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Sum rules, Regge trajectories and relativistic quark models

Siniša Veseli and M.G. Olsson

Department of Physics, University of Wisconsin, Madison, WI 53706

Abstract

We present an analysis which uses Regge structure and the Bjorken and Voloshin HQET sum rules to restrict the choice of parameters of a relativistic quark model.

1 Introduction

It is a well known fact that almost any relativistic quark model which involves a reasonable quark-antiquark potential will adequately reproduce the spin-averaged spectrum of the heavy-heavy and heavy-light states. Almost all successful potentials are based on some variant of a one gluon exchange (Coulomb) part plus a confining part expected from QCD. However, the specific choice of parameters of the model is usually based only on their ability to reproduce data. That and the fact that the parameters of the model are correlated as far as meson spectrum is concerned,[1] is the main reason why one can find nearly as many different sets of parameters for the same model, as there are papers using that particular model.

In this letter we advocate an analysis which uses the linear Regge structure of a given relativistic quark model in the light-light limit [1, 2], together with sum rules of the heavy quark effective theory in the heavy-light limit [3, 4], in order to constrain the parameters of a given potential. For definiteness, we use the simplest and the most widely used generalization of the nonrelativistic Schrödinger equation [5]-[9], the so called spinless Salpeter or the square root equation. It is clear though that the same reasoning can be applied to any relativistic quark model which exhibits linear Regge behavior in the light-light limit.

The aim of this paper is not to say whether a particular model is right or wrong in terms of the type of confinement or the particular potential it uses. Instead, our goal is to show how one can construct a model which is consistent with experiment in terms of its Regge structure in the light-light limit, and at the same time self-consistent with respect to the sum rules in the heavy-light limit.

In Section 2 we give a brief description of the model we are using as an example of our analysis. Section 3 discusses the implications of the linear Regge structure of

[1]By this we mean that changing one parameter in the model inevitably leads to changes in other parameters. For example, it is a well known fact that it is much easier to determine the difference between b and c quark mass from the meson spectrum, than it is to determine either mass.

the model in the light-light limit. Sum rule constraints on the model are presented in Section 4, while our conclusions are summarized in Section 5.

2 Relativistic quark model

As already mentioned, for definiteness we use the relativistic quark model with Hamiltonian given by [5]-[9]

$$H = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2} + V(r) \ , \tag{1}$$

where $p^2 = p_r^2 + l(l+1)/r^2$, $V(r) = V_{conf}(r) + V_{oge}(r)$, and

$$V_{oge}(r) = -\frac{4}{3}\frac{\alpha_s}{r} \ , \tag{2}$$

$$V_{conf}(r) = br + c \ . \tag{3}$$

The necessity of adding a (negative) constant to the usual linear confining potential was shown in [10], from the non-relativistic limit of the Bethe-Salpeter equation. There it was emphasized that c is a parameter which is as fundamental and indispensable as the quark masses, slope of the linear potential b, and the strong coupling constant α_s.[2]

One can find many papers[3] in the literature which use the above model with or without relativistic corrections (e.g. the spin-orbit and color hyperfine interaction) which are completely specified in terms of quark masses and parameters of $V_{conf}(r)$ and $V_{oge}(r)$, with the fixed effective or with some sort of running coupling constant. For the sake of simplicity, we use an effective short range coupling constant, since there is little this analysis can say about $V_{oge}(r)$.

[2]Under additional assumptions an even stronger result $c \simeq -2\sqrt{b}\exp\left(-\gamma_E + 1/2\right)$ was obtained in [10]. We shall, however, consider c as an independent parameter.

[3]References [5]-[9] and [11]-[14] are just a few of them. Probably the most ambitious and the most sophisticated version of the model is due to Godfrey and Isgur [8].

As one can see from (1), (2) and (3), in order to completely specify the model (which should be able to reproduce spin-averaged heavy-heavy and heavy-light meson states) we need seven parameters (we assume that the u and d quarks have the same mass): four constituent quark masses ($m_{u,d}, m_s, m_c$ and m_b), α_s, and two parameters specifying the confining part of the potential, b and c. Again, there is little one can say about quark masses or α_s, since these are expected to run as one goes from the light-light to the heavy-light and heavy-heavy systems. However, we can try to constrain the confining part of the potential so that the model is at least self-consistent in the light-light and the heavy-light limits.

3 Regge trajectories

Let us first consider the effective string tension b. It is a well known experimental fact that all light-quark hadrons lie upon linear Regge trajectories with a universal slope [15]. While it may have been already pointed out in the past [1, 2] we would like to reemphasize here the relation between linear trajectories, linear confinement, and relativistic dynamics. It seems inescapable that massless quarks bound by a linear confinement potential generate a family of parallel linear Regge trajectories, whose slopes depend on the Lorentz nature and other properties of the interaction.

The leading Regge slope follows from the correspondence (classical) limit. The lowest energy state of (1) for a given large angular momentum results for circular orbits at large r and p. The minimal energy condition $\frac{\partial H}{\partial r}\Big|_L \equiv 0$ implies that ($p_r \to 0,\ p \to L/r$)

$$Lp(\frac{1}{E_1} + \frac{1}{E_2}) = br^2 \ , \tag{4}$$

where $E_i = \sqrt{p^2 + m_i^2}$. For a light-light meson $m_{1,2} \to 0$, $E_{1,2} \to p$, and hence (4) gives $2L = br^2$. This, together with $H \to 2L/r + br$, implies that Regge slope is given by [2]

$$\alpha'_{LL} = \frac{L}{H^2} = \frac{1}{8b} \ . \tag{5}$$

Combining this with the observed slope of the leading ρ trajectory [15, 16],

$$\alpha'^{exp}_{LL} = 0.88 \ GeV^{-2} \ , \tag{6}$$

we see that in order for the model to be consistent with experiment in the light-light limit, we have to require

$$b = 0.142 \ GeV^2 \ . \tag{7}$$

This requirement is often overlooked in the literature,[4] and b is usually fixed to be about 0.18 GeV^2 (see Table 1 for a few examples), which is a value suitable for models with the Nambu string slope of $\alpha'_{LL} = 1/(2\pi b)$ (e.g. flux tube models [17]).

A similar result for the effective string tension b can also be obtained from the Regge behavior of the model in the heavy-light limit, since it is expected that the Regge slope for the energy of the light degrees of freedom is twice the slope for the light-light case [2], i.e. $\alpha'_{HL} = 2\alpha'_{LL}$. Indeed, in our case for a heavy-light meson we have $E_2 \to m_2 \to \infty$ and $E_1 \to p$ ($m_1 \to 0$), and hence (4) gives $L = br^2$. Together with $H \to m_2 + L/r + br$, this yields

$$\alpha'_{HL} = \frac{L}{(H - m_2)^2} = \frac{1}{4b} \ . \tag{8}$$

We should also note that the results (6) and (8) depend on the nature of the confining potential used with (1), and that the above arguments cannot be used with models which do not exhibit linear Regge behavior (e.g. non-relativistic quark model).

[4] A most recent example is Ref. [14]. There, (1) was used with several different potentials, and they were all inconsistent with the Regge structure of the model in the light-light limit.

4 HQET sum rules

Let us now consider the heavy-light limit of (1),

$$H \overset{m_2\to\infty}{\longrightarrow} m_2 + \sqrt{m_1^2 + p^2} + V(r) \ . \tag{9}$$

It is clear that in this case the constant c can be reabsorbed into the heavy quark mass by $m_2 \to m_2 - c$, and that the heavy-light spin-averaged meson spectrum by itself does not contain enough information to determine c. However, additional constraints on the value of c can be obtained from the Bjorken [3] and Voloshin [4] sum rules.

In the heavy-light limit the angular momentum of the light degrees of freedom (LDF) decouples from the spin of the heavy quark, and both are separately conserved by the strong interaction. Therefore, the total angular momentum j of the LDF is a good quantum number. For each j there are two degenerate heavy-light states ($J = j \pm \frac{1}{2}$), which can be labeled as J_j^P. Let us denote the $L = 0$ doublet ($0^-_{\frac{1}{2}}$ and $1^-_{\frac{1}{2}}$ states) by (C, C^*), and by (E, E^*) and (F, F^*) the two $L = 1$ doublets (($0^+_{\frac{1}{2}}$, $1^+_{\frac{1}{2}}$) and ($1^+_{\frac{3}{2}}$, $2^+_{\frac{3}{2}}$), respectively). Correspondingly, the unknown form factors for the semileptonic decays of the B (or B_s) meson are ξ_C (for $C \to (C, C^*)$), ξ_E (for $C \to (E, E^*)$), and ξ_F (for $C \to (F, F^*)$ transitions). They are defined within the covariant trace formalism [18], and are functions of the four-velocity transfer $\omega = v \cdot v'$, where v and v' denote four-velocities of the initial and final heavy-light state. In terms of these form factors the Bjorken sum rule [3] is given by

$$-\xi_C'(1) = \frac{1}{4} + \frac{1}{4}\sum_i \left|\xi_E^{(i)}(1)\right|^2 + \frac{2}{3}\sum_j \left|\xi_F^{(j)}(1)\right|^2 \ . \tag{10}$$

The sums here are understood in a generalized sense as sums over discrete states and integrals over continuum states. Similarly, the Voloshin sum rule [4] can be written as

$$\frac{1}{2} = \frac{1}{4}\sum_i (\frac{E_E^{(i)}}{E_C} - 1) \left|\xi_E^{(i)}(1)\right|^2 + \frac{2}{3}\sum_j (\frac{E_F^{(j)}}{E_C} - 1) \left|\xi_F^{(j)}(1)\right|^2 \equiv \Delta \ . \tag{11}$$

In the above expressions $E_E^{(i)}$ and $E_F^{(j)}$ denote energies of the LDF in the i-th excited state with quantum numbers of the (E, E^*) doublet, and j-th excited state with quantum numbers of the (F, F^*) doublet, respectively. E_C is the LDF energy in the lowest (C, C^*) doublet (corresponding to (D, D^*) or (D_s, D_s^*) mesons). The energy of the LDF in any heavy-light state is defined as the state mass minus the heavy quark mass.

Since in our model we cannot say anything about continuum contributions to these two sum rules, and since all contributions in the sums of (10) and (11) are positive definite, one can argue that resonant contributions to the right-hand sides of (10) and (11) should be smaller than direct calculation of $-\xi_C'(1)$ and $1/2$, respectively, if a model is to be consistent with these two sum rules. This will in fact be the key argument of the sum rule part of the model analysis.

In the valence quark approximation one can find precise definitions of the form factors which appear in expressions (10) and (11) (consistent with the covariant trace formalism [18]), in terms of the wave functions and energies of the LDF [19]. We reproduce below only the expressions for $\xi_C'(1)$, $\xi_E(1)$ and $\xi_F(1)$, which are necessary for the sum rule analysis of a model:

$$
\begin{aligned}
\xi_C'(1) &= -\frac{1}{2} - \frac{1}{3}E_C^2\langle r^2\rangle_{00} \ , &(12)\\
\xi_E(1) &= \frac{1}{3}(E_C + E_E)\langle r\rangle_{10} \ , &(13)\\
\xi_F(1) &= \frac{1}{2\sqrt{3}}(E_C + E_F)\langle r\rangle_{10} \ . &(14)
\end{aligned}
$$

Since the model we are considering is spinless, the E and F doublets are degenerate and the expectation values in (12)-(14) are defined as

$$
\langle F(r)\rangle_{L'L}^{\alpha'\alpha} = \int r^2 dr R^*_{\alpha' L'}(r) R_{\alpha L}(r) F(r) \ , \tag{15}
$$

where $R_{\alpha L}(r)$ are radial wave functions of a heavy-light system. Also, (13) and (14) in the spinless case imply that

$$\xi_E(1) = \frac{2}{\sqrt{3}}\xi_F(1) \ . \tag{16}$$

Using (13) and (14), one can simplify the expressions for both sum rules. The Bjorken sum rule (10) becomes

$$-\xi_C'(1) \geq \frac{1}{4} + \frac{1}{12}\sum_i \left[(E_C + E_{E,F}^{(i)})\langle r\rangle_{10}^{(i)}\right]^2 \ , \tag{17}$$

while the Voloshin sum rule (11) is now given by

$$\frac{1}{2} \geq \frac{1}{12}\sum_i (\frac{E_{E,F}^{(i)}}{E_C} - 1)\left[(E_C + E_{E,F}^{(i)})\langle r\rangle_{10}^{(i)}\right]^2 \equiv \Delta \ . \tag{18}$$

In these two equations the sums are only over resonances, since we are neglecting continuum contributions.

To understand the essence of the sum rule constraint we first consider a slightly oversimplified example. We assume that non-resonant contributions are small, and then the equality sign holds in (17) and (18). We also assume that the sums in these two equations are saturated by the lowest P-wave doublets, then (17) and (18) imply

$$-\xi_C' \simeq \frac{1}{4} + \frac{1}{12}(2E_C + \Delta E)^2\langle r\rangle_{10}^2 \ , \tag{19}$$

$$\frac{1}{2} \simeq \frac{1}{12}\frac{\Delta E}{E_C}(2E_C + \Delta E)^2\langle r\rangle_{10}^2 \ . \tag{20}$$

In the above equations we have written $E_{E,F}^{(1)} = E_C + \Delta E$, where ΔE is just the difference between the spin-averaged masses of the S-wave and the lowest P-wave doublet.[5] From (20) one obtains

$$(2E_C + \Delta E)^2\langle r\rangle_{10}^2 \simeq 6\frac{E_C}{\Delta E} \ . \tag{21}$$

[5]If one assumes that the spin-averaged mass of the lowest (E, E^*) doublet in the D systems (corresponding to D_0 and D_1 mesons), is about 2350 MeV, then this difference is about 440 MeV.

Using the above expression in (19), together with the bound $-\xi_C'(1) \geq 1/2$ from (12), we find

$$E_C \geq \frac{1}{2}\Delta E \ . \tag{22}$$

This implies that energy of the LDF in the lowest S-wave mesons cannot be smaller than one half of the difference between the spin-averaged masses of the lowest P-wave and the S-wave. In particular, this would also set a lower bound on the constant c, which is usually assumed to be negative, or an upper bound on the constituent heavy quark masses, since the energy of the LDF is defined as the state mass minus the heavy quark mass.

We now go back to the model analysis of (17) and (18). As already mentioned, for the description of heavy-light mesons we need seven parameters. We have already determined the effective string tension b from the consistency of the model with the experimental Regge slopes of light-light mesons. In order to determine all other parameters except c, we can use the observed heavy-light spectrum. Any change in c effectively just changes the heavy constituent quark masses by the same amount, and an equivalent description of the spectrum is obtained. It is evident that the heavy-light wave functions are not affected by this change. However, changing c does affect the energies of the LDF in the heavy-light mesons. This in turn affects the form factors predicted by the model. The idea is that by examining the model predictions for the right-hand sides of (17) and (18), one can determine physically acceptable values of c, and other parameters of the model. As explained earlier, the main requirement here is self-consistency of the model in the sense that its predictions for the right-hand sides of (17) and (18) are smaller than its direct calculation of $-\xi_C'(1)$, and 1/2, respectively. However, in order to account for all possible uncertainties in our calculations (e.g. effects of spin-averaging of P-wave states, assumption of the unknown P-wave masses, etc.), we relax the sum rule constraints by 5%. This means that the sum rule calculation of Δ should yield result smaller than 0.525 (instead of 0.5), and that the sum rule calculation of $-\xi_C'(1)$ should yield a result which is at most 5% larger than the result obtained from the direct calculation. In this way a more conservative bounds on c will be obtained.

In order to illustrate the above ideas, we consider two sets of parameters:

- Set 1: we fix $c = 0$ and $m_{u,d} = 350\ MeV$, and a fit to the spin-averaged heavy-light meson spectrum then yields $m_s = 542\ MeV$, $m_c = 1366\ MeV$, $m_b = 4703\ MeV$, and $\alpha_s = 0.390$.

- Set 2: we fix $c = 0$ and $m_{u,d} = 300\ MeV$, and from the fit to the heavy-light data we then obtain $m_s = 503\ MeV$, $m_c = 1390\ MeV$, $m_b = 4726\ MeV$, and $\alpha_s = 0.390$.

For both of these two sets we used $b = 0.142\ GeV^2$ from (7), and both of them yield an excellent description of the known heavy-light (spin-averaged) meson masses, with errors less than 5 MeV.[6] Using the parameters which reproduce spin-averaged data, and also an effective string tension consistent with experiment, gives us confidence that the unknown spin-averaged meson masses for radial excitations are reproduced reasonably well.

For both of these two sets, and for c ranging from 0 to $-600\ MeV$, we have evaluated $-\xi'_C$ directly using (12). Using 5 lowest P-waves we also evaluated the right-hand sides of (17) and (18). The results of our calculations (for $B \to D, D^*$ semileptonic decays) are shown in Figures 1 (for the Bjorken sum rule) and 2 (for the Voloshin sum rule). As one can see in Figure 1, both sets of parameters can satisfy the Bjorken sum rule in the sense that the direct calculation of $-\xi'_C(1)$ yields a larger result than the sum rule approach. Imposing the weaker requirement, i.e. that the sum rule result is at most 5% larger than the direct result, for set 1 we

[6] We have assumed that D_0 ($0^+_{\frac{1}{2}}$ state) and D_1 ($1^+_{\frac{1}{2}}$ state) have spin-averaged mass of 2350 MeV, which together with the known P waves $D_1(2425)$ and $D_2^*(2459)$ leads to the spin-averaged mass of 2414 MeV for the lowest P-wave in the D systems. Heavy quark symmetry arguments then imply a spin-averaged mass of 2523 MeV for the corresponding lowest P-wave in the D_s systems.

find $c \leq -180\ MeV$, while for set 2 we get $c \leq -185\ MeV$. We also note that for these values of c, $-\xi_C'(1) \simeq 0.71$ (for set 1) and $-\xi_C'(1) \simeq 0.70$ (for set 2). For the Voloshin sum rule (Figure 2) we find that parameter set 1 can satisfy the stronger constraint ($\Delta \leq 0.5$) for some values of c, which is not the case for parameter set 2. However, imposing the 5% relaxed bound ($\Delta \leq 0.525$), we find that model is self-consistent with

$$200\ MeV \leq -c \leq 470\ MeV\ , \tag{23}$$

for parameter set 1, and with

$$265\ MeV \leq -c \leq 410\ MeV\ , \tag{24}$$

for parameter set 2. Therefore, in the two cases considered, we found the Voloshin sum rule to be more restrictive than the Bjorken sum rule. It is also interesting to observe that the minimum of the function $\Delta(c)$ always occurs close to the point at which Bjorken sum rule approach yields $-\xi_C'(1) = 1/2$. That can be shown analytically from (19) and (20), and can be seen from Figures 1 and 2.[7] For the constituent heavy quark masses values of c given in (23) imply $1566\ MeV \leq m_c \leq 1836\ MeV$, and $4903\ MeV \leq m_b \leq 5173\ MeV$ (set 1), while the ones given in (24) yield $1655\ MeV \leq m_c \leq 1800\ MeV$, and $4991\ MeV \leq m_b \leq 5136\ MeV$ (set 2). Also, the range of energy of the LDF in the S-wave (C, C^*) heavy-light meson corresponding to (23) is $140\ MeV \leq E_C \leq 410\ MeV$ (set 1), while the one corresponding to (24) is $175\ MeV \leq E_C \leq 320\ MeV$ (set 2). These values for E_C are to be compared with the ones obtained in [20], where it was found $266\ MeV \leq E_C \leq 346\ MeV$. This result followed from the analysis of the recent data on semileptonic B decays [21] using the $1S$ lattice QCD heavy-light wave function [22]. We have also repeated the same calculation for the $B_s \to D_s, D_s^*$ semileptonic decays with the same sets of parameters (the difference is basically

[7] This is not necessarily the case with models based on the Dirac (or Salpeter) equation.

only the light quark mass), and found that these decays are much less restrictive than corresponding $B \to D, D^*$ decays.

The most serious concern which one might have about our sum rule analysis of a model, is the issue of degeneracy of the two P-wave doublets, which is due to the spinless nature of the particular model we considered in this paper.[8] While it is certainly true that the uncertainties introduced in this way should partly cancel out (due to spin-averaging, contributions of some states to sum rules will be overestimated, and for other states they will be underestimated), we have still tried to account for possible theoretical errors by relaxing the sum rule constraints by 5%. Also note that including more radially excited states would yield more strict bounds on the acceptable values of c, than are the ones we quote. Given all that, we believe that (23) and (24) represent reasonably conservative estimates for the two different parameter sets of the model considered in this paper. This conclusion is also supported by the comparison of the values for E_C obtained here with the ones obtained in [20].

5 Conclusion

In the construction of potential models of mesons a confining interaction of the form $V_{conf}(r) = br + c$ plays an important role. The additive constant c in a number of cases just renormalizes quark masses, but has very little effect on meson wave function. On the other hand, form factors for the semileptonic decays of B (or B_s) meson in the heavy quark limit depend sensitively on this constant through the energies of the light degrees of freedom. The result is a wide variety of form factor predictions from a class of very similar models.

[8]For models based on the Dirac or Salpeter equation this issue vanishes, since these models distinguish between (E, E^*) and (F, F^*) doublets. Because of that, sum rule analysis of these models should be more reliable than the one appropriate for the spinless models.

Using the simplest and the most widely used generalization of the nonrelativistic Schrödinger equation [5]-[9], we have presented an analysis, based on the linear Regge structure of the model in the light-light limit, and its predictions for the two HQET sum rules in the heavy-light limit, which can constrain parameters of the confining part of the potential. Even though we have used a particular relativistic quark model as an example, it is clear that a similar analysis could be repeated for any model with linear Regge behavior.

The aim of this analysis is not to show whether a particular model is correct in terms of a given equation, a type of confinement, or a specific potential it uses. We have instead outlined a method to restrict the confining parameters of the model in order to make it consistent with experiment in the light-light limit (Regge structure), and also self-consistent with respect to HQET sum rules in the heavy-light limit. Once these parameters are fixed, other parameters (e.g. constituent quark masses and short range coupling constant) can be determined from the requirement that model should reproduce various experimental data (e.g. meson masses).

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References

[1] J. S. Kang and H. Schnitzer, Phys. Rev. D **12**, 841 (1975); J. Dias deDeus and J. Pulido, Z. Phys. C **28**, 413 (1985); P. Cea, P. Colangelo, G. Nardulli, G. Paiano, and G. Preparata, Phys. Rev. D **26**, 1157 (1982); P. Cea, G. Nardulli, and G. Paiano, Phys. Rev. D **28**, 2291 (1983); A. Martin, Z. Phys. C **32**, 359 (1986).

[2] C. Goebel, D. LaCourse, and M. G. Olsson, Phys. Rev. D **41**, 2917 (1990).

[3] J. D. Bjorken, *New Symmetries in Heavy Flavor Physics*, in *Results and Perspectives in Particle Physics*, Proceedings of the 4$^{\text{th}}$ Recontres de Physique de la Vallée D'Aoste, La Thuile, Italy, 1990, edited by A. Greco, Editions Frontières, Gif-sur-Yvette, France, 1990; N. Isgur and M. B. Wise, Phys. Rev. D **43**, 819 (1991).

[4] M. B. Voloshin, Phys. Rev. D **46**, 3062 (1992).

[5] B. Durand and L. Durand, Phys. Rev. D **25**, 2312 (1982).

[6] D. B. Lichtenberg, W. Namgung, E. Predazzi, and J. G. Wills, Phys. Rev. Lett. **48**, 1653 (1982).

[7] B. Durand and L. Durand, Phys. Rev. D **30**, 1904 (1984).

[8] S. Godfrey and N. Isgur, Phys. Rev. D **32**, 189 (1985).

[9] S. Jacobs, M. G. Olsson, and C. J. Suchyta III, Phys. Rev. D **33**, 3338 (1986).

[10] D. Gromes, Z. Phys. C **11**, 147 (1981).

[11] W. Lucha, H. Rupprecht, and F. Schöberl, Phys. Rev. D **46**, 1088 (1992).

[12] L. P. Fulcher, Z. Chen, and K. C. Yeong, Phys. Rev. D **47**, 4122 (1993).

[13] L. P. Fulcher, *Matrix Approach to Solution of the Spinless Salpeter Equation*, in Proceedings of the International Conference on *Quark Confinement and the Hadron Spectrum*, Como, Italy, 20-24 June 1994, edited by N. Brambilla and G. M. Prosperi, World Scientific, 1994.

[14] D. S. Hwang and G.-H. Kim, Phys. Rev. D **53**, 3659 (1996).

[15] V. D. Barger and D. B. Cline, *Phenomenological Theories of High Energy Scattering*, W. A. Benjamin Inc., New York, 1969.

[16] L. Montanet et al., Particle Data Group, Phys. Rev. D **50**, 1173 (1994).

[17] D. LaCourse and M. G. Olsson, Phys. Rev. D **39**, 2751 (1989); M. G. Olsson and S. Veseli, Phys. Rev. D **51**, 3578 (1995).

[18] H. Georgi, Nucl. Phys. B **348**, 293 (1991); J. G. Körner and G. A. Schuler, Z. Phys. C **38**, 511 (1988); A. F. Falk, Nucl. Phys. B **378**, 79 (1992).

[19] S. Veseli and M. G. Olsson, Phys. Lett. B **367**, 302 (1996).

[20] M. G. Olsson and S. Veseli, Phys. Lett. B **353**, 96 (1995).

[21] B. Barish et al., CLEO Collaboration, Phys. Rev. D **51**, 1014 (1995).

[22] A. Duncan, E. Eichten, and H. Thacker, Phys. Lett. B **303**, 109 (1993).

TABLE

Table 1: Parameters of the confining part of the potential used in several papers which employed the relativistic generalization of the Schrödinger equation discussed in the text, with the same type of the confining potential (3). The one gluon exchange potentials used in these papers are not necessarily the same as (2). Reference [12] used two different values for c for description of D and B mesons. To obtain the universal Regge slope model requires $b = 0.142\ GeV^2$, while the range of the c values for which model is consistent with HQET sum rules depends on other assumptions and parameters of the model.

Author(s)	Reference	$b\ [GeV^2]$	$-c\ [MeV]$
Durand and Durand (1984)	[7]	0.180	0
Godfrey and Isgur (1985)	[8]	0.180	253
Jacobs et. al. (1986)	[9]	0.192	0
Lucha et. al. (1992)	[11]	0.211	850
Fulcher et. al. (1993)	[12]	0.191	246 (214)
Fulcher (1994)	[13]	0.219	175
Hwang and Kim (1996)	[14]	0.183	0

FIGURES

Figure 1: Comparison of the direct calculation of $-\xi'_C(1)$ (full lines) with the Bjorken sum rule result obtained with 5 lowest P-wave states (dashed lines). 1 and 2 denote two different sets of parameters, as explained in the text. The dotted line is the bound $-\xi'_C(1) \geq \frac{1}{2}$ coming from (12). The results shown are for $B \to D, D^*$ semileptonic decays.

Figure 2: Voloshin sum rule calculation of Δ with 5 lowest P-wave states (full lines), for the two different sets of parameters 1 and 2. The expected upper bound of 0.5 and the 5% relaxed bound of 0.525 are shown with dotted and dashed line, respectively. The results shown are for $B \to D, D^*$ semileptonic decays.

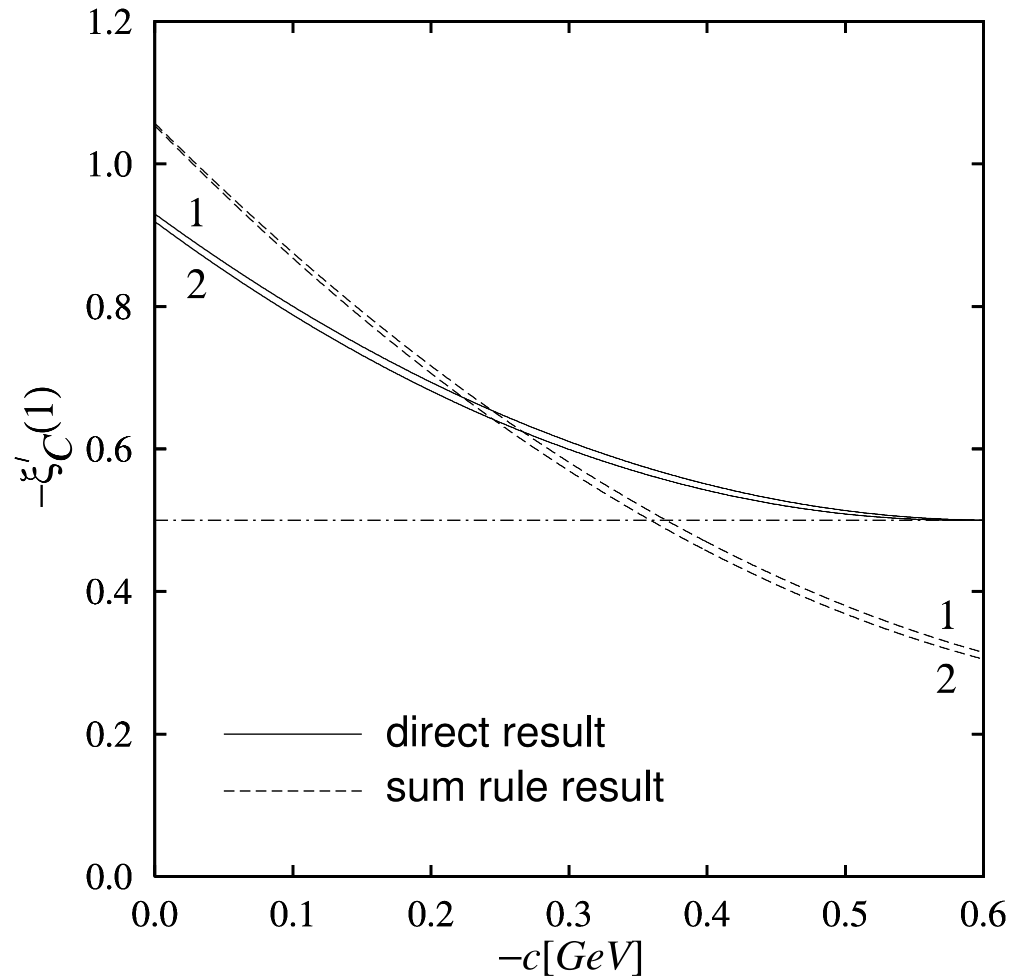
1.2
1.0
0.8
0.6
0.4
0.2
0.0
$-\xi'_C(1)$
1
2
1
2
direct result
sum rule result
0.0
0.1
0.2
0.3
0.4
0.5
0.6
$-c[GeV]$

Figure 1

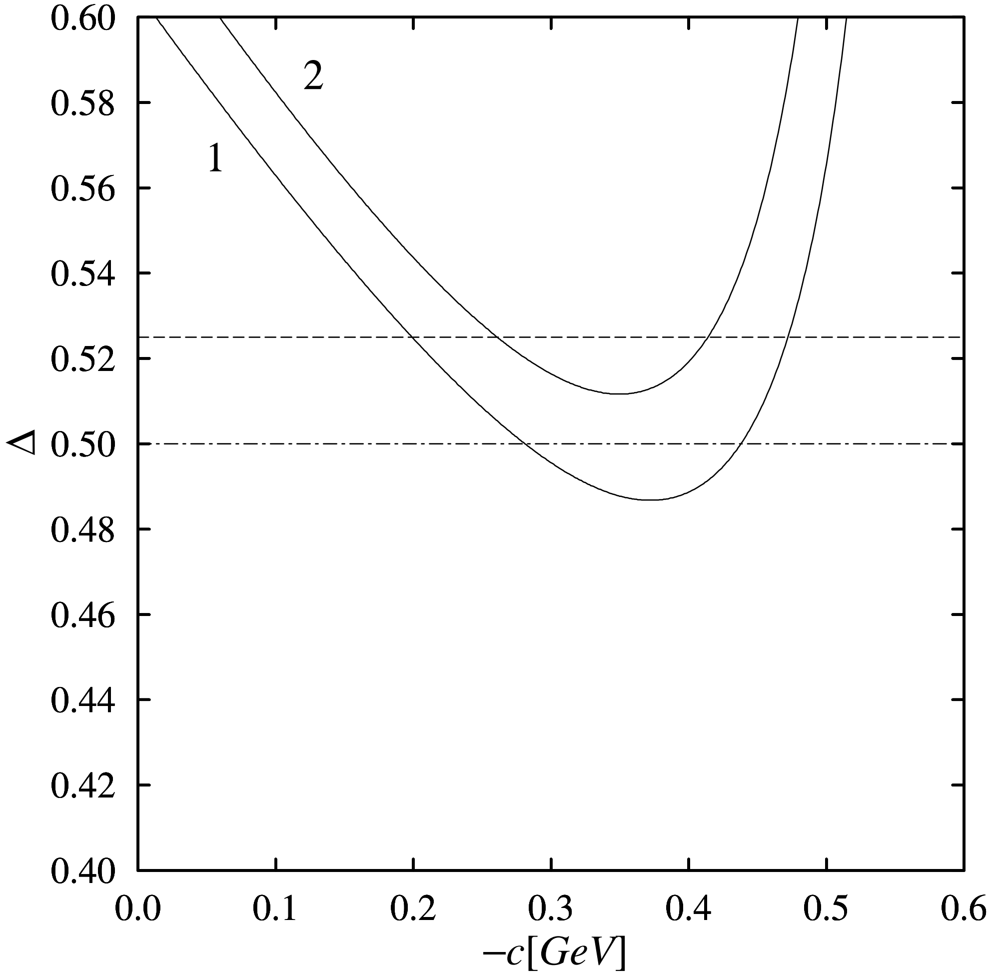
0.60
0.58
0.56
0.54
0.52
0.50
0.48
0.46
0.44
0.42
0.40
Δ
0.0
0.1
0.2
0.3
0.4
0.5
0.6
−c[GeV]
1
2

Figure 2