NIST Digital Library of Mathematical Functions

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Abstract. The National Institute of Standards and Technology is preparing a Digital Library of Mathematical Functions (DLMF) to provide useful data about special functions for a wide audience. The initial products will be a published handbook and companion Web site, both scheduled for completion in 2003. More than 50 mathematicians, physicists and computer scientists from around the world are participating in the work. The data to be covered include mathematical formulas, graphs, references, methods of computation, and links to software. Special features of the Web site include 3D interactive graphics and an equation search capability. The information technology tools that are being used are, of necessity, ones that are widely available now, even though better tools are in active development. For example, LaTeX files are being used as the common source for both the handbook and the Web site. This is the technology of choice for presentation of mathematics in print but it is not well suited to equation search, for example, or for input to computer algebra systems. These and other problems, and some partially successful work-arounds, are discussed in this paper and in the companion paper by Miller and Youssef.

Keywords: Mathematics, special functions, handbook, digital library, knowledge management

AMS(MOS) Codes: 33-00, 68T30

1. Introduction

Scientific and engineering professionals who are are conducting research or developing applications frequently need ready access to specific technical data. Traditionally, applicable technical data has been gathered into handbooks, and often these publications become indispensable aids that persist through many editions. Accuracy—freedom from errors—is critical for any handbook to be successful, and also relevance to a significant community of users. Successful mathematics handbooks that come to mind are certain tables of integrals—and the ubiquitous *National Bureau of Standards Handbook of Mathematical Functions* [1].

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The handbook [1] is the motivation and model for a large project, headed by the National Institute of Standards and Technology[1], to develop a *Digital Library of Mathematical Functions (DLMF)*.

The NBS handbook was published in 1964, and except for correction of errata through the 10th printing in 1972, it has never been revised. The DLMF is currently under development in a large project headed by NIST. It will be published as a traditional handbook of about 1000 pages, but there will also be a hypertext version that will be made available on CD-ROM and from a public Web site at NIST.

The NBS handbook and the new DLMF have many features in common, of which two will be mentioned here. First, subject matter. The general goal of each is to provide carefully selected and validated technical data about the elementary and higher mathematical functions[2]. The main criterion for inclusion of data is proven or potential use in disciplines outside mathematics. Information of interest only within pure mathematics is excluded. The majority of chapters deal with individual groups of functions that share a common characteristic such as the property of being a solution of a given differential equation. The remaining chapters are methodological, providing concentrated accounts of mathematical techniques that are indispensable for working with special functions.

The second feature in common is intended audience. The level and presentation of the technical data are directed toward experienced professionals. Thus extended pedagogical descriptions are excluded so that the space saved can be used for additional technical data. It might be thought that space is not an issue for the DLMF hypertext version. However, there is benefit in requiring the authors of the chapters to consider carefully which data are paramount and to include only the most important data. Also, at least for the first edition, the project aims for maximum overlap between the typeset and hypertext versions. Offsetting this requirement to some extent, DLMF authors are allowed considerable freedom to point users toward references where additional data can be found.

Despite these similarities, the hypertext version of the DLMF is fundamentally different from the old NBS handbook because it comes with hyperlinks, interactive graphics, and tools for downloading and searching. An online record of feedback from users is planned, and

[1] In 1988 the National Bureau of Standards (NBS) was renamed the National Institute of Standards and Technology (NIST).

[2] The higher mathematical functions are also known as special functions. Examples are Bessel, Legendre and elliptic functions. The term higher transcendental functions has also been used.

tools for generating tables and graphs on demand may be added in the future. Prototypes for all these capabilities exist now.

The print and hypertext versions of the DLMF issue from the same source, which is a *mathematical database* consisting (at present) of LaTeX computer files. Together with a special LaTeX class and software tools developed at NIST, the database preserves limited mathematical semantics of the objects it contains; see the companion paper [6] by the author's colleagues B. Miller and A. Youssef. Managing the mathematical knowledge in this database so that it will remain accessible and usable into the indefinite future is a serious concern. The data is being collected at great financial cost. Though it will be preserved in conventional form as a published handbook, it would be very unfortunate if the database were lost due to future software and hardware incompatibilities.

2. 1964 NBS Handbook

The handbook [1] quickly became the reference of choice for applications of special functions, and it remains so today. This success was due to (i) its concise style that provided users with concentrated and relevant information, (ii) its orientation toward the large community of users in disciplines, such as the physical sciences and engineering, in which mathematics is an essential tool, and (iii) the care that was taken to engage the best available authors and editors.

It was conceived as early as 1952, active work was begun in 1956, and the technical parts were largely complete by 1960. The delay in publication to 1964 was largely due to the death in 1958 of its chief architect, Milton Abramowitz. It supplies the most important data relating to special functions (formulas, tables for computing by interpolation, and graphs) as determined by the needs of its time.

The success of the NBS handbook is exhibited by its sales and citations. The official government edition has been in print continuously and has sold more than 150,000 copies. Dover Publications has marketed identical low-price printings in paperback since 1965; Dover's sales are estimated to be 4 or 5 times greater than the government sales. A study of *Science Citation Index* shows the handbook was cited over 7000 times between 1992 and 1996, or approximately once every 1.5 hours of every working day. Furthermore, the citation rate was increasing even more rapidly than the *Index* as a whole. Among journals that cited the NBS handbook, journals in chemistry, earth sciences, electrical engineering, optics, physics and statistics ranked higher in frequency of citation than did any primarily mathematical journal, thereby illustrating the broad impact of the NBS handbook.

3. The DLMF Project

3.1. Motivation and Goals

The continued appeal of the old NBS handbook is due to the permanence of its collection of formulas, which take up about half of its 1046 pages. Mathematical formulas that are found to be useful in applications hold their value for a long time, even forever. However, this collection is showing its age in some respects. For example, since 1960

- new properties have been found for many of the functions, such as integral representations, integrals, addition formulas and generating functions;

- additional functions have gained practical importance, such as Carlson's symmetric form of elliptic integrals, discrete orthogonal polynomials, new statistical distribution functions, Painlevé transcendents and basic hypergeometric functions;

- new fields of application of special functions have appeared, such as soliton theory and nonlinear dynamics; and

- developments in asymptotic analysis have led to improved analytical approximation of special functions, for example through use of uniform asymptotics, asymptotics via distribution theory, and reexpansion of remainder terms.

Because of these developments there is an increasingly critical need for a major expansion of the core content of the old handbook.

The other half of the old handbook is irrelevant today. This part is devoted to massive tables of values and descriptions of how to calculate values of special functions, often with the aid of interpolation in the tables. This old method has been superseded by newer, more effective methods that depend upon the speed with which computers can perform arithmetic. In effect, a data-intensive and operation-conserving method has been replaced by a data-conserving and operation-intensive one. Developments since 1960 that have contributed to this obsolescence include

- significant improvements in methods for accurate and efficient computation of special functions that have been obtained through advances in stability analysis, Padé approximation theory, boundary-value techniques for differential and difference equations, and application of the new developments in asymptotics;
- the rapid development of computing technology, both in hardware and in software; and
- the emergence of powerful scientific software, such as subroutine libraries and computer algebra systems, which enable users to generate sophisticated symbolic and numerical solutions for a wide range of problems involving special functions.

Accordingly, a new reference work should essentially eliminate tables and pay attention to software.

The main goal of the DLMF project is to meet the need for an up-to-date replacement for the old handbook. A secondary goal is to develop foundations for constructing mathematical databases in general, together with tools for their convenient and effective use in computers and over computer networks. The combination of a database and associated tools is what we mean by a *digital library*. The experience gained in the construction of a nontrivial digital library in an important area of mathematics will be of great value in demonstrating feasibility, comparing alternative technologies, exposing weaknesses of current technologies, and suggesting avenues for future research. Also, because of the expected long life and wide community of users, the DLMF will advance the penetration of digital library technology into the daily practice of scientific researchers and technical professionals in all science-based fields of work.

3.2. Beginnings

The first discussions took place within NIST in October 1996. These led to a conference presentation [3] and a workshop at NIST with invited external participants in summer 1997; see also [2]. The workshop resulted in the organizational structure for the project and an initial list of chapters with a brief synopsis for each chapter.

A difficult question at this early stage was how such a large project could be financed. NIST was making a large commitment but not enough to complete the project without obtaining additional funds. In 1998 a sample chapter was written [7], a prototype Web site[3] was constructed, a funding proposal was submitted to the National Science

[3] http://dlmf.nist.gov

Foundation, and a contingency plan involving royalties on sales was prepared. The proposal failed but by that time NIST had named a board of associate editors and obtained tentative agreement from well-known researchers to write about half the chapters, if funding arrangements could be worked out. Two conference papers describing the project at this stage in more detail are [4] and [5].

In 1999 a second proposal to the National Science Foundation resulted in funding at the level requested for three years starting in September of that year.

3.3. Organizational Structure

The structure identified in 1997 is in use today. There are five classes of project contributors: principal editors, associate editors, authors, validators, and project staff.

The principal editors and areas of responsibility are Dr. D. W. Lozier, NIST, General Editor; Prof. F. W. J. Olver, University of Maryland and NIST, Mathematics Editor; Dr. C. W. Clark, NIST, Physical Sciences Editor; and Dr. R. F. Boisvert, NIST, Information Technology Editor. They form the executive committee, directing and bearing responsibility for the whole project.

The (unpaid) associate editors and areas of responsibility are Prof. R. A. Askey, University of Wisconsin, Special Functions; Prof. Sir M. V. Berry, University of Bristol, Physics; Prof. W. Gautschi, Purdue University, Numerical Methods; Prof. L. C. Maximon, George Washington University, Physics; Prof. M. Newman, University of California at Santa Barbara, Combinatorics and Number Theory; Prof. I. Olkin, Stanford University, Statistics; Prof. Dr. P. Paule, Johannes Kepler University, Symbolic Computing; Prof. W. P. Reinhardt, University of Washington, Chemistry; and Dr. N. M. Temme, CWI Amsterdam, Special Functions. The associate editors assist the principal editors in choice of technical content, selection of authors and validators, resolution of technical questions, and review of chapter drafts.

The authors and validators are drawn from around the world based on their research accomplishments. They are compensated by contracts with NIST. The author's contracts have two parts, the first requiring delivery of an initial draft that meets the strict editorial guidelines of the project, and the second requiring the author to coordinate his or her efforts with all the other project contributors in completing the DLMF Web site. The validation contracts, one for each chapter, require verification of all formulas and other mathematical material.

Finally, the project staff with major responsibilities include J. Conlon, NIST, Web Development; M. McClain, NIST, Web Development;

Dr. B. R. Miller, NIST, Information Architecture; Dr. B. V. Saunders, NIST, Graphics; Q. Wang, NIST, Graphics; and Prof. A. Youssef, George Washington University, Search Engine. They are assisted by summer student employees and, on occasion, other NIST staff members.

3.4. Time Frame and Current Status

The DLMF handbook and Web site are being developed simultaneously. The former is scheduled to be in print in 2003, with the public announcement of the Web site soon after. Like the NBS handbook, the DLMF handbook will be priced low enough for purchase by individuals. The Web site will be free.

4. DLMF Chapters

4.1. Chapter Contracts

The chapter contracts include detailed instructions for writing the main text and for preparing associated material such as Web-only sections, metadata, and graphics. These instructions are necessary to ensure that the resulting handbook and Web site will have consistent coverage and a uniform style across all chapters. A sample chapter, written by the Mathematics Editor, is provided as a model for the writing. To keep the printed handbook to a reasonable size and to avoid the temptation to include material of less than central importance, a target length is stipulated for each chapter. The author's sense of what is most important in applications of the mathematics of the chapter, subject to review and approval by the principal and associate editors, determines the coverage.

The style of the main text is that of a concise reference manual, with a minimum of explanation and other verbiage. Also, it will not be written in the typical mathematician's style, in which general theory is treated first and special cases (if any) are considered later. Rather, DLMF chapters will list the most important cases of formulas and other properties as determined by usage in applied mathematics, together with sufficient conditions for mathematical validity. The reason for this is that users will refer to the DLMF to find specific facts, not to peruse a general treatment of the subject of the chapter. Space permitting, brief indications of important extensions and generalizations, with references, are allowed.

For the Web version of the DLMF, limited additional material is allowed in Web-only sections. This can include additional members of a family of formulas or graphs when the first few are given in the main text. Modest extensions of main text tables, such as tables of zeros or expansion coefficients, are allowed also, but extensive tables will not be included anywhere in the DLMF. Authors are required to list relevant software references, and these will appear in a special Web-only section.

4.2. Current Status

The current list of chapters, 40 in number and subject to change, is Mathematical and Physical Constants; Algebraic and Analytical Methods; Asymptotic Approximations; Numerical Methods; Computer Algebra; Elementary Functions; Gamma Function; Exponential, Logarithmic, Sine and Cosine Integrals; Error and Related Functions; Incomplete Gamma Function and Generalized Exponential Integral; Airy and Related Functions; Bessel Functions; Struve Functions and Anger-Weber Functions; Confluent Hypergeometric Functions; Coulomb Wave Functions; Parabolic Cylinder Functions; Legendre Functions and Spherical Harmonics; Hypergeometric Functions; Generalized Hypergeometric Functions; q-Hypergeometric Functions; Classical Orthogonal Polynomials; Other Orthogonal Polynomials; Elliptic Integrals; Theta Functions; Multidimensional Theta Functions; Jacobian Elliptic Functions; Weierstrass Elliptic and Modular Functions; Bernoulli and Euler Numbers and Polynomials; Zeta and Related Functions; Combinatorial Analysis; Functions of Number Theory; Probability Functions and Statistical Distributions; Mathieu Functions and Hill's Equation; Lamé Functions; Spheroidal Wave Functions; Heun Functions; Painlevé Transcendents; Integrals with Coalescing Saddles; $3j$, $6j$, $9j$ Symbols; and Random Numbers and Monte Carlo Simulation. Authors had been selected and contracts were in force for all chapters in the summer of 2001. For most chapters, original drafts were received by the first quarter of 2002.

These drafts are subjected to a rigorous review by the principal editors, especially the Mathematics Editor. Detailed reports, which sometimes run to 30 pages, are written to guide authors in preparing revisions. Only when a revision is received that is deemed initially acceptable does an author receive a first payment under the contract. Further revisions will take place as a result of an independent validation, which is obtained under a separate NIST contract, and as a result of final editing of the entire DLMF. During the final editing stage, for example, cross-references among the chapters will be made. The second (final) payments under the contracts will be made when the DLMF handbook is complete and ready for publication.

4.3. Function Chapters

The bulk of DLMF chapters will supply properties of individual classes of special functions, for example the solutions of a specific differential equation. As an example, Figure 1 shows the title page of the handbook version of the chapter on Airy functions.

As illustrated in the figure, the sections of a function chapter are divided into five parts. The first part, Notation, establishes the DLMF notation for the functions of the chapter. Important alternative notations are listed also, and their relations to the DLMF notation are stated precisely. Often the DLMF notation will coincide with customary usage, but for some of the less common special functions no widely accepted usage is in existence. In such cases, the notation used in the DLMF is expected to lead to customary usage in the future.

The second part, Mathematical Properties, is the core of the chapter. This starts with a section defining the functions of the chapter, using the notation established previously. Graphical depictions of the functions of the chapter—line graphs, contour diagrams, surface plots—are given next, followed by sections that list the important formulas as determined by frequency of use in applications. The part ends with sections on any important related functions and generalizations.

The third part, Applications, presents a small number of typical or illustrative problems in which the functions of the chapter play an important role. Details are omitted but references are made to places where full treatments can be found. Often problems from both physics and mathematics are included in this part.

In the fourth part, Computation, effective approaches to computing the functions of the chapter—series expansions, numerical quadrature, numerical solution of differential and difference equations, representations in terms of other functions—are described briefly, with references to full treatments. A similar treatment of approximations, such as polynomial or rational approximations, is also given in this part. Next, a section on tables consisting of references to published tables, but not actual tables, is given. The computation part concludes with a Web-only section on software. This refers to software libraries and computer algebra packages that provide numerical computations or symbolic manipulations relevant to the functions of the chapter. When such software is nonproprietary and available free from a public Web site, as is sometimes the case, live links will be embedded to assist the user in obtaining the software.

In the finished DLMF the references from all chapters will be gathered together into a reference list for the whole DLMF. However, each chapter will also have a References part that contains one or two types

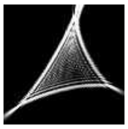
Chapter AI:

Airy and Related Functions

Frank W. J. Olver[1]

August 30, 2001 16:52

Gallery



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Acknowledgments: This chapter is based in part on Abramowitz and Stegun (1964, Chapter 10) by H. A. Antosiewicz. The graphs and visualizations in §AI.3 were constructed by R. R. Lipman, D. W. Lozier, B. V. Saunders, Q. Wang with the aid of the software package of Amos (1986). In §AI.9, B. R. Fabijonas supplied higher terms in some of the asymptotic expansions and computed the numerical tables. D. W. Lozier assisted with the compilation of §AI.21. The Web site was prepared by R. F. Boisvert and B. R. Miller.

Figure 1. Title page of DLMF chapter on Airy and Related Functions. This page is not in final form. The chapter code, AI, will be changed to a number, and other changes may also occur. Left and right gallery pictures used by permission of Royal Society of London and M. V. Berry, respectively. ©National Institute of Standards and Technology.

of reference information. The first type, which is present in the handbook and Web versions of the DLMF, is a list of the main resources for the chapter. Often a small number of important texts or monographs exist to which DLMF users can be directed for a comprehensive account of the mathematics contained in the chapter. The second type

Chapter AS:

Asymptotic Approximations

Frank W. J. Olver and Roderick Wong

August 30, 2001 17:04

Contents

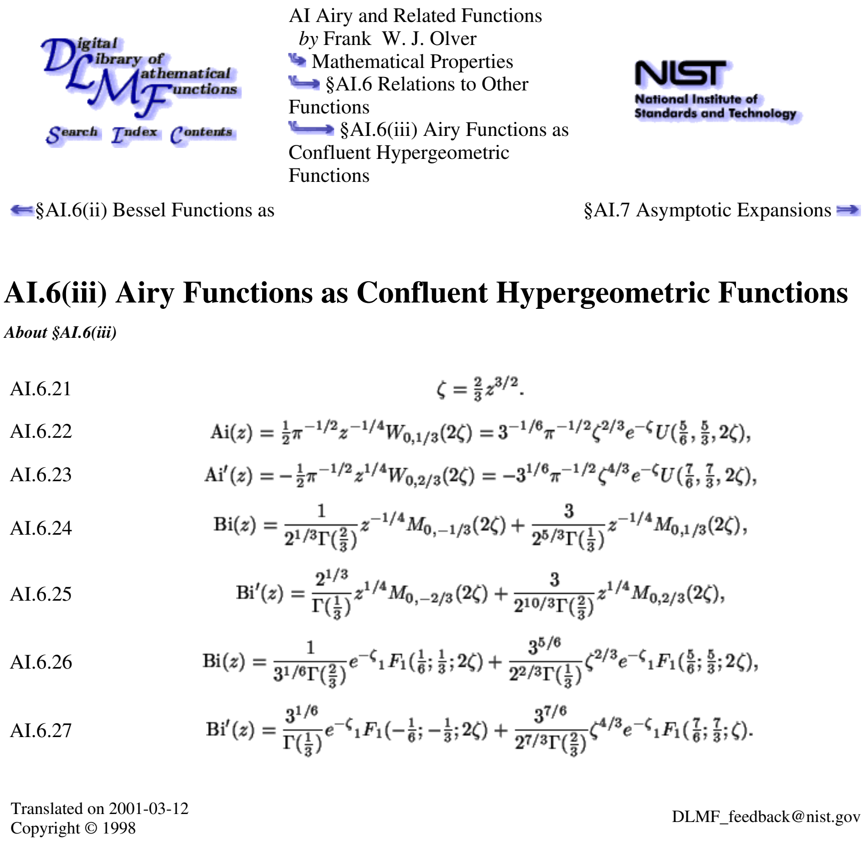
Figure 2. Title page of DLMF chapter on Asymptotic Approximations. This page is not in final form. The chapter code, AS, will be changed to a number, and other changes may also occur. ©National Institute of Standards and Technology.

of reference information will appear only in the handbook version. For each section of a chapter, it provides a list of specific references, with page numbers where appropriate, where proofs can be found for all the properties in the section. For the Web version, the same information is provided by hypertext links in the sections themselves.

Most DLMF chapters will have a place on the title page where interesting images relevant to the chapter are displayed. This is called the Gallery. In the Web version of the DLMF a link associated with each image pops up a brief description of the phenomenon depicted.

4.4. Methodology Chapters

Certain areas of mathematics are central to any serious work with special functions, or promise to be so in the future. The methodology chapters are Algebraic and Analytical Methods, Asymptotic Approximations, Numerical Methods, and Computer Algebra. These are difficult chapters to construct because each is a topic that could justify a digital library of its own. A standard pattern, such as the one developed for function chapters, cannot be devised. Therefore authors are given general admonitions to avoid textbook style and to select only those parts of the subject matter that directly bear on special functions. A target page limit is stipulated. For a glimpse of the level of presentation, see Figure 2.



AI Airy and Related Functions
by Frank W. J. Olver
Mathematical Properties
§AI.6 Relations to Other Functions
§AI.6(iii) Airy Functions as Confluent Hypergeometric Functions

Search Index Contents

NIST National Institute of Standards and Technology

§AI.6(ii) Bessel Functions as

§AI.7 Asymptotic Expansions

AI.6(iii) Airy Functions as Confluent Hypergeometric Functions

About §AI.6(iii)

AI.6.21 $$\zeta = \tfrac{2}{3} z^{3/2}.$$

AI.6.22 $$\mathrm{Ai}(z) = \tfrac{1}{2}\pi^{-1/2} z^{-1/4} W_{0,1/3}(2\zeta) = 3^{-1/6}\pi^{-1/2}\zeta^{2/3} e^{-\zeta} U(\tfrac{5}{6}, \tfrac{5}{3}, 2\zeta),$$

AI.6.23 $$\mathrm{Ai}'(z) = -\tfrac{1}{2}\pi^{-1/2} z^{1/4} W_{0,2/3}(2\zeta) = -3^{1/6}\pi^{-1/2}\zeta^{4/3} e^{-\zeta} U(\tfrac{7}{6}, \tfrac{7}{3}, 2\zeta),$$

AI.6.24 $$\mathrm{Bi}(z) = \frac{1}{2^{1/3}\Gamma(\frac{2}{3})} z^{-1/4} M_{0,-1/3}(2\zeta) + \frac{3}{2^{5/3}\Gamma(\frac{1}{3})} z^{-1/4} M_{0,1/3}(2\zeta),$$

AI.6.25 $$\mathrm{Bi}'(z) = \frac{2^{1/3}}{\Gamma(\frac{1}{3})} z^{1/4} M_{0,-2/3}(2\zeta) + \frac{3}{2^{10/3}\Gamma(\frac{2}{3})} z^{1/4} M_{0,2/3}(2\zeta),$$

AI.6.26 $$\mathrm{Bi}(z) = \frac{1}{3^{1/6}\Gamma(\frac{2}{3})} e^{-\zeta} {}_1F_1(\tfrac{1}{6}; \tfrac{1}{3}; 2\zeta) + \frac{3^{5/6}}{2^{2/3}\Gamma(\frac{1}{3})} \zeta^{2/3} e^{-\zeta} {}_1F_1(\tfrac{5}{6}; \tfrac{5}{3}; 2\zeta),$$

AI.6.27 $$\mathrm{Bi}'(z) = \frac{3^{1/6}}{\Gamma(\frac{1}{3})} e^{-\zeta} {}_1F_1(-\tfrac{1}{6}; -\tfrac{1}{3}; 2\zeta) + \frac{3^{7/6}}{2^{7/3}\Gamma(\frac{2}{3})} \zeta^{4/3} e^{-\zeta} {}_1F_1(\tfrac{7}{6}; \tfrac{7}{3}; \zeta).$$

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Figure 3. Subsection of DLMF chapter on Airy and Related Functions. This page is not in final form. The chapter code, AI, will be changed to a number, and other changes may also occur. ©National Institute of Standards and Technology.

5. Metadata

For every subsection of the DLMF a collection of metadata is constructed. Figure 3 shows a typical subsection from the Web version of the Airy functions chapter, and Figure 4 shows its metadata. A hypertext link (About §Ai.6(iii)) in the subsection depicted in Figure 3 leads the Web user to Figure 4..

One purpose of the metadata is to record information that leads to a proof for every formula in the subsection. This can be a reference to a publication or, as in the case of the third Note in Figure 4, a brief indication of how the formulas may be derived. In the handbook version, the references are aggregated to the section level and listed in the References part of each DLMF chapter.

Digital Library of Mathematical Functions
Search Index Contents

AI Airy and Related Functions
by Frank W. J. Olver
Mathematical Properties
§AI.6 Relations to Other Functions
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About §AI.6(iii) Airy Functions as Confluent Hypergeometric Functions

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About §AI.6(iii) Airy Functions as Confluent Hypergeometric Functions

Also see *About Mathematical Properties* and *About AI*.

Notes

- See (AI.6.21) for definition of ζ.
- See Chapter CH (Confluent Hypergeometric Functions) for Hypergeometric functions $W_{\kappa,\mu}$, U, $M_{\kappa,\mu}$ and ${}_pF_q$.
- These formulas are derivable from (AI.6.1)-(AI.6.5) and those given in CH (Confluent Hypergeometric Functions).

Internal Metadata

Label

- `sec:AI.RL.HY`

Indexing Data

- Airy functions > relation to > confluent hypergeometric functions
- Confluent hypergeometric functions > relation to > Airy functions

Translated on 2001-03-12

DLMF_feedback@nist.gov

Figure 4. Metadata for a DLMF subsection. This page is not in final form. The chapter code, AI, will be changed to a number, and other changes may also occur. ©National Institute of Standards and Technology.

Another use of metadata is to clarify notation and conditions. In conventional mathematical writing it is customary to rely heavily on notation that may have been introduced much earlier in the work or, in the case of very common notation, simply assume the user will understand. Similar comments hold also for conditions, such as sectors of the complex plane in which an asymptotic expansion is valid. But conventional practice is inappropriate for a handbook. Users may not be aware of undefined notation and unstated conditions, particularly so when they use Web capabilities such as query-based search to find their way directly to an individual formula. Therefore, the metadata may be used to restate definitions that can be found elsewhere in the main text.

In the DLMF great care is being taken to define notation and specify conditions completely. The approach taken is informal, following accepted standards of good taste in mathematical writing, though handbooks are an obvious place where formal techniques could perhaps be put to good use. For example, complete semantics are necessary to successfully download formulas into a computer algebra system. Only an incomplete capability will exist in the first public release of the DLMF. Providing a full capability is a likely subject for a follow-up project.

The remaining, more mundane, type of metadata is used to construct indexes and to facilitate the construction of the Web site. For examples, see Figure 4.

6. Graphics

Special functions exhibit many different behaviors as arguments and parameters vary. These behaviors are fully described by the mathematical properties but often it is helpful to visualize them with graphics. The DLMF will contain 2D and 3D graphs and surfaces. Color is being used on 2D plots to distinguish curves and on 3D surfaces to represent height or, occasionally, phase angles. Surfaces on the Web are being represented in VRML (Virtual Reality Modeling Language) format, a computer graphics file format designed for use with a special viewer that allows the user to rotate and zoom the surface. The VRML format also has a program capability that is being used to construct 2D moving sections of a surface. These sections are displayed in a 2D frame alongside the surface, as illustrated in Figure 5. A description of the DLMF VRML capabilities is contained in [8].

The VRML format is also being used to generate fixed views for the handbook version of the DLMF. An example is shown in Figure 6.

7. Search Engine

A search engine capable of searching for equations based on mathematics-like queries is being constructed. It is based on an available public-domain text-based search engine, PLWeb. In most cases the target of a search is a DLMF subsection that contains a formula that satisfies a query. For example, the search string `gamma(1/3)` locates all formulas in which $\Gamma(\frac{1}{3})$ appears.

A variety of innovative techniques are being built into PLWeb to facilitate searching in equations. One is the concept of search depth.

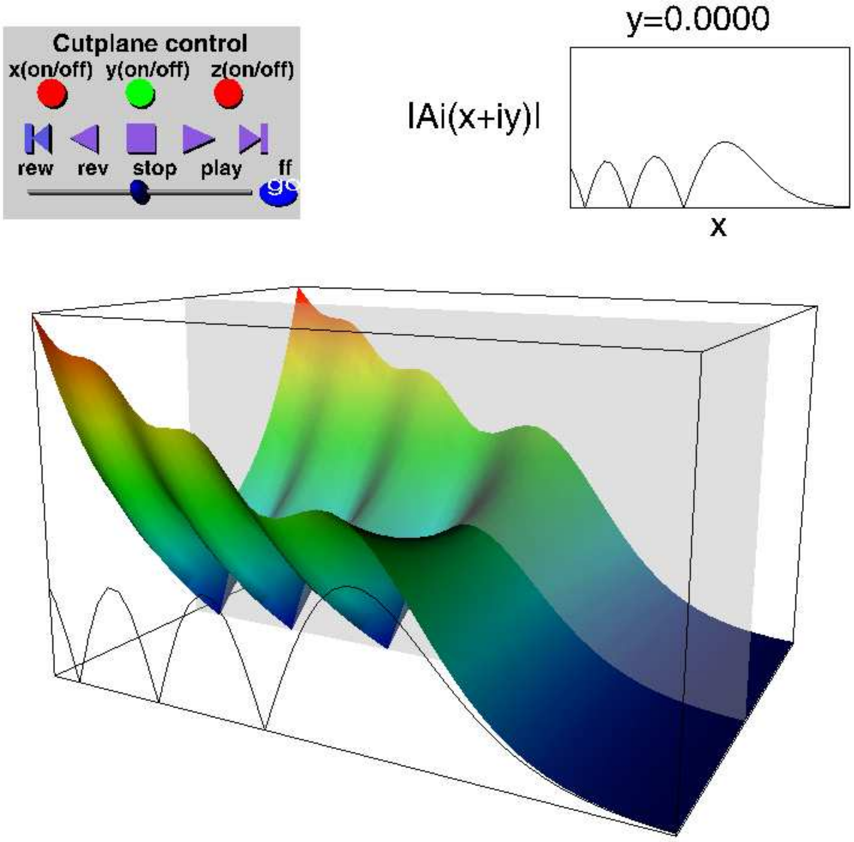

Figure 5. View of the Airy function $z = |\mathrm{Ai}(x+iy)|$. The surface in xyz-coordinates is cut by the xz-plane to display oscillatory behavior along the negative real axis. The zeros, to 2 decimals, are $-2.34, -4.09, -5.52$. Exponential growth and decay in different parts of the complex plane are visible also. ©National Institute of Standards and Technology.

For example, the string `Ai^2+Bi^2` with the search depth set to `medium` locates all occurrences of $\mathrm{Ai}^2(z) + \mathrm{Bi}^2(z)$. Occurrences with z replaced with any other symbol, or no symbol, are found also. To locate this same expression with the search depth set to `low` would require the explicit search string `Ai^2(z)+Bi^2(z)`.

Another innovation is use of surrogate subsections. A surrogate containing mathematics keywords for every formula is created for every subsection. The surrogates are searched but the original subsections are returned to the user. For details, see [6].

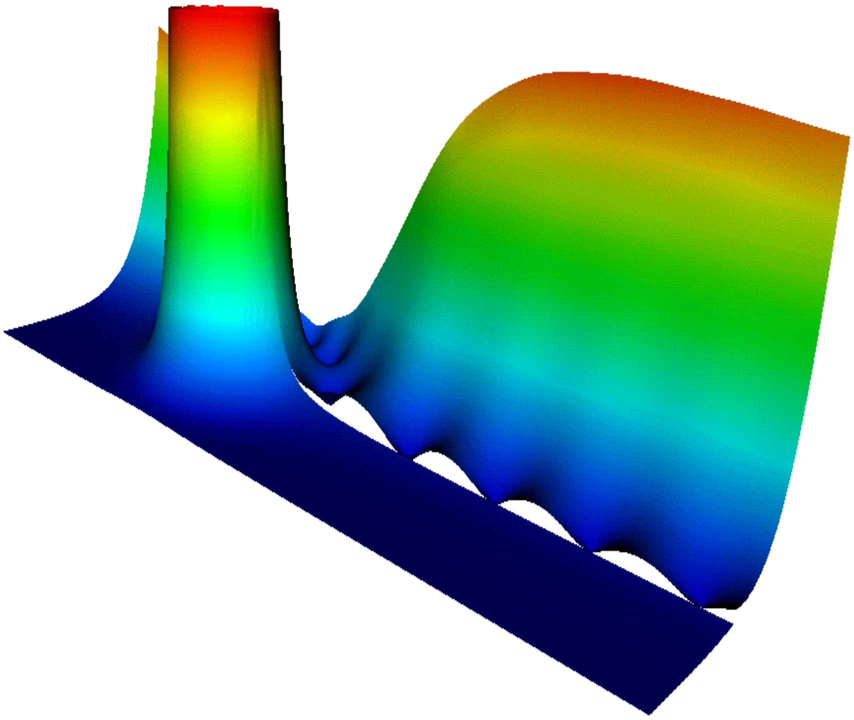

Figure 6. View of principal branch of the Hankel function $|H_5^{(1)}(x+iy)|$ showing pole at the origin, branch cut, location of zeros near the cut, and exponential growth and decay in different parts of the complex plane. Five zeros around the pole are not fully visible in this view. In the Web version of the DLMF, this view may be rotated and seen from any angle.

8. Summary and Conclusions

The NIST DLMF Project is a major attempt, funded by the U.S. National Institute of Standards and Technology and the U.S. National Science Foundation, to construct *ab initio* an up-to-date reference database of mathematical properties of special functions for use by scientists, engineers, statisticians, and applied mathematicians. The project participants are conducting a thorough assessment of current needs, led by a prestigious board of editors and associate editors. The immediate products will be a 1000-page conventional hardcover handbook and a free public Web site, constructed and maintained by NIST. The actual writing is being done by recognized world experts who are being paid by contract with NIST. The new work is expected to serve as the standard reference for many years to come.

From the standpoint of MKM (Mathematical Knowledge Management), the DLMF Project is significant because it is not an academic exercise. Its main purpose is to provide comprehensive standard reference information in a particular field of mathematics. NIST is the U.S. government agency that is charged with responsibility for exactly this kind of work, and also with the long-term maintenance, enhancement and dissemination of the resulting databases. No purely academic endeavor has this kind of responsibility and long-term commitment.

Because of the need to produce a tangible product within a specific time frame, it is necessary to make use of the generally available tools of today. Superior techniques are being investigated in academic settings. Eventually these may lead to considerably improved tools that are widely accessible. NIST will be there to assess them and adopt them when the time is right. If MKM proves its worth in the DLMF Project, it will undoubtedly find many future applications in the provision of scientific and technical databases.

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