Capacity of PPM on Gaussian and Webb Channels*

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Abstract — **We compare the capacities of M-ary pulse position modulation (PPM) on Gaussian and Webb channels, which are often used to model optical channels with avalanche photodiode (APD) detectors. Both types of channels exhibit the same brickwall thresholds on minimum signal-to-noise ratio per information bit (bit-SNR) for different values of M.**

Consider a symmetric channel with input signals $\mathbf{x}$ restricted to an M-ary orthogonal constellation (such as PPM) and no restriction on the channel outputs $\mathbf{y}$. The maximum mutual information between $\mathbf{x}$ and $\mathbf{y}$ is achieved with an equiprobable distribution on the inputs, and the channel capacity can be evaluated as

$$C = \log_2 M - E_{\mathbf{v}|\mathbf{x_1}} \log_2 \sum_{j=1}^{M} \frac{p(\mathbf{v}|\mathbf{x}_j)}{p(\mathbf{v}|\mathbf{x}_1)} \qquad (1)$$

where $\mathbf{v}$ is any random vector obtained from $\mathbf{y}$ via an invertible transformation.

For a standard additive white Gaussian noise channel (AWGN-1), the components of the channel output vector $\mathbf{y}$, given one of the orthogonal inputs $\mathbf{x}_j$, are conditionally independent Gaussian random variables, identically distributed except for y_j: y_i is $N(0, \sigma^2)$, $i \neq j$, and y_j is $N(m, \sigma^2)$. The capacity is evaluated from (1), using $v_j \stackrel{\triangle}{=} y_j/\sigma$ and $\rho \stackrel{\triangle}{=} m^2/\sigma^2$:

$$C(\rho) = \log_2 M - E_{\mathbf{v}|\mathbf{x_1}} \log_2 \sum_{j=1}^{M} \exp\left[\sqrt{\rho}(v_j - v_1)\right] \qquad (2)$$

A "double" AWGN channel (AWGN-2) adds greater noise to the orthogonal component in the direction of the signal. The components of the channel output $\mathbf{y}$, given one of the orthogonal inputs $\mathbf{x}_j$, are conditionally independent Gaussian random variables, identically distributed except for y_j: y_i is $N(m_0, \sigma_0^2)$, $i \neq j$, and y_j is $N(m_1, \sigma_1^2)$, with $m_1 > m_0$ and $\sigma_1 > \sigma_0$. The capacity evaluated from (1) is

$$C(\rho, \gamma) = \log_2 M - E_{\mathbf{v}|\mathbf{x_1}} \log_2 \sum_{j=1}^{M} \exp\left[\gamma\sqrt{\rho}(v_j - v_1) + (1-\gamma)(v_j^2 - v_1^2)/2\right] \qquad (3)$$

where the (conditional) statistics of $v_j \stackrel{\triangle}{=} (y_j - m_0)/\sigma_0$, and hence the capacity, depend on two parameters $\rho \stackrel{\triangle}{=} (m_1 - m_0)^2/\sigma_0^2$ and $\gamma \stackrel{\triangle}{=} \sigma_0^2/\sigma_1^2 < 1$, rather than on four parameters $m_0, \sigma_0, m_1, \sigma_1$.

An optical channel with APD detectors can be modeled as a "double" Webb channel (Webb-2), plus additional Gaussian thermal noise [1]. A Webb random variable $W(m, \sigma^2, \delta^2) = m + w\sigma$ is a scaled-and-translated version of a standardized Webb random variable $w \stackrel{\triangle}{=} W(0, 1, \delta^2)$ having probability density $p(w; \delta^2) = \frac{1}{\sqrt{2\pi}}(1 + w/\delta)^{-3/2} e^{-w^2/2(1+w/\delta)}$, $w > -\delta$. For a pure Webb-2 channel, the components of the channel output $\mathbf{y}$, given one of the orthogonal inputs $\mathbf{x}_j$, are conditionally independent Webb random variables, identically distributed except for y_j: y_i is $W(m_0, \sigma_0^2, \delta_0^2)$, $i \neq j$, and y_j is $W(m_1, \sigma_1^2, \delta_1^2)$, with $m_1 > m_0$, $\sigma_1 > \sigma_0$, and $\delta_1 > \delta_0$. The optical APD channel model imposes an additional interrelationship $\gamma = \delta_0^2/\delta_1^2$. The capacity is then evaluated from (1) in terms of $\Delta \stackrel{\triangle}{=} \delta_1^2 - \delta_0^2$ as

$$C(\rho, \gamma, \Delta) = \log_2 M - E_{\mathbf{v}|\mathbf{x_1}} \log_2 \sum_{j=1}^{M} \frac{p\left(\sqrt{\gamma}(v_j - \sqrt{\rho}); \frac{\Delta}{1-\gamma}\right) p(v_1; \frac{\gamma\Delta}{1-\gamma})}{p\left(\sqrt{\gamma}(v_1 - \sqrt{\rho}); \frac{\Delta}{1-\gamma}\right) p(v_j; \frac{\gamma\Delta}{1-\gamma})} \qquad (4)$$

where v_j, ρ, and γ have the same definitions (in terms of the Webb-2 channel variables) as for the AWGN-2 model.

We evaluated the M-dimensional expectations in (2), (3), and (4) accurately via Monte Carlo simulation. Some results are plotted in Fig. 1 for the AWGN-1 and Webb-2 channels for different PPM orders M. The abscissa in this figure is a normalized bit-SNR, $\rho_b \stackrel{\triangle}{=} \rho/(2C)$. Along each Webb-2 curve, the two independent variables held constant are $\Delta = 60.8$ and $\rho\gamma/(1-\gamma) = 17.6$, which correspond to a representative optical APD problem with $\eta n_s = 38$ detected signal photons per PPM word and an excess noise factor $F = 2.16$. The Webb-2 capacity curves for each M exhibit the same brickwall thresholds on minimum ρ_b as the AWGN-1 capacity curves. For different M, these thresholds are offset from each other by a factor $M/(M-1)$, representing the penalty for using orthogonal signals instead of a simplex signal set. In the limit as $M \to \infty$, the minimum ρ_b approaches (for both AWGN-1 and Webb-2) the well-known bit-SNR threshold of -1.59 dB for a standard AWGN channel with no restriction on the channel inputs.

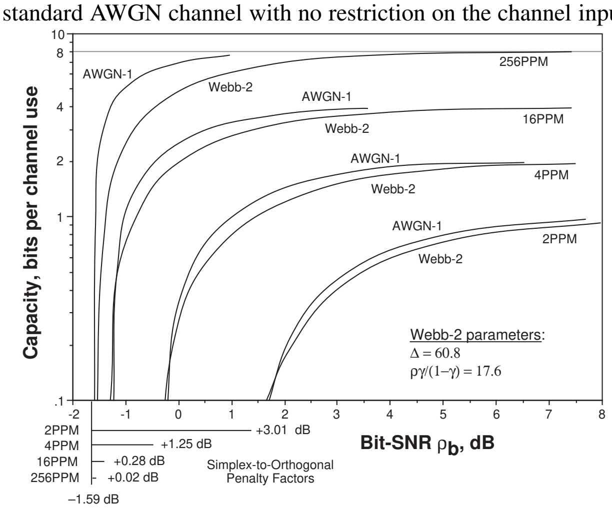

Fig. 1: Capacity of M-ary PPM on AWGN-1 and Webb-2 channels.

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References

[1] P. P. Webb, R. J. McIntyre and J. Conradi, "Properties of Avalanche Photodiodes," RCA Review, vol. 35, June, 1974, pp. 234-278.