$$\rho = \rho_0 - \alpha T + \beta S, \tag{2.32}$$

with α and β constants, this is emphatically not accurate enough for serious modeling or observational studies. The entropy does not have a simple expression in terms of the other state variables either, but one can still define a potential temperature as that temperature obtained by the adiabatic change of the pressure to a reference value (typically 1 bar) or, equivalently, a potential density. See oceanography texts for a thorough discussion -- nothing that we do in this course will be dependent on the details of the equation of state of seawater. For our purposes, in fact, we can think of seawater as incompressible, with no distinction between the heat capacities at constant pressure or constant volume. The "thermodynamics" then reduces to the simple heat equation $c dT/dt = Q$, where Q is the heating rate. We also have an equation for conservation of salt, $dS/dt = R$, where R are the sources and sinks of salt, from which we can then obtain an equation for $d\rho/dt$.

2.6 Energy conservation

Multiplying 2.1 by $\boldsymbol{V}$, we obtain an equation for the rate of change of the kinetic energy density

$$\begin{aligned}\rho\frac{d}{dt}\frac{|\boldsymbol{V}|^2}{2} &= -\boldsymbol{V}\cdot\nabla p - \rho\boldsymbol{V}\cdot\nabla\Phi \\ &= -\nabla\cdot(p\boldsymbol{V}) + p\nabla\cdot\boldsymbol{V} - \rho\boldsymbol{V}\cdot\nabla\Phi\end{aligned} \tag{2.33}$$

To obtain an equation for the conservation of energy, this equation must be considered in conjunction with the changes in potential and internal energy:

$$\rho\frac{d\Phi}{dt} = \rho\boldsymbol{V}\cdot\nabla\Phi \tag{2.34}$$

$$\rho\frac{de}{dt} = -p\rho\frac{d\alpha}{dt} = -p\nabla\cdot\boldsymbol{V} \tag{2.35}$$

Therefore,

$$\rho\frac{d}{dt}\left(\frac{|\boldsymbol{V}|^2}{2} + e + \Phi\right) = -\nabla\cdot(p\boldsymbol{V}), \tag{2.36}$$

or, using 1.7,

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{|\boldsymbol{V}|^2}{2} + e + \Phi\right)\right] = -\nabla\cdot\left[\rho\boldsymbol{V}\left(\frac{|\boldsymbol{V}|^2}{2} + e + p\alpha + \Phi\right)\right]. \tag{2.37}$$

If our domain is bounded by rigid surfaces, the normal component of the velocity must vanish at these surfaces. The application of Gauss's Theorem then demonstrates that the energy density, $\rho(|\boldsymbol{V}|^2/2 + e + \Phi)$, is conserved when integrated over the domain.

The energy flux, $\rho\boldsymbol{V}(|\boldsymbol{V}|^2/2 + e + p\alpha + \Phi)$, contains the term $\rho\boldsymbol{V}p\alpha = p\boldsymbol{V}$ resulting from the energy transfer that occurs when work is done by the fluid against the pressure force. The presence of this term complicates the local energetics of fluids. The quantity $\sigma \equiv e + p\alpha + \Phi \equiv h + \Phi$ is often referred to, in meteorology, as the static energy, a confusing terminology since only the flux of this quantity is relevant to the energetics. The static energy equals $c_pT + \Phi$ for an ideal-gas atmosphere with constant heat capacity.

Problems

2.1 Show that the vorticity of a fluid in solid body rotation with angular velocity Ω is 2Ω.

2.2 Confirm that for a zonally symmetric flow (a flow independent of longitude), conservation of angular momentum (Eq. 2.16) is consistent with the equation of motion in spherical coordinates.

2.3 For an ideal gas in hydrostatic balance, prove that:

a) $\frac{d\sigma}{dz} = c_p \frac{T}{\Theta} \frac{d\Theta}{dz}$, where σ is the dry static energy and Θ is the potential temperature;

b) the specific enthalpy in an atmospheric column, mass integrated from the surface to the top of the atmosphere, is equal to the potential plus internal energy of the column; and

c) the following expressions for the pressure-gradient force are all equivalent (even without hydrostatic balance):

$$-\rho^{-1}\nabla p = -c_p \Theta \nabla \Pi = -c_s^2 (\rho\Theta)^{-1} \nabla(\rho\Theta).$$

2.4 Consider an exactly spherical Earth with radius $a = 6.4\times10^6$ m and an exactly spherically symmetric gravitational field that produces a gravitational acceleration $g = 9.8$ m/s^2 near the surface. Consider an ocean (approximated as an incompressible, homogeneous fluid) whose total mass is such that its depth would be $H = 5$ km if spread uniformly over the planet. Assume that the system is rotating with the angular velocity of the Earth, and that the ocean is at rest in this rotating frame. What is the depth of the ocean as a function of latitude? (Hint #1: the ocean surface will be a surface of constant geopotential. Hint #2: do not solve this exactly -- take advantage of small non-dimensional numbers.)