THE GENERALIZATION OF MATHEMATICAL DESCRIPTION[0]

Kevin R. Vixie
Departments of System Science and Mathematics
Portland State University, P.O. Box 751
Portland , Oregon , 97207-0751
vixiek@sysc.pdx.edu

Abstract

Mathematics is treated as a source of descriptive archetypes. The unavoidable incompleteness of descriptions of reality is suggested to be inherently intertwined with the existence of fundamental objects which define the implicit boundaries in the within/between distinctions. Local and global descriptions are complimentary ... the inescapable incompleteness of each is explored as a general problem in knowing. Descriptions are susceptible to different degrees of precision suggesting generalized series expansions as a pervasive archetype.

Keywords: Epistemology, series expansion, global and local, manifolds, mathematics.

1 Motivation

At least three mathematical [1] concepts hover beneath ... above ... behind ... the ideas presented in this paper.

One concept is that of a state space and the flow in that space that results from a given vector field ... i.e. a dynamical system. The state space can be thought of as the set of all possible configurations and the flow can be thought of as lawful connections between configurations at different times. We can actually think of connection as transformational law.

The next concept is that of local and orthogonal approximation of functions ... of asymptotic expansions that unfold a particular function at a point of its domain and orthogonal function expansions revealing the spectral components of a function at all points of the domain.

The last concept is that of local versus global illustrated, for example, by the difference between Fourier Series and Taylor Series or by the differences and similarities between the 2-sphere and the Euclidean 2-space.

[0] This article is a corrected version of the article that appeared under the same name in a 1997, special issue of "Advances in Systems Science and Applications". This is the official journal of the International Institute for General Systems Studies and the special issue contained the proceedings of the Second Workshop of the IIGSS held in San Marcos, Texas, January 9-11, 1997

[1] I would suggest that a natural definition of mathematics is "the careful study and collection of patterns and connections".

2 Fundamental Notes

Reality as an infinite whole comprehended in increasing accuracy by consecutive feature abstractions.

Understanding is a process of successive abstraction of features from reality and the study of those features. The requisite observation is made possible by a connection with that reality by means of acceptance. At some level, the acceptance becomes very simple ... that of opening to receive and confidence in the reality of the gift. We know by faith.

To think about is to consider facets (features). To see, hear, touch, taste, smell, intuit ... is to receive facets. Reason enables connections to be seen but cannot give us these facets. Reason positions the mind, enabling further enlightenment. Reason enables the mind to see patterns in more precise detail. Reason is, as it were, a magnifying glass with which to study intuitive realizations.

We shall call the sum of these facets, configuration ... i.e. the observed patterns of shapes and pictures and sounds, etc. As configuration is seen, heard, experienced ... patterns of relation begin to emerge. Configuration leads to configuration and an underlying simplicity ... a Law of connection ... emerges. This simplicity, the elementary particle of the connection, is an invariant throughout the connection and can be seen as the generator of the connection. Connection is therefore the action of law on configuration. In fact, we may think of connection as the embodiment of law ... different configurations giving rise to different paths in the space of configurations ... each the embodiment of the law given the particular configuration. To understand is therefore to grasp both configuration and transformational law [2].

The seemingly inescapable incompleteness at any level of configuration description together with the fact that knowledge of further facets of configuration increases completeness, suggests the series expansion as a fundamental archetype of configurational description[3]. Because of their importance to the description of configuration and the fact that law is connection within and between configuration, we must conclude that series expansions are indeed a basic achetype in the entire process of understanding.

The fundamental element as a pervasive archetype of composition and therefore of description.

Classification of elementary patterns and elementary particles of a particular descriptive type is both a prerequisite to, and a product of, the process of understanding. Elementary particles as a basic mode of composition is universal. Chemical elements, letters of the alphabet, musical tones, 20 amino acids, 4 basic units on a DNA... The finite generating the infinite[4]. Examination of these examples will reveal a cascade of increasing variety.

Another example is the construction of arbitrary functions from sets of simple functions ... polynomials, sine's and cosines, Bessel functions, etc. In this case we have countable infinity generating uncountable infinity[5]. But actually the countable infinity can be finitely characterized if one includes one infinite set, the integers. The description would be comprised of one sine wave and the process of scaling by the different integers (changing the frequency).

Incompleteness of descriptions and the emergence of fundamental scales.

Configurational descriptions of a particular type are often incomplete in their descriptions of configuration i.e. there is some scale below which the description breaks down. One common instance are those series expansion descriptions that make an assumption of endless divisibility ... of continuum. These descriptions typically work well down to the point at which the elementary particles of the object are encountered. Therefore the point of optimal truncation of a particular description may in fact be the scale at which the object becomes atomistic. Conversely, one can define the scale at which a particular typed expansion breaks down to be the a fundamental scale of the object (w.r.t. the descriptor type). At this level a fundamental particle of the object is resolvable and is a level at which a process or configuration becomes atomistic[6].

Descriptive incompleteness implies nontrivial description, while completeness is namelike.

Fundamental particles of descriptors that correspond with fundamental particle of objects are namelike. Descriptors which are not namelike are often incomplete, but it is the very characteristic of not being namelike, that leads to its ability to unfold the object it is describing[7]. If the descriptor is namelike, it does not unfold the object (because it leads to a description that is more a labeling than an unfolding).

Local description, global description and the inadequacy of hierarchy as a descriptive archetype.

Feature abstractions often focus on the local structure and therefore our ultimate goal ... global understanding ... must concern itself with how the pieces fit together to make the big picture. Local and global ... local vs. global ... therefore raises many issues that are intimately involved with any attempt at description and understanding[8]. Global is composed of the local and gluing of the local. This gluing is characterized by not only samescale gluing of the local but also by transcale connections that add further gluing maps to constrain the global structure. Global structure, in it's difference from the extension of local structure, is discretely characterizable. Global structure, if attempted to be made local, produces contradictions and nonsense. Global structure alone, is compatible with many varied local complexions and therefore imparts a sense of indeterminacy to the understanding. Hierarchy is characterized by a specific treelike global structure, and therefore is unable to capture other global structures obtainable through other schemes of connection. Consequently hierarchy must be generalized to a more general local-global schema. A universal approach to configuration must involve generalized multiple scale local-global schema.

3 Generalized Series Expansions

In the study of functions from one abstract space to another, a very fruitful description is that provided by the series expansion. There are many types, but the idea behind them all is that of

[2] e.g. ... boundary conditions and Differential Equations, Initial Conditions and Flow, Shape and fundamental law

[3] The observation of multiple scales of configuration also suggests series expansions

[4] Here is variety arising from constraint ... the elementary few bringing a harmony of sorts to the variety.

[5] In the case of a bounded domain.

[6] The emergence of a fundamental scale due to the unavoidable incompleteness of the description can be viewed as a generic property of typed descriptions.

[7] Not namelike implies structurelike and is an effective descriptor because it captures structure

[8] Local and global scales both exhibit fundamental problems with comprehensive completeness. Godel's theorem essentially says that becoming complete on a global scale forces inconsistency on a local scale and it is the global completeness which forces the inconsistency on us.(Analogy: the "hairy ball" theorem)

successively sharpened approximations of the function by a complete set of simple functions[9]. Taylor series expansions, Fourier expansions, Bessel function expansions, and Laurant serires expansions are all well known examples.

3.1 The Generalization

An alternative to the metric based series is one based on nested sequences of subsets of some typed set of descriptives. A description, consisting of selections from this ordered sequence of sets, would provide a sequence of descriptions whose accuracy increases as the point of truncation is moved further out on the sequence.

Suppose that some descriptive type α has a set of descriptors $\aleph$ and that the sequence of subsets of $\aleph$, name them A_n for n = 0,1,2,3 ... satisfy

$$A_0 \subset A_1 \subset A_2 \subset A_3 \subset \ldots$$

and

$$\bigcup_n A_n = \aleph$$

The decomposition of $\aleph$ into a nested sequence, orders the elements of $\aleph$ in terms of decreasing importance to description. For example, elements that are in the 5th subset make contributions to the description that would typically be larger or more important in some sense than any element of the descriptive sequence that is in the 6th and not in the 5th subset.

Some examples are

1. $\aleph$ = { set of all polynomials } , A_n = { set of all nth order polynomials}

2. $\aleph$ = { set of all 2-D patterns with finite resolution } , A_n = { set of all 2-D patterns with minimal resolution of $1/2^n$ }[10]

3. $\aleph$ = { set of all emotional descriptives} , A_n = {set of nth order emotional states $\equiv$ states which would support all mth order states, m>n}

e.g.

A_1 = { happy, sad }

$A_2 \setminus A_1$ = { euphoric, bubbly, excited, depressed, lonely, nostalgic, ... }

$A_3 \setminus A_2$ = { etc. }

4. $\aleph$ = {the set of all finite Fourier expansions on the unit interval [0,1]}, A_n = { set of Fourier expansions including terms with frequency = 0,1,2, ... n-1,n }

... in this case the set of descriptors can be used to perfectly describe the set of all continuous functions from the closed unit interval - [0,1], to the real numbers, $\Re$

A key point is that this generalization covers both the orthogonal expansion idea and the asymptotic expansion idea. Note also that the descriptions actually work to describe something like the closure of the span of $\aleph$. For example, in the case of $\aleph = \{$ set of all 2-D patterns with finite resolution $\}$, $\aleph$ can be used to descibe patterns that are continous even though they do not belong to $\aleph$.

3.2 When to Truncate: Emergence of Fundamental Scales

The inherent incompleteness of many descriptors[11], leads to a point at which the descriptor is no longer able to describe. The fundamental scale of the object ... with respect to the descriptor ... has been reached. One descriptor defines configuration down to islands of uncertainty. These islands, unfolded under the influence of other typed descriptors, are the fundamental pieces of reality that form the distinctions which yield the within and between (or without)[12]. The islands, (fundamental particles of reality) which never yield to a complete unfolding of themselves, serve as endpoints of typed descriptions of the without or beginning points of descriptions unfolding the within.

4 LOCAL VERSUS GLOBAL: extension, scale, and generalized manifolds

4.1 Preface : Scale and transcale effects ... and implications on global structure.

The fact that reality "happens" at many scales of time and space, has implications for understanding. First, it makes it possible to decompose systems into subsystems, sub-subsystems, etc. and supersystems, super-supersystems, etc. Different length-time[13] scales are nearly decoupled by the periodic structure of natural phenomenon. This tends to isolate a time length scale from level above or below it. Energy and information from the levels above are ignored because they are being "ridden" (i.e. they are seen as a slowly varying bias which is ignored), while those from the levels below are averaged out. The idea of reality as a hierarchy is therefore canonical ... i.e. natural.

But hierarchy is not a complete descriptive modality. This is due to transcale effects ... flow of matter, energy, and information across boundaries of scale or to be more precise, from one length-time scale to another.

4.2 Extension and Global Structure

Localization in length-time scale (as well as in space-time) occurs naturally in the study of real objects. The local structure that is observed is almost certainly not a complete description of the global object, i.e. the naive extension of the local structure to a global one almost always yields the wrong global structure[14].

---

[9] complete in that the closure of their span equals some functional space ... e.g. the set of all continuous functions on [0,1]

[10] note that $\aleph$ is not simply the set of all 2-D patterns since this would not equal the union of the A_n's.

[11] Notice that this is not referring to unfoldings of descriptors by descriptors (we will say primary by secondary)... these often lead to convergent series, with no need of truncation. But the primary descriptor to which that secondary descriptor converges will be incomplete in it's description of real objects. Therefore the secondary description can inherit an optimal truncation level by its connection to the primary descriptive process (that is, w.r.t. the real object being described).

[12] note that the between can be defined as the relations that are without the within

4.3 Generalized Manifolds

At this point it is natural to introduce the concept of a generalized manifold. First a brief look at manifolds[15].

An n-dimensional manifold M, is a topological space that is locally Euclidean (and Hausdorff and 2nd countable) ... i.e. each point of M has an open neighborhood that is homeomorphic to the unit ball in R^n. If in addition to this local structure, we can construct the whole manifold by gluing these smaller pieces together smoothly, then it is a differentiable n-dimensional manifold. The gluing maps are derived from the local homeomorphisms and their overlaps, and tell us how the "little" pieces of R^n fit or glue together to yield M. Put a little more intuitively, manifolds are topological spaces that can be seen as locally identical to Euclidean spaces of some fixed dimension, but which are glued together in a manner permitting them to be non-Euclidean. For instance a circle is a 1-dim manifold, a torus is a 2-dim manifold, etc. A key observation is that while there are infinitely many 2-manifolds, they all have identical local structures[16].

Generalizing this to the description of natural objects, the typed descriptors originating from some archetype (the "flat space") are used to describe the object in a way that can be thought of as local (in scale or position or functional description). The description becomes more complete as these descriptions reveal connections due to the structure and dynamics of the natural object. These connections ... the "gluing maps" ... constrain the description to yield global structure that is qualitatively different than any of the naive extensions of any of the local descriptions. The descriptive structure begins to encompass the infinite complexity the characterizes all natural objects.

Because of the trans-scale coupling, some same-scale components are connected only through the trans-scale couplings. Therefore the structure that is built by looking only at the same-scale components and gluings is not fully constrained. Through the trans-scale couplings the global structure finds further constraint and refinement[17].

[13] Length-time scales are naturally length-time scale rather than time only or length only for the simple reason that dynamics are usually coupled with vibrations of one sort or another ... and vibrations typically have constant or quasi-constant s peeds of propagation (e.g. speed of light, speed of sound) . Hence, a typical length of a natural space-time object would have a corresponding natural time interval = {length scale } / { speed of propagation }

[14] The ability to see the limits of a particular mode of understanding ... to not try to push a descriptor beyond its validity ... to not try to fit reality into one mold as it were, is the difference between an understanding that continues to grow and expand and one that becomes concerned (after the initial insight) with increasingly diminutive bits and minutia. Realizing this promotes robustness of understanding and breadth of viewpoint.

[15] see for instance the first part of chapter 5 of V.I. Arnold's "Ordinary Differential Equations" (1991, Springer-Verlag) for a brief introduction to manifolds or Boothby's " An Introduction to Differentiable Manifolds and Riemannian Geometry" (1975, Academic Press) for a much more complete introduction

[16] The difference between the Global Nature of R^2 and that of S^2 is discrete in nature since it can be completely described in terms of finite complexes. While this delineation doesn't need the Euclidean nature to be unfolded to understand the global nature ... it also doesn't say anything about what it means to be locally Euclidean! This suggests an explanation of the indeterminate feeling that one gets when considering wholistic descriptions ... why wholistic explanations many times seem to nail nothing down (which causes many to consider them flawed descriptions). And yet, like the descriptions that lack local detail there is an essential truth or completenessor verity to the description!

[17] transcale objects would also contribute to the gluing of local pieces together to make a complete global one.