RESEARCH REPORT SERIES
(Statistics #2007-10)

A Mean Squared Error Criterion for Comparing X-12-ARIMA and Model-Based Seasonal Adjustment Filters

Yea-Jane Chu[1]
George C. Tiao[2]
William R. Bell

SPSS, Inc.[1]

University of Chicago[2]

Statistical Research Division
U.S. Census Bureau
Washington, D.C. 20233

Report Issued: September 17, 2007

A Mean Squared Error Criterion for Comparing X-12-ARIMA and Model-Based Seasonal Adjustment Filters

Yea-Jane Chu, George C. Tiao, and William R. Bell

SPSS, Inc., University of Chicago, and U.S. Census Bureau

contact author: William R. Bell, U.S. Census Bureau, 4600 Silver Hill Road,

Room 5K142A, Washington, DC 20233, USA

e-mail: William.R.Bell@census.gov

September 13, 2007

Abstract: Various authors – Cleveland and Tiao (1976), Burridge and Wallis (1984), and Depoutot and Planas (1998) – have compared weight functions from X-11 versus model-based seasonal adjustment filters. We suggest a different approach to comparing filters by computing the mean squared error (MSE) when using an X-12-ARIMA filter for estimating the underlying seasonal component from an ARIMA model-based decomposition, and comparing this to the MSE of the optimal model-based estimator. This provides a criterion for choosing an X-12 filter for a given series (model the series and pick the X-12 filter with lowest MSE), and also provides results on how much MSE increases when using an X-12 filter rather than the optimal model-based filter. Calculations for monthly time series following the airline model with various parameter values show little increase in MSE for estimating the canonical seasonal component by using the best X-12 filter instead of the optimal model-based filter, particularly for concurrent adjustment. The results are much less favorable to the X-12 filters with a uniform prior distribution on the white noise allocation in the seasonal model decomposition. Examinations of simulated series show that, for the canonical decomposition, automatic filter choices of the X-12-ARIMA program sometimes use shorter seasonal moving averages than is desirable.

Keywords: Census X11, concurrent adjustment, moving averages, seasonal decomposition.

Disclaimer: This paper is released to inform interested parties of ongoing research and to encourage discussion. The views expressed on statistical, methodological, technical, or operational issues are those of the authors and not necessarily those of SPSS, the University of Chicago, or the U.S. Census Bureau.

1 Introduction

The fixed filtering approach to seasonal adjustment, as implemented in the original Census X-11 program (Shiskin, Young, and Musgrave 1967) and its successors, X-11-ARIMA (Dagum 1975) and X-12-ARIMA (Findley, Monsell, Bell, Otto, and Chen 1998), has been widely used by government and industry. This approach relies on a finite set of empirically developed moving averages. The user can either specify the particular moving averages used for a time series or let the program choose them automatically according to some empirical criteria. An advantage to this approach is that it is relatively easy to use even for people with limited statistical background. A disadvantage is that the reliance on a limited set of filters raises the possibility of cases arising that are not well-handled by the available filters. Another disadvantage is that the empirical criteria used by the program to automatically select filters do not follow from standard statistical principles that would lead to certain optimality properties such as minimum mean squared error (MMSE).

In contrast, a model-based approach to seasonal adjustment specifies stochastic models for the observed series and underlying components, and derives seasonal adjustment filters from optimal signal extraction theory. The filters used are thus determined by the model form specified, by assumptions made about the component decomposition, and by estimates of the model parameters. (See Bell and Hillmer (1984) for discussion.) The model-based approach offers more flexibility in determining filters than the empirical filtering approach, as well as providing for determination of filters according to standard statistical principles.

To relate these two approaches, Cleveland and Tiao (1976) and Burridge and Wallis (1984) proposed stochastic models leading to seasonal adjustment filters close to filters in the Census X-11 program. This line of work is further extended in Chu, Tiao, and Bell (2002) to provide models for 24 X-12 symmetric filters.[1] These results can provide a model-based foundation for use of X-12 filters. However, the models developed to approximate the X-12 filters are rather complex, more complex than models used in practice, making this approach rather cumbersome and vague in practice as a means of evaluating and choosing X-12 filters. Depoutot and Planas (1998), and Planas and Depoutot (2002), avoid complex approximating models by restricting consideration to the popular "airline" ARIMA model (Box and Jenkins 1976), using the ARIMA model-based seasonal decomposition approach of Hillmer and Tiao (1982), hereafter HT, and Burman (1980). They specifically focus on matching the weights of X-12 filters with weights from model-based optimal filters under the "canonical decomposition."

In this paper we suggest a different approach to comparing X-12 filters to model-based filters. More specifically, for a given ARIMA model we compute the mean squared error (MSE) when a specific X-12 filter is used to estimate the underlying seasonal component from the model-based decomposition. This approach provides results on how much accuracy is lost (in terms of increased MSE) by using an X-12 filter rather than the optimal model-based filter. The approach also provides an objective means of choosing an X-12 filter, namely, pick the filter that minimizes the MSE.

An issue that arises in the ARIMA model-based approach concerns uncertainty about the model decomposition in regard to allocation of white noise between the seasonal and nonseasonal components. We consider two options for dealing with this uncertainty. One is to assume a particular white noise allocation, such as the canonical decomposition of HT and Burman (1980), which allocates all the white noise to the nonseasonal component. The other option considered is to allow for uncertainty by putting a prior distribution on the white noise allocation and examining the average MSE over the prior. Here we obtain results for both the canonical decomposition (which can be viewed as corresponding to a particular degenerate prior) and for a uniform prior over the admissible range of the white noise allocation. Bell and Otto (1992) also used these two cases in a Bayesian approach to treating ARIMA model-based seasonal adjustment.

In Section 2, we briefly review the ARIMA model-based approach to seasonal adjustment and the white noise allocation issue. This sets up the framework for developing our approach to comparing X-12 and model-based filters in Section 3. Sections 4 and 5 then present results of such comparisons for a monthly time series following the airline model for various combinations of the airline model parameters. Section 4 presents results for symmetric filters, and Section 5 for concurrent filters. The results show which X-12 filters fare best in this comparison for series following various airline models, and how much accuracy is lost in terms of increased MSE from using the best X-12 filter instead of the optimal model-based filter. In many cases little accuracy is lost in estimating the canonical model-based seasonal component by using the best X-12 filter, particularly for concurrent adjustment. The results are much less favorable to the X-12 filters when we assume a uniform prior distribution on the white noise allocation. Additional results presented in Section 4 use simulated series from the airline model to compare our best X-12 filter selections with those from the automatic filter selection procedure of the X-12-ARIMA program. Finally, Section 6 summarizes the results and raises some questions for future research.

2 The ARIMA Model-Based Approach and the White Noise Allocation Issue

In this section, we briefly review the ARIMA model-based approach to seasonal adjustment and the white noise allocation issue. Following HT and Burman (1980), we suppose that an observable time series, Z_t, where in this paper t denotes the month, can be decomposed as

$$Z_t = S_t + N_t, \tag{1}$$

where S_t and N_t are unobservable seasonal and nonseasonal components that follow the ARIMA models

$$U(B)S_t = \eta_S(B)b_t, \qquad \text{and} \tag{2}$$

$$(1-B)^d\phi_N(B)N_t = \eta_N(B)c_t, \tag{3}$$

respectively. In (2) and (3) B is the backshift operator such that $BS_t = S_{t-1}$, $U(B) = (1 + B + \cdots + B^{s-1})$, and s denotes the number of time periods per year (here $s = 12$). Further, $\phi_N(B)$ is a polynomial in B of degree p with its zeros lying outside the unit circle, while $\eta_S(B)$ and $\eta_N(B)$ are polynomials of degrees $s-1$ and $p+d$, respectively, with zeros lying on or outside the unit circle. (Note: These assumptions effectively impose only upper limits on the degrees. If $\eta_S(B)$ has lower degree than $s-1$ we can append additional terms with zero coefficients to raise its degree to $s-1$, and similarly if $\eta_N(B)$ has lower degree than $p+d$.) We also assume that $U(B)$ and $\eta_S(B)$ have no common zeros, and that $(1-B)^d\phi_N(B)$ and $\eta_N(B)$ have no common zeros. The innovation series b_t and c_t are mutually independent Gaussian white noises with variances σ_b^2 and σ_c^2, respectively.

Overall model implied by component models: Let $A_Z(z)$, $A_S(z)$, and $A_N(z)$ denote the "pseudo" autocovariance generating functions (ACGFs) of Z_t, S_t, and N_t, respectively. We then have from (1)–(3) that

$$A_Z(z) = A_S(z) + A_N(z) \tag{4}$$

where

$$A_S(z) = \frac{\eta_S(z)\eta_S(z^{-1})}{U(z)U(z^{-1})}\sigma_b^2 \qquad \text{and} \tag{5}$$

$$A_N(z) = \frac{\eta_N(z)\eta_N(z^{-1})}{(1-z)^d(1-z^{-1})^d\phi_N(z)\phi_N(z^{-1})}\sigma_c^2. \tag{6}$$

It follows that $A_Z(z)$ can be written in the form

$$A_Z(z) = \frac{\theta(z)\theta(z^{-1})}{\varphi(z)\varphi(z^{-1})}\sigma_a^2 \tag{7}$$

where $\varphi(z) = U(z)(1-z)^d\phi_N(z)$ and $\theta(z)$ both have degree $p+s+d-1$. Thus, the *overall* model for Z_t is the ARIMA model

$$\varphi(B)Z_t = \theta(B)a_t, \tag{8}$$

The innovation series a_t is Gaussian white noise with variance σ_a^2. We assume that all the zeros of $\theta(B)$ are outside of the unit circle.

Decomposition of an overall model: On the other hand, given an overall model in the form of (8), which can be verified from observable data Z_t, we can proceed to use the results in HT and Burman (1980) to obtain a decomposition of Z_t into seasonal and nonseasonal components S_t and N_t as follows.

Note first that, given the ARIMA model (8) for Z_t, any choice of $\eta_S(B)$, $\eta_N(B)$, σ_b^2, and σ_c^2 satisfying (4)–(7) gives what is termed an "*acceptable*" decomposition of $A_Z(z)$ into seasonal and nonseasonal component ACGFs, corresponding to an acceptable decomposition of Z_t into seasonal and nonseasonal component series as in (1). Now if $A_S(z)$ and $A_N(z)$ represent an acceptable decomposition, then $A_S(z)+\tau$ and $A_N(z)-\tau$, where τ is a constant, represent another acceptable decomposition provided that $A_S(e^{-i\lambda})+\tau$ and $A_N(e^{-i\lambda})-\tau$ are nonnegative for all $\lambda \in [0,\pi]$. Thus, in general there are an infinite number of ways one can decompose a series corresponding to a given overall model.

Now we can represent the range of acceptable decompositions in terms of one unidentified parameter. Specifically, writing $\Phi(B) \equiv \varphi(B)/U(B) = (1-B)^d \phi_N(B)$, and following HT, we perform a (unique) partial fraction decomposition of $A_Z(z)$ in (7) into

$$A_Z(z) = A_S(z) + A_N(z) + \kappa$$

where

$$\begin{aligned} A_S(z) &= \frac{Q_S(z)}{U(z)U(z^{-1})} \qquad \text{with } Q_S(z) = q_{0S} + \sum_{i=1}^{s-2} q_{iS}(z^i + z^{-i}), \\ A_N(z) &= \frac{Q_N(z)}{\Phi(z)\Phi(z^{-1})} \qquad \text{with } Q_N(z) = q_{0N} + \sum_{i=1}^{p+d-1} q_{iN}(z^i + z^{-i}), \end{aligned}$$

and κ is a constant. Let

$$\begin{aligned} A_{SC}(z) &= \frac{Q_S(z)}{U(z)U(z^{-1})} - \varepsilon_s, \qquad \text{and} \\ A_{NC}(z) &= \frac{Q_N(z)}{\Phi(z)\Phi(z^{-1})} - \varepsilon_n \end{aligned}$$

where

$$\begin{aligned} \varepsilon_s &= \min_{\lambda\in[0,\pi]} \frac{Q_S(e^{-i\lambda})}{U(e^{-i\lambda})U(e^{i\lambda})}, \qquad \text{and} \\ \varepsilon_n &= \min_{\lambda\in[0,\pi]} \frac{Q_N(e^{-i\lambda})}{\Phi(e^{-i\lambda})\Phi(e^{i\lambda})}. \end{aligned}$$

Then, we can write

$$A_Z(z) = A_{SC}(z) + A_{NC}(z) + \varepsilon_s + \varepsilon_n + \kappa.$$

As shown in HT, an acceptable decomposition exists if and only if $\gamma_{\max} \equiv \varepsilon_s + \varepsilon_n + \kappa \geq 0$. When this is so, acceptable decompositions $A_Z(z) = A_S^\gamma(z) + A_N^\gamma(z)$ can be indexed by $\gamma \in [0, \gamma_{\max}]$, and the range of acceptable seasonal and nonseasonal components must correspond to

$$A_S^\gamma(z) = A_{SC}(z) + \gamma \tag{9}$$

$$A_N^\gamma(z) = A_{NC}(z) + (\gamma_{\max} - \gamma). \tag{10}$$

We shall let S_t^γ denote the seasonal component corresponding to $A_S^\gamma(z)$ and $N_t^\gamma = Z_t - S_t^\gamma$ the nonseasonal component corresponding to $A_N^\gamma(z)$. The constant $\gamma_{\max}$, when positive, can be viewed as corresponding to unobservable white noise in the series Z_t. We see from equations (9) and (10) that the value specified for $\gamma \in [0, \gamma_{\max}]$ thus determines an allocation of this white noise between the unobserved seasonal and nonseasonal components.

Canonical Decomposition: In (9) it is easy to see that setting $\gamma = 0$ will minimize the innovation variance σ_b^2 of the seasonal component. HT call this the "canonical decomposition" and discuss its properties. Fundamentally, the canonical decomposition provides the most stable seasonal component, i.e., the one that shows the least variation over time from a fixed seasonal pattern. At the other extreme, setting $\gamma = \gamma_{\max}$ will maximize the innovation variance of the seasonal component and provide the most variation over time from a fixed seasonal pattern. The only information provided about γ by the model (8) for the observed series Z_t, and hence by the data, is the range $\gamma \in [0, \gamma_{\max}]$. This means that given the ARIMA model for Z_t in (8), unobserved seasonal components for all values of $\gamma \in [0, \gamma_{\max}]$ are equally consistent with the data. Dealing with uncertainty about γ is discussed next in Section 3.

3 An Approach to Comparing X-12 and Model-Based Seasonal Filters

In this section we develop an approach to comparing any given linear seasonal filter with the optimal model-based seasonal filter based on comparing their MSEs when estimating the seasonal component S_t^γ. We then apply the results to comparing X-12 and optimal model-based seasonal filters. We first discuss the case where γ is assumed to be known (Section 3.1), and then the case where γ is unknown (Section 3.2).

Let $w_S(B) = \sum_i w_{Si} B^i$ be a specific linear filter to be used for estimating any seasonal component S_t, i.e., $\hat{S}_t = w_S(B) Z_t$, and let $w_N(B) = 1 - w_S(B)$ be the corresponding linear filter for estimating N_t. For a given value of γ, the error in estimating S_t^γ by $\hat{S}_t$, which we shall denote by $g_t^\gamma = S_t^\gamma - \hat{S}_t$, is

$$g_t^\gamma = w_N(B) S_t^\gamma - w_S(B) N_t^\gamma. \tag{11}$$

Given that S_t^γ and N_t^γ are assumed to follow models of the form of (2) and (3), it is easy to see from (11) that the error series g_t^γ will be stationary if $w_N(B)$ contains $U(B)$ as a factor and $w_S(B)$ contains $(1-B)^d$ as a factor. This will be true for all the filters considered here. When this is true the ACGF of g_t^γ is

$$A_g^\gamma(z) = w_N(z) w_N(z^{-1}) A_S^\gamma(z) + w_S(z) w_S(z^{-1}) A_N^\gamma(z) \tag{12}$$

and the corresponding MSE is

$$\mathrm{MS}(g_t^\gamma) = (2\pi)^{-1} \int_{-\pi}^{\pi} A_g^\gamma(e^{-i\lambda}) d\lambda. \tag{13}$$

The above results were given by Pierce (1979). Since (12) shows $A_g^\gamma(z)$ to be symmetric (coefficients of z^k and z^{-k} are equal for all k) we can compute $\mathrm{MS}(g_t^\gamma)$ by expanding (12) and taking the constant term (coefficient of z^0 in the expansion). Before doing so, however, we must cancel the unit root factors $U(z)U(z^{-1})$ that appear in $w_N(z) w_N(z^{-1})$ and in the denominator of $A_S^\gamma(z)$, and similarly cancel $(1-z)^d (1-z^{-1})^d$ that appears in $w_S(z) w_S(z^{-1})$ and in the denominator of $A_N^\gamma(z)$.

We now discuss an interesting and important property of $\mathrm{MS}(g_t^\gamma)$. Watson (1987, eq. (3.9)) showed that (allowing for differences in notation)

$$\mathrm{MS}(g_t^\gamma) = \mathrm{MS}(g_t^0) + \gamma(1 - 2w_{S0}) \tag{14}$$

where g_t^0 is the estimation error for S_t^γ at $\gamma = 0$, i.e., the error in estimating the canonical seasonal S_t^0, and w_{S0} is the "center weight" of the filter $w_S(B)$, that is, the weight that $w_S(B)$ applies to Z_t (at the time point at which we are estimating S_t). In (14) when $w_{S0} < 0.5$, $\mathrm{MS}(g_t^\gamma)$ is an increasing linear function of γ that is thus bounded below by $\mathrm{MS}(g_t^0)$ and bounded above by $\mathrm{MS}(g_t^{\gamma_{\max}})$.

Equations (12)–(14) apply whether $w_S(B)$ is a model-based or X-12 filter, symmetric or asymmetric. When $w_S(B)$ is the (symmetric or asymmetric) model-based filter corresponding to the true value of γ we get the optimal (MMSE) signal extraction estimate, which we shall denote as $\tilde{S}_t^\gamma$. In the symmetric case the optimal signal extraction filter is $w_S^\gamma(B) = A_S^\gamma(B)/A_Z(B)$ and (12) simplifies to $A_g^\gamma(z) = A_S^\gamma(z)A_N^\gamma(z)/A_Z(z)$ (Bell 1984). Bell and Martin (2004) discuss optimal asymmetric signal extraction, and computation of the resulting MSE, with ARIMA component models.

3.1 Comparing X-12 and Model-Based Filters When γ Is Known

Now let j index the filters within a relevant set J of X-12 filters, such as the symmetric X-12 seasonal filters. We write $x_S^j(B)$ for a particular X-12 seasonal filter, with corresponding estimated seasonal component $\hat{S}_t^j = x_S^j(B)Z_t$. Letting $g_t^{\gamma,j} = S_t^\gamma - \hat{S}_t^j$ be the error series, for each $j \in J$ we can expand (12) as discussed above to compute $\mathrm{MS}[g_t^{\gamma,j}]$. We can then pick the best X-12 filter, $x_S^{j*}(B)$, to achieve the minimum MSE, i.e.,

$$\mathrm{MS}[g_t^{\gamma,j*}] = \min_{j \in J}\{\mathrm{MS}[S_t^\gamma - \hat{S}_t^j]\}. \tag{15}$$

Stationarity of the error series $g_t^{\gamma,j}$ for any γ and any symmetric X-12 filter follows from (11) since, according to Bell (1992), any symmetric X-12 seasonal filter $x_S^j(B)$ contains $(1-B)^6$ and any symmetric X-12 seasonal adjustment filter, $x_N^j(B) = 1 - x_S^j(B)$, contains $U(B)$. Asymmetric filters obtained by applying symmetric X-12 filters to series extended by a sufficient number of forecasts and backcasts from the ARIMA model (8) will also contain the needed differencing operators.

We remark here that the center weight $w_{S0}^j < 0.5$ for all the symmetric X-12 seasonal filters (note, e.g., Bell and Monsell 1992). Hence, for X-12 symmetric seasonal filters the minimum of $\mathrm{MS}(g_t^{\gamma,j})$ as a function of γ always occurs at $\gamma = 0$, i.e., at the canonical decomposition. Thus, any X-12 symmetric seasonal filter will better estimate the canonical seasonal component for a given model than any other admissible seasonal component for that model.

3.2 Comparing X-12 and Model-Based Filters When γ Is Unknown

When the value of γ in the model-based decomposition is regarded as unknown, we assign to γ a prior probability density, $p(\gamma)$, over the acceptable range $[0, \gamma_{\max}]$, and then compute, for a given filter, the average MSE over $p(\gamma)$. That is, we compute $\mathrm{E}_\gamma \, \mathrm{MS}(g_t^\gamma) = \int \mathrm{MS}(g_t^\gamma) p(\gamma) d\gamma$. Computation of this average MSE is greatly aided by the linearity of $\mathrm{MS}(g_t^\gamma)$ in γ as shown in (14). We thus have the following lemma.

Lemma: Let $Z_t = S_t^\gamma + N_t^\gamma$ be an acceptable decomposition of Z_t following the model (8), where S_t^γ and N_t^γ follow models given by (2) and (3) corresponding to ACGFs as given by (5) and (6). Let $w_S(B)$ be a linear seasonal filter such that the error series g_t^γ in (11) is stationary. Then the average MSE over the distribution of γ with density $p(\gamma)$ for estimating S_t^γ by $w_S(B)Z_t$ is

$$\mathrm{E}_\gamma \, \mathrm{MS}(g_t^\gamma) = \mathrm{MS}(g_t^{\mu_\gamma})$$

where $\mu_\gamma = \int \gamma p(\gamma) d\gamma$ is the mean of γ.

Proof: The result follows immediately from (14) by noting that

$$\mathrm{E}_\gamma \, \mathrm{MS}(g_t^\gamma) = \mathrm{MS}(g_t^0) + \mu_\gamma(1 - 2w_{S0}) = \mathrm{MS}(g_t^{\mu_\gamma}).$$

Note: In the special case of a uniform prior for γ, $\mu_\gamma = \frac{1}{2}\gamma_{\max}$. The Lemma also applies to the canonical decomposition by setting $\mu_\gamma = 0$, since the canonical decomposition can be viewed as corresponding to a degenerate prior of $\gamma = 0$ with probability one.

Note that $w_S(B)$ could be either a symmetric filter or an asymmetric filter, so the Lemma applies to both symmetric and concurrent seasonal adjustment. It could also be either a finite or an infinite filter. The only requirements are that (i) $w_S(B)$ and $w_N(B)$ contain the needed operators $(1-B)^d$ and $U(B)$, respectively, so the error series g_t^γ is stationary, and (ii) the time point for which we are estimating S_t^γ lies within the span of the observed data. The second requirement excludes forecasting of S_t^γ, since, for forecasting, $w_N(B) \neq 1 - w_S(B)$.

The Lemma can be used to compute the average MSE for a given filter, X-12 or model-based, and in each case we need only compute the MSE when the filter is used to estimate $S_t^{\mu_\gamma}$. Given a set J of X-12 filters, the best X-12 filter in terms of average MSE is thus the one that achieves the minimum average MSE over the set. For additional related results and further insights, see Watson (1987) and Bell and Otto (1992).

3.3 MSE Comparison Measures

In Sections 4 and 5 we compare MSEs of X-12 and optimal model-based filters. For the case of γ unknown we compare the average MSEs (over $p(\gamma)$) assuming a uniform prior for γ on $[0, \gamma_{\max}]$, which, from the Lemma, reduces to comparing the MSEs for estimating $S_t^{\mu_\gamma}$ with $\mu_\gamma = \gamma_{\max}/2$. We can also think of the canonical decomposition comparisons as "average MSE comparisons" under a degenerate prior of $\gamma = 0$ with probability one, which reduces to comparing the MSEs for estimating $S_t^{\mu_\gamma}$ with $\mu_\gamma = 0$.

We present the average MSE comparisons as percentage differences, using the average MSE for the best model-based estimator, $\tilde{S}_t^{\mu_\gamma} = w_S^{\mu_\gamma}(B)Z_t$, as a base value. So the percentage difference for a given X-12 filter $x_s^j(B)$ is

$$100 \times \left\{ \frac{\mathrm{MS}[S_t^{\mu_\gamma} - x_S^j(B)Z_t] - \mathrm{MS}[S_t^{\mu_\gamma} - \tilde{S}_t^{\mu_\gamma}]}{\mathrm{MS}[S_t^{\mu_\gamma} - \tilde{S}_t^{\mu_\gamma}]} \right\}. \tag{16}$$

When $x_s^j(B)$ in (16) is $x_s^{j^*}(B)$ satisfying (15) (with the γ in (15) fixed at μ_γ), then (16) gives the percentage increase in MSE from using the best X-12 filter instead of the best model-based filter for estimating $S_t^{\mu_\gamma}$. The denominator of (16) can be computed from standard signal extraction results. In the numerator we can write

$$S_t^{\mu_\gamma} - x_S^j(B)Z_t = [S_t^{\mu_\gamma} - \tilde{S}_t^{\mu_\gamma}] + [(w_S^{\mu_\gamma}(B) - x_S^j(B))Z_t]. \tag{17}$$

The first term on the right hand side of (17), the error in the optimal estimate $\tilde{S}_t^{\gamma}$, is orthogonal to all linear functions of Z_t. Thus, the two terms in (17) are orthogonal and the numerator of (16) immediately reduces to $\mathrm{MS}[(w_S^{\gamma}(B) - x_S^j(B))Z_t]$, the MS of the difference of the two estimators $\tilde{S}_t^{\gamma}$ and $x_S^j(B)Z_t$. For the airline model, which we use here, $w_S^{\mu_\gamma}(B) - x_S^j(B)$ always contains $U(B)(1-B)^2 = (1-B)(1-B^s)$, so that $(w_S^{\mu_\gamma}(B) - x_S^j(B))Z_t$ is stationary. Thus, the ACGF of $(w_S^{\mu_\gamma}(B) - x_S^j(B))Z_t$, and hence its MS, can be calculated.

Use of (16) thus implies that, apart from the normalization by the denominator, we measure the distance between the X-12 and model-based filters by comparing the mean squared difference of their seasonal component estimators. In contrast Depoutot and Planas (1998), hereafter DP, directly compared filter weights from X-12 and (canonical) model-based filters. Their criterion for comparing an X-12 filter, $x_S^j(B)$, with a canonical model-based filter, $w_S^0(B)$, can be written

$$\sum_h (w_{S,h}^0 - x_{S,h}^j)^2 = (2\pi)^{-1} \int_{-\pi}^{\pi} \left| w_S^0(e^{-i\lambda}) - x_S^j(e^{-i\lambda}) \right|^2 d\lambda. \tag{18}$$

Equation (18) can be thought of as measuring the mean squared difference of two "seasonal component estimators" obtained by applying the X-12 and canonical model-based filters to a white noise series (with variance 1). Our criterion measures the mean squared difference of the two estimators of the canonical seasonal component of the series Z_t. This can be written as

$$\mathrm{MS}[(w_S^0(B) - x_S^j(B))Z_t] = \int_{-\pi}^{\pi} \left| w_S^0(e^{-i\lambda}) - x_S^j(e^{-i\lambda}) \right|^2 f(\lambda) d\lambda \tag{19}$$

where $f(\lambda) = (2\pi)^{-1}A_Z(e^{-i\lambda})$ is the spectral density of Z_t. This weights the squared difference of the X-12 and canonical model-based filters at each frequency λ by the value of the spectral density $f(\lambda)$ at that frequency. (DP actually write their comparison criterion as $\pi^{-1}\int_0^\pi \left|w_S^0(e^{-i\lambda}) - x_S^j(e^{-i\lambda})\right|^2 d\lambda$, a form that is equivalent to (18) for symmetric filters but not for asymmetric filters. They also start with $d\lambda$ replaced by $dm(\lambda)$, where $m(\lambda)$ is a general measure on $[0, \pi]$, though they explicitly consider only Lebesgue measure, i.e., $d\lambda$. Note that setting $dm(\lambda) = f(\lambda)d\lambda$ yields our criterion (19). Finally, despite the difference between (18) and (19), DP's choices of seasonal moving averages (for symmetric filters and the canonical decomposition), are essentially in agreement with those that we report in the next section. Our focus here is not just on the best X-12 filter choices, but also on the MSEs related to the X-12 filters, and particularly on how these compare, via (16), to the MSEs of the model-based filters.)

4 Symmetric Filter Comparisons

In this and the next section we apply the results of Section 3 to the airline model with various parameter values for monthly time series Z_t,

$$(1 - B)(1 - B^{12})Z_t = (1 - \theta_1 B)(1 - \theta_{12}B^{12})a_t, \tag{20}$$

where a_t is a normally distributed white noise series with variance that we set to unity. Our objectives are to (i) compare the average MSEs for various X-12 filters to the average MSE of the optimal model-based filter, and (ii) determine which of a set of J X-12 seasonal filters minimizes the average MSE, $\text{MS}[g_t^{\mu_\gamma, j}]$. We focus on the cases of the canonical decomposition ($\mu_\gamma = 0$) and the uniform prior on γ ($\mu_\gamma = \gamma_{\max}/2$).

We study the airline model (20) because it is probably the most commonly used model for analyzing seasonal time series. We restrict consideration to nonnegative values of θ_1 and θ_{12}. One reason for this is that the condition $\theta_{12} \geq 0$ is needed for an acceptable decomposition to exist (HT, p 67). A second reason is that, in practice, estimated models tend to satisfy these constraints. DP modeled over 7000 series with the airline model and found that about 97 percent of the estimates of the model parameters (θ_1, θ_{12}) were positive.

The symmetric X-12 filters are determined by the choices of seasonal and trend moving averages (MAs) that are applied in X-12's iterative filtering calculations. See Bell and Monsell (1992), Findley et al (1998), or Chu, Tiao and Bell (2002) for details. As notation for the X-12 filters we write, for example, S3335H13 to denote the X-12 seasonal filter that results when the first seasonal MA is the 3×3 ($(1/9)(F^{12}+1+B^{12})(F^{12}+1+B^{12})$), the second seasonal MA is the 3×5 ($(1/15)(F^{12}+1+B^{12})(F^{24}+F^{12}+1+B^{12}+B^{24})$), and the trend MA is the 13-term symmetric Henderson MA. Findley et al (1998, pp. 149-151) and Dagum (1985, p. 634) discuss the Henderson trend MAs.

For the set J of X-12 symmetric seasonal filters that we consider here it would be desirable, in principle, to include all the possibilities, i.e., those resulting from all possible combinations of X-12's seasonal and trend

MAs. This would involve, however, a large number of combinations, and would include several sets of filters not appreciably different from one another. (For example, Bell and Monsell (1992) note that filters with S3335 versus S3535, and having the same Henderson trend MA, are not appreciably different.) We thus restrict J to contain the 20 X-12 seasonal filters generated from the combinations of five different seasonal MAs (S3131, S3333, S3335, S3339, and S315315) and 4 Henderson trend MAs (H9, H13, H17, and H23). As will be noted in Section 4.3, the S3333, S3335, and S3339 seasonal MAs, as well as the H9, H13, and H23 Henderson trend MAs, are possibilities that can arise from the X-12-ARIMA automatic filter selection scheme. The S3131 and S315315 seasonal MAs, and the H17 Henderson trend MA, are available as user-specified options.[2]

Section 4.1 following gives MSE comparisons between X-12 and model-based symmetric filters for the canonical decomposition, while Section 4.2 gives MSE comparisons for the case of the uniform prior on γ. MSEs for the symmetric model-based filters were computed from standard signal extraction results (Bell 1984), while those for the symmetric X-12 filters were computed by expanding (12) as discussed in the first part of Section 3. Section 4.3 examines automatic X-12 filter selections for time series simulated from the airline model and notes how these selections compare to the "best selections" as determined in Sections 4.1 and 4.2. One point to note for practical application of the results in this section is that there is an implicit assumption that the time series under consideration is "sufficiently long" for the filters being compared. That is, we implicitly assume the series is long enough and the filter weights die out sufficiently quickly as they reach forward and backward through the series so that the weights that would be applied before the beginning and after the end of the observed series are essentially negligible.

4.1 Comparisons for the Canonical Decomposition

We consider first the case where the true seasonal component is from the canonical decomposition ($\gamma = 0$) of the airline model. Table 1 below shows the best X-12 filter, the minimum MSE in (15), and the best X-12 filter's MSE percentage difference (16) for the combinations where the parameter θ_1 takes one of the values {0.9, 0.7, 0.5, 0.3, 0.1} and the parameter θ_{12} takes one of the values {0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1}. We use fewer values of θ_1 because the variation in the results across different θ_1 values is not that large. We note the following from the results in the table:

1. The percentage increase in MSE from using the best X-12 symmetric filter rather than the optimal model-based filter is generally small for estimating the canonical seasonal. It is generally less than or equal to about 12 percent, except for large values of θ_{12} (.9) or small values of θ_{12} (.1, .2).

Table 1: Symmetric Filter Estimation of the Canonical Seasonal for the Airline Model

(Choices of the best symmetric X-12 filters, their MSE values, and the

percentage increases in MSE over those of the optimal model-based filters)

	$\theta_{12} = 0.9$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
$\theta_1 = 0.9$	S315315-H9	S315315-H9	S3339-H9	S3335-H9	S3333-H9	S3333-H23	S3333-H23	S3333-H23	S3131-H23
	.053235	.079359	.108168	.128642	.142494	.144434	.149712	.158752	.161449
	37.10%	10.68%	9.94%	8.48%	8.04%	4.86%	10.59%	28.24%	59.00%
0.7	S315315-H9	S315315-H9	S3339-H9	S3335-H9	S3335-H23	S3333-H23	S3333-H23	S3333-H23	S3131-H23
	.045764	.067915	.091772	.108704	.118918	.122222	.125617	.132100	.131979
	36.09%	10.45%	9.65%	8.36%	6.69%	4.39%	7.22%	18.42%	31.97%
0.5	S315315-H9	S315315-H9	S3339-H9	S3335-H9	S3335-H23	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.043036	.064493	.086877	.103016	.112077	.117828	.121617	.127458	.126960
	33.35%	9.94%	9.27%	8.30%	5.89%	5.25%	6.91%	14.30%	20.42%
0.3	S315315-H9	S315315-H9	S3339-H9	S3335-H9	S3335-H23	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.045520	.070974	.097316	.117043	.130572	.138997	.148252	.157756	.163579
	29.12%	9.23%	8.95%	7.95%	6.69%	5.89%	9.77%	18.02%	28.91%
0.1	S315315-H9	S315315-H9	S3339-H9	S3335-H9	S3335-H23	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.052639	.085050	.118390	.144067	.163967	.176238	.192599	.207177	.220763
	25.40%	8.63%	8.69%	7.69%	7.37%	6.50%	12.16%	21.22%	36.08%

In each cell, 1st row : the chosen X-12 filter, i.e., j^* as defined by eq. (15);

2nd row: the MMSE value from eq. (15) when t is in the middle of a sufficiently long series;

3rd row: the percentage increase in MSE (X-12 filter compared to optimal model-based filter) from eq. (16).

2. Larger values of θ_{12} imply longer seasonal MAs for the best X-12 filter. This is, of course, to be expected, because as θ_{12} approaches 1 the stochastic seasonal component will tend to become deterministic, to be estimated by the mean for each month. We can roughly summarize the best seasonal MA choices corresponding to given values of θ_{12}. This is done in Table 2.

Table 2: Best Choices of X-12 Seasonal MAs for Estimating the Canonical Seasonal

value of θ_{12}	.1 − .2	.3 − .4	.5 − .6	.7	.8 − .9
best seasonal MA	S3131	S3333	S3335	S3339	S315315

Exceptions to the above choices occur for $(\theta_1, \theta_{12}) = (.9, .2)$, $(.7, .2)$, and $(.9, .5)$, for which the S3333 seasonal MA is best. Figure 1 below shows, though, that in these cases the MSEs with the seasonal MAs shown in Table 2 are only slightly higher. (As noted earlier, DP arrived at essentially these same choices of seasonal MAs.)

3. With one exception the Henderson trend MA chosen is the 9-term for $\theta_{12} \geq .6$ and the 23-term for $\theta_{12} \leq .5$. The one exception is that the 9-term Henderson is chosen for $(\theta_1, \theta_{12}) = (.9, .5)$. Figure 1 shows, though, generally little dependence of the MSEs on the choice of Henderson trend MA.

4. We see that the value of θ_1 has little effect on the choice of seasonal or trend MA for determining the best X-12 filter for estimating the canonical seasonal. More effect from the value of θ_1 would be expected for estimation of the canonical trend component.

5. The X-12 MSEs tend to increase as θ_1 and θ_{12} decrease. The largest MSE shown in Table 1 (.220763) is about five times the smallest (.043036).

The general conclusion from Table 1 is that, except for the largest and smallest values of θ_{12}, little is lost by using the best X-12 symmetric filter instead of the optimal model-based symmetric filter for estimating the canonical seasonal from a series that follows the airline model.

Figure 1 shows how the MSE of the X-12 estimated seasonal varies across different X-12 filters. The figure consists of two columns with three plots each. The first column of plots shows results for the canonical decomposition, and the second column of plots, to be discussed later in Section 4.2, shows results for the uniform prior on γ. The three rows of plots correspond to the values $\theta_{12} = .8$, .5, and .2, respectively. We use these three values to generically represent high, medium, and low values of θ_{12}. Within each plot are sets of results for θ_1 values .8, .5, and .2, as noted. For each θ_1 value the MSEs as plotted are seen to fall into five groups of four values, each group corresponding to a particular choice of X-12 seasonal MA (S3131, S3333, S3335, S3339, and S315315). The four values within each group correspond to the four choices of Henderson trend MAs considered (9-term, 13-term, 17-term, and 23-term, in that order). Two general results are evident from the plots of Figure 1 for the canonical decomposition:

- MSEs are generally insensitive to the choice of Henderson trend MA. Some exceptions occur when a very poor choice is made for the seasonal MA (e.g., with the S3131 seasonal MA when $\theta_{12} = .8$). Keep in mind that these results are for estimation of the canonical seasonal (equivalently, the canonical nonseasonal). We would expect more sensitivity to the choice of Henderson trend MAs in MSEs for X-12 trend estimates.

- Choice of the best seasonal MA is not crucial. For $\theta_{12} = .8$ the S3339 seasonal MA does about as well as the S315315, for $\theta_{12} = .5$ the S3333 does about as well as the S3335, and for $\theta_{12} = .2$ the S3333 does about as well as the S3131. Straying further than this from the best choice of seasonal MA entails a more substantial increase in MSE.

4.2 Comparisons for the Uniform Prior on γ

We now consider the case where γ is unknown and with a uniform prior distribution over $[0, \gamma_{\max}]$. From the Lemma of Section 3.2, the average MSEs, for both the X-12 and model-based filters, are the MSEs of the filters for estimating $S_t^{\mu_\gamma}$. With the uniform prior, or indeed with any symmetric prior for γ in $[0, \gamma_{\max}]$, $\mu_\gamma = \gamma_{\max}/2$. Table 3 gives MSE results analogous to those in Table 1 except that we include more θ_1 values, $\{0.9, 0.7, 0.6, 0.5, 0.4, 0.3, 0.1\}$, because the results here depend more on θ_1. Comparing the results of Table 3 with those of Table 1, we observe the following:

1. The average MSE values in Table 3 are higher than the corresponding values in Table 1, as are the percentage increases. In particular, for large values of θ_{12} the MSEs are much higher, while for small values of θ_{12} the percentage increases are much larger. The higher MSEs in Table 3 could be expected due to the result noted at the end of Section 3.1 that for X-12 filters the MSE in estimating S_t^γ is an increasing function of γ.

2. Much shorter seasonal MAs are chosen as best X-12 filters in comparison to the choices in Table 1: the S3333 seasonal MA is generally best for $\theta_{12} \geq .3$, and the S3131 seasonal MA is generally best for $\theta_{12} \leq .2$. Exceptions are that the S3333 seasonal MA is chosen for $(\theta_1, \theta_{12}) = (.9, .2)$ and $(.7, .2)$, the

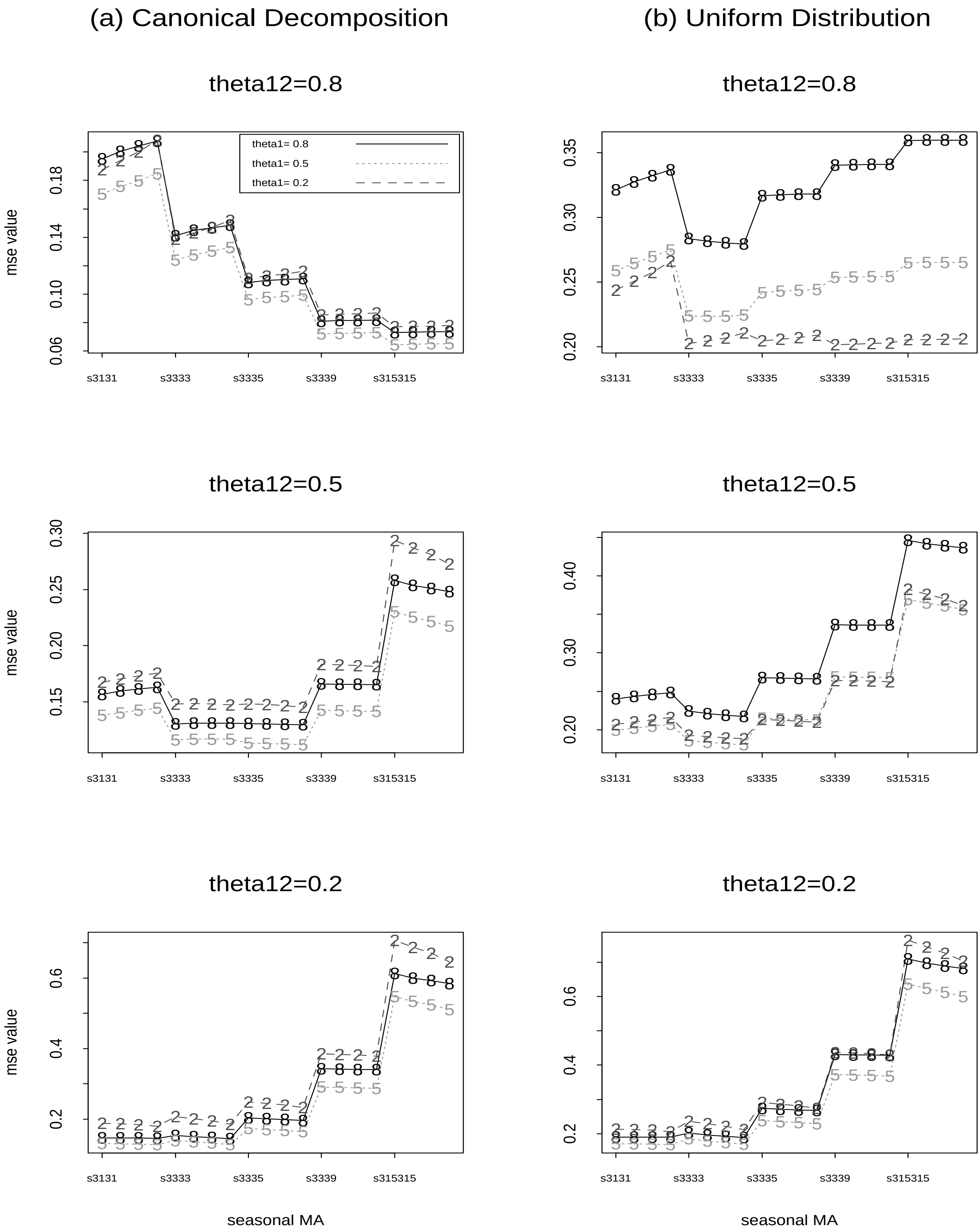

Figure 1: The MSEs when using various X-12 symmetric seasonal filters to estimate the seasonal component of the airline model with various parameter values ((a) canonical decomposition, (b) uniform prior on γ)

S315315 for $(.3, .9)$ and $(.1, .9)$, and the S3339 for $(.1, .8)$. Apart from the cases where the S315315 and the S3339 seasonal MAs are chosen, we see that for estimating the seasonal component $S_t^{\mu_\gamma}$ with $\mu_\gamma = \gamma_{\max}/2$ (which contains half the available white noise), short X-12 seasonal MAs are generally the best.

3. There is more variation in the best choice of Henderson trend MA in Table 3 than in Table 1. For $\theta_{12} \leq .5$ the 23-term is always chosen, but for $\theta_{12} \geq .6$ the Henderson trend MA chosen ranges from the 9-term to 23-term as θ_1 increases from .1 to .9. Figure 1 shows, though, that for a given seasonal MA the MSEs for estimating $S_t^{\mu_\gamma}$ with an X-12 filter over the different choices of Henderson trend MAs usually don't vary much.

4. Contrary to the results of Table 1, in Table 3 the X-12 filter MSEs generally increase consistently with θ_{12}, except for $\theta_1 = .1$ or .3.

The second column of 3 plots in Figure 1 shows how the MSEs under the uniform prior on γ vary across alternative X-12 seasonal filters. As with the canonical decomposition, for a given seasonal MA the MSEs usually don't vary much over the different choices of Henderson trend MAs, except in a few cases where the seasonal MA is badly chosen. Results on the MSEs with alternative choices of the seasonal MAs are more mixed. For $\theta_{12} = .8$ and $\theta_1 = .2$, MSEs for seasonal MA choices other than the S3131 are not very different, but for $\theta_1 = .5$ or .8 (especially) more is lost by not choosing the best seasonal MA (S3333). For $\theta_{12} = .5$ and $\theta_1 = .5$ or .2, MSEs with the S3131, S3333, and S3335 seasonal MAs are similar, but with other seasonal MAs the MSEs are higher. For $\theta_{12} = .5$ and $\theta_1 = .8$, the MSEs with the S3333 seasonal MA are lower than those with the S3131 and much lower than the MSEs with other seasonal MAs. Finally, for $\theta_{12} = .2$ and any value of θ_1, the MSEs with the S3131 and S3333 seasonal MAs are similar while the MSEs with the other seasonal MAs are higher.

Table 3: Symmetric Filter Estimation of the Seasonal Component for the Airline Model with a Uniform Prior on γ

(Choices of the best symmetric X-12 filters, their MSE values, and the percentage increases in MSE over those of the optimal model-based filters)

	$\theta_{12} = 0.9$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
$\theta_1 = 0.9$	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3131-H23
	.337827	.306658	.279582	.256596	.237706	.222907	.212203	.205590	.192544
	51.23%	46.79%	44.57%	45.19%	49.66%	59.76%	78.87%	114.37%	171.27%
0.7	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3131-H23
	.282400	.256544	.234234	.215491	.200263	.188600	.180487	.175922	.164770
	42.99%	27.40%	34.61%	34.02%	36.62%	43.53%	56.63%	79.50%	106.86%
0.6	S3333-H17	S3333-H17	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.261487	.237973	.217683	.200714	.187153	.177003	.170262	.166434	.156362
	41.59%	35.07%	30.66%	28.67%	29.56%	34.04%	43.25%	58.66%	73.28%
0.5	S3333-H13	S3333-H13	S3333-H17	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.244459	.223376	.206016	.191643	.180633	.173212	.169378	.167617	.160713
	41.89%	33.46%	27.88%	24.82%	24.83%	28.63%	37.24%	51.07%	65.30%
0.4	S3333-H9	S3333-H9	S3333-H13	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.231099	.212460	.198132	.187121	.178919	.174656	.174335	.175212	.172117
	44.00%	32.81%	25.61%	21.69%	17.23%	24.20%	32.77%	46.00%	61.59%
0.3	S315315-H9	S3333-H9	S3333-H9	S3333-H17	S3333-H23	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.213244	.205383	.193803	.186636	.181507	.180613	.184146	.187946	.188966
	42.72%	33.29%	23.78%	19.01%	17.29%	20.34%	28.76%	41.44%	58.27%
0.1	S315315-H9	S3339-H9	S3333-H9	S3333-H9	S3333-H23	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.172726	.191630	.197003	.197020	.199585	.206058	.218331	.228835	.238992
	32.71%	29.98%	22.10%	14.75%	12.02%	14.21%	22.69%	25.67%	53.28%

In each cell, 1st row : the chosen X-12 filter, i.e., j^* as defined by eq. (15);

2nd row: the MMSE value from eq. (15) when t is in the middle of a sufficiently long series;

3rd row: the percentage increase in MSE (X-12 filter compared to optimal model-based filter) from eq. (16).

4.3 Comparisons to X-12-ARIMA Automatic Filter Selections

In this section we compare the previous results on the "best" X-12 filter selections to automatic filter selections that are made by the X-12-ARIMA program (when the user does not select a specific filter). According to the X-12-ARIMA reference manual (U.S. Census Bureau, 2002), a 3×3 MA is used to calculate the initial seasonal estimate, then the program chooses whether to use a 3×3, 3×5, or 3×9 seasonal MA based on the size of the "moving seasonality ratio." In our notation, either the S3333, S3335, or S3339 will be selected for the X-12 seasonal MA. Also, for monthly series, either a 9-, 13-, or 23-term Henderson trend MA will be selected based on the size of the $\bar{I}/\bar{C}$ ratio, where $\bar{I}$ and $\bar{C}$ are the average absolute month-to-month changes (percent changes for a multiplicative decomposition) of the estimated irregular and trend-cycle components, respectively. In our notation, either H9, H13, or H23 will be selected for the X-12 Henderson trend MA. For further discussion of the automatic filter selection procedure, see Ladiray and Quenneville (2001).

To examine the automatic filter selection procedure, we simulated 100 time series of length 660 from the airline model with $N(0,1)$ innovations for each of various (θ_1,θ_{12}) combinations. We used long time series so that symmetric filters are effectively applied in the middle of the time series. We list the relative frequency of the seasonal MAs from the automatic selection procedure in the top panel, (I), of Table 4 for comparison to the best X-12 filters listed in panel (II). The latter are obtained from Tables 1 and 3 for the canonical decomposition and for the uniform prior on γ, respectively. The values in parentheses are the proportions of times the various filters were selected over the 100 simulated series. To simplify the table, only the selections for θ_1 and θ_{12} taking on the values (0.9, 0.7, 0.5, 0.3, 0.1) are listed. (Selections of the Henderson trend MAs were also examined, but as Figure 1 shows that these choices rarely have much effect on the MSE, these results are not shown.)

For estimating the canonical seasonal, Table 4 shows that for $\theta_{12} = .7$ or .9, the automatic selection procedure tended to select seasonal MAs which are shorter than optimal. For $\theta_{12} = .9$ the automatic selection procedure mostly selected the S3335, and occasionally selected the S3339, instead of the best choice S315315, a choice not considered by the automatic selection procedure. For $\theta_{12} = .7$ the automatic selection procedure almost always selected the S3335 instead of the best choice S3339. In Figure 1, for $\theta_{12} = .8$ we see that selecting the S3339 instead of the S315315 seasonal MA doesn't increase the MSE very much, but selecting the S3335 instead of the S315315 or S3339 does significantly worse. For $\theta_{12} \leq .5$ the automatic selection procedure tends to do a better job of selecting the seasonal MA, and Figure 1 shows in these cases that even when the best X-12 filter is not chosen the automatic selection procedure generally chooses one with only a slightly higher MSE.

For the uniform prior on γ (equivalently, for estimating $S_t^{\mu_\gamma}$), Table 4 shows that the automatic selection procedure tends to select longer seasonal MAs than the best choices, though Figure 1 shows that sometimes these choices do not increase the MSEs very much. In particular, for $\theta_{12} = .8$ or .5, the automatic procedure's usual choice of the S3335 does poorly for $\theta_1 = .8$, but not as badly for $\theta_1 = .5$ or .2. For $\theta_{12} = .5$ and low

Table 4: Selections of Seasonal MAs for Series Following the Airline Model:
(I) from the X-12-ARIMA automatic filter selection procedure applied to 100 simulated series;
(II) the choices that minimize the MSE as in eq. (15) for either the canonical decomposition or the uniform prior on γ

Panel	$\theta_1 \setminus \theta_{12}$	0.9	0.7	0.5	0.3	0.1
	0.9	S3339(.29)†				
		S3335(.71)	S3335(1.0)	S3335(.96)	S3335(.29)	
				S3333(.04)	S3333(.71)	S3333(1.0)
	0.7	S3339(.23)				
		S3335(.77)	S3335(1.0)	S3335(1.0)	S3335(.16)	
					S3333(.84)	S3333(1.0)
	0.5	S3339(.12)				
(I)		S3335(.88)	S3335(1.0)	S3335(.82)	S3335(.06)	
				S3333(.18)	S3333(.94)	S3333(1.0)
	0.3	S3339(.05)				
		S3335(.95)	S3335(1.0)	S3335(.52)	S3335(.01)	
				S3333(.48)	S3333(.99)	S3333(1.0)
	0.1					
		S3335(1.0)	S3335(.97)	S3335(.22)		
			S3333(.03)	S3333(.78)	S3333(1.0)	S3333(1.0)
(II)	Canonical‡	S315315	S3939	S3335	S3333	S3131
	Uniform‡	S3333→S315315	S3333→S3939	S3333	S3333	S3131

† The values in parentheses are the proportions of times over the 100 simulated series that the given seasonal MA was chosen.

‡ Canonical means canonical decomposition; Uniform means uniform prior on γ.

values of θ_1, the automatic selection procedure frequently makes the best choice of S3333. For $\theta_{12} = .2$, the automatic selection procedure's usual choice of S3333 generally does well, with only slightly higher MSE than the best choice of S3131, a choice not considered by the automatic selection procedure.

To summarize these results, under either the canonical or uniform priors the automatic filter selection procedure tends to make better choices for small than for large values of θ_{12}. For $\theta_{12} \geq .7$, X-12-ARIMA tends to pick seasonal MAs shorter than the best for estimating the canonical seasonal, and longer than the best for estimating under the uniform prior for γ. We must also keep in mind, though, that under the uniform prior even the best X-12 filter choices usually do not do very well. Thus, the best case for the automatic selection procedure's filter choices appears to be estimating the canonical seasonal with a value of $\theta_{12} \leq .5$. One qualification to note is that these results were obtained for time series sufficiently long (660 months) that results for the symmetric X-12 and model-based filters are relevant. It is possible that use of shorter seasonal MAs may do relatively better for estimating the canonical seasonal with time series of shorter lengths more typically encountered in practice (e.g., 10-25 years), even for large values of θ_{12}. Study of this question requires results for finite signal extraction filters and is a topic for future research.

5 Concurrent Filter Comparisons

In this section we provide results analogous to those of Section 4 but for concurrent filters instead of symmetric filters. That is, for series Z_t following airline models with various parameter values, we show the best X-12 concurrent seasonal filter, its average MSE (over $p(\gamma)$), and the percentage increase of this MSE relative to that of the optimal model-based concurrent filter. We present these results both for $p(\gamma)$ corresponding to the canonical decomposition (Table 5) and to the uniform prior (Table 6). MSEs for the model-based concurrent filters were computed as in Watson (1987, Section A.4), an approach also given by Pierce (1980, pp. 99 and 104). Bell and Martin (2004) give a more general discussion offering several alternative approaches to this calculation. MSEs for the X-12 concurrent filters were computed by expanding (12) as discussed in Section 3.

The "concurrent X-12 filters" considered here are those obtained by full forecast extension of the observed series Z_t using the airline model, followed by application of the same set of 20 symmetric X-12 filters considered in Section 4. Note that we could not evaluate MSEs for concurrent seasonal filters obtained from X-12-ARIMA without full forecast extension because without full forecast extension the concurrent X-12 filters contain only the factor $(1 - B)$, not the factor $(1 - B)^2$ required for the error series in (11) to be stationary. For both the X-12 and model-based concurrent filters we assume that the series is long enough so that the filter weights that would be applied prior to the start of the series are negligible, i.e., backcast extension is unnecessary.

Examining the results for the canonical decomposition presented in Table 5 yields mostly similar conclusions to the conclusions drawn for symmetric filters from Table 1. The increases in MSE from using the best X-12 filter are small ($<$ 11 percent) with one exception ($(\theta_1, \theta_{12}) = (.9, .1)$). Longer seasonal MAs are best for larger values of θ_{12}. The 9-term Henderson trend MA is generally best for $\theta_{12} \geq .5$ (with two exceptions) and the 23-term is best otherwise. Finally, the MSEs for the X-12 concurrent seasonal filters tend to increase with decreasing θ_{12}, although exceptions occur in the upper right of the table (small θ_{12}, large θ_1).

Comparing Tables 1 and 5 we note that, for given values of (θ_1, θ_{12}), the MSEs for the best X-12 concurrent filters are larger, often substantially larger, than those for the X-12 symmetric filters. The increases in MSE come, of course, from error due to forecast extension. The MSEs for the model-based concurrent filters are similarly larger than those for the model-based symmetric filters, since we use the same forecast extension for both the X-12 and the model-based filters. An important consequence of this increase in the MSEs for the X-12 and model-based filters is that the percentage increases in MSE from using the best X-12 filters rather than the best model-based filters are less for concurrent filters than they are for symmetric filters. In fact, except for perhaps $(\theta_1, \theta_{12}) = (.9, .1)$, it appears that the increases in MSE from using the best X-12 concurrent seasonal filters rather than the best model-based concurrent seasonal filters are small enough to be ignored.

Table 6 presents results for X-12 concurrent filters when there is a uniform prior on γ. The first three conclusions reached from Table 3 for the symmetric filters generally hold for the concurrent filters: MSE values tend to be higher than for the corresponding results with the canonical decomposition (Table 5); mostly short seasonal MAs are chosen; and the 23-term Henderson trend MA is chosen for $\theta_{12} \leq .5$, with variation in the choice of Henderson trend MAs for other θ_{12} values. In Table 6 the MSEs tend to increase with θ_{12} for $\theta_1 \geq .6$ while the reverse holds for $\theta_1 \leq .4$.

In comparing MSEs between Tables 3 and 6 we generally see an increase in MSE due to error from forecast extension. This increase is substantial in some cases (particularly in the lower right hand corner of the table), but not very large in others. Since the forecast extension error similarly inflates the MSEs for X-12 and model-based concurrent filters, under the uniform prior for γ the increases in MSE from using the best X-12 concurrent filter rather than the optimal model-based concurrent filter are not as large as are the corresponding increases for the symmetric filters. In fact, for $\theta_1 \leq .6$ all the increases are less than 20 percent, and for $\theta_1 \geq .7$ most are not much larger than 20 percent.

There are a few cases in Table 6 where the MSEs for the X-12 concurrent filters are slightly lower than those for the corresponding X-12 symmetric filters (from Table 3). While this cannot happen with the optimal model-based filters (optimal symmetric filters always have lower MSEs than optimal concurrent filters), it can happen with the X-12 filters since they are not optimal under the model. This phenomenon is most likely to occur when the X-12 symmetric filter is short and far from optimal, and when short-run forecast error is low (which happens with large values of θ_1).

Similar to Figure 1 for symmetric filter comparisons, Figure 2 shows how the MSEs vary across alternative X-12 seasonal filters for concurrent filter comparisons. The same conclusions drawn from Figure 1 for both the canonical decomposition and the uniform prior on γ still generally apply to Figure 2: choice of Henderson trend MA has little effect on the MSEs (with only a few exceptions), and choice of the best seasonal MA is usually not crucial (one or more alternative seasonal MA choices give MSEs close to the best). One interesting observation in Figure 2 is that MSEs for $\theta_1 = 0.2$ in most cases are notably larger than those for $\theta_1 = 0.5$ and $= 0.8$, except in the uniform prior case with $\theta_{12} = 0.8$, and for the canonical decomposition with $\theta_{12} = 0.8$ when using the S3131 and S3333 seasonal MAs.

An important question is whether it is important to make different choices of X-12 filters for the symmetric and concurrent cases. That is, does the best X-12 filter for the symmetric case perform well or poorly in the concurrent case and vice versa? We find that most of the chosen X-12 filters in Table 1 and Table 5 for given (θ_1, θ_{12}) values are the same, and similarly for Table 3 and Table 6. For cases where there is a difference in the filter choices, if we compare the MSE when the best X-12 filter for concurrent seasonal adjustment is applied to symmetric adjustment, with the MSE from the best X-12 filter for symmetric adjustment, or vice versa, we find there is little increase in the MSE from not picking the best X-12 filter for the given situation. Thus, we believe we can get by with a single choice of X-12 filter as the basis for both symmetric and concurrent seasonal adjustment (and presumably for everything in between). A fortunate consequence of taking this approach is that, if the model used for forecast extension is correct, this will minimize mean squared revisions (Geweke 1978, Pierce 1980).

Table 5: Concurrent Filter Estimation of the Canonical Seasonal for the Airline Model

(Choices of the best concurrent X-12 filters, their MSE values, and the percentage increases in MSE over those of the optimal model-based filters)

	$\theta_{12} = 0.9$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
θ_1 =0.9	S315315-H9	S315315-H9	S3339-H9	S3335-H9	S3333-H9	S3333-H23	S3333-H23	S3333-H23	S3131-H23
	.082786	.138064	.182530	.210536	.224784	.224975	.219123	.208031	.186888
	10.44%	3.18%	3.09%	2.39%	2.04%	1.42%	3.87%	10.38%	20.52%
0.7	S315315-H9	S315315-H9	S3339-H9	S3335-H9	S3333-H9	S3333-H23	S3333-H23	S3333-H23	S3131-H23
	.072343	.121951	.162764	.190721	.208636	.214769	.216230	.214057	.203812
	9.90%	2.93%	2.72%	2.07%	1.93%	0.93%	1.94%	5.21%	8.61%
0.5	S315315-H9	S315315-H9	S3339-H9	S3335-H9	S3335-H23	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.069984	.119986	.162464	.194115	.217026	.231001	.240300	.246507	.245520
	9.17%	2.59%	2.35%	1.77%	1.44%	0.78%	1.19%	2.92%	3.97%
0.3	S315315-H9	S315315-H9	S3339-H9	S3335-H9	S3333-H9	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.075999	.133932	.185155	.225895	.258629	.281711	.301301	.316547	.325150
	8.15%	2.32%	2.15%	1.57%	1.45%	0.81%	1.63%	3.25%	4.79%
0.1	S315315-H9	S315315-H9	S3339-H9	S3335-H9	S3333-H17	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.089834	.161806	.227141	.281296	.326676	.362381	.394746	.421064	.440916
	7.25%	2.10%	2.00%	1.43%	1.23%	0.89%	1.99%	3.56%	5.45%

In each cell, 1st row : the chosen X-12 filter, i.e., j^* as defined by eq. (15);

2nd row: the MMSE value from eq. (15) when t is at the end of a sufficiently long series;

3rd row: the percentage increase in MSE (X-12 filter compared to optimal model-based filter) from eq. (16).

Table 6: Concurrent Filter Estimation of the Seasonal Component for the Airline Model with a Uniform Prior on γ

(Choices of the best concurrent X-12 filters, their MSE values, and the percentage increases in MSE over those of the optimal model-based filters)

	$\theta_{12} = 0.9$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
$\theta_1 = 0.9$	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3131-H9	S3131-H23
	.317675	.304292	.289951	.274452	.257931	.240530	.222385	.203146	.179097
	29.06%	26.29%	24.89%	24.76%	26.07%	29.18%	34.82%	44.03%	56.24%
0.7	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3131-H23
	.283655	.278119	.271240	.263141	.253966	.243916	.233150	.221936	.205134
	20.44%	17.47%	15.68%	14.90%	15.11%	16.39%	18.94%	23.06%	26.23%
0.6	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.269679	.268923	.266662	.263057	.258316	.252650	.246267	.239193	.226977
	18.37%	14.76%	12.46%	11.18%	10.81%	11.34%	12.80%	15.23%	16.41%
0.5	S3333-H13	S3333-H17	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.257526	.262848	.266514	.268462	.269118	.268658	.267264	.264490	.256699
	17.38%	12.97%	10.24%	8.65%	8.08%	8.46%	9.77%	11.79%	12.96%
0.4	S3333-H9	S3333-H13	S3333-H17	S3333-H23	S3333-H23	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.247233	.259567	.270192	.278837	.285688	.291210	.295537	.297664	.294783
	17.33%	11.80%	8.59%	6.78%	6.06%	6.39%	7.69%	9.56%	10.95%
0.3	S3333-H9	S3333-H9	S3333-H9	S3333-H17	S3333-H23	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.239179	.259120	.277561	.294016	.307844	.320031	.330762	.338384	.340944
	18.22%	11.09%	7.31%	5.38%	4.53%	4.81%	6.14%	7.90%	9.49%
0.1	S315315-H9	S3939-H9	S3333-H9	S3333-H9	S3333-H23	S3333-H23	S3333-H23	S3131-H23	S3131-H23
	.210889	.261332	.304870	.338923	.369571	.397021	.422458	.442695	.457558
	12.11%	8.05%	5.81%	3.47%	2.60%	2.82%	4.19%	5.83%	7.75%

In each cell, 1st row : the chosen X-12 filter, i.e., j^* as defined by eq. (15);

2nd row: the MMSE value from eq. (15) when t is at the end of a sufficiently long series;

3rd row: the percentage increase in MSE (X-12 filter compared to optimal model-based filter) from eq. (16).

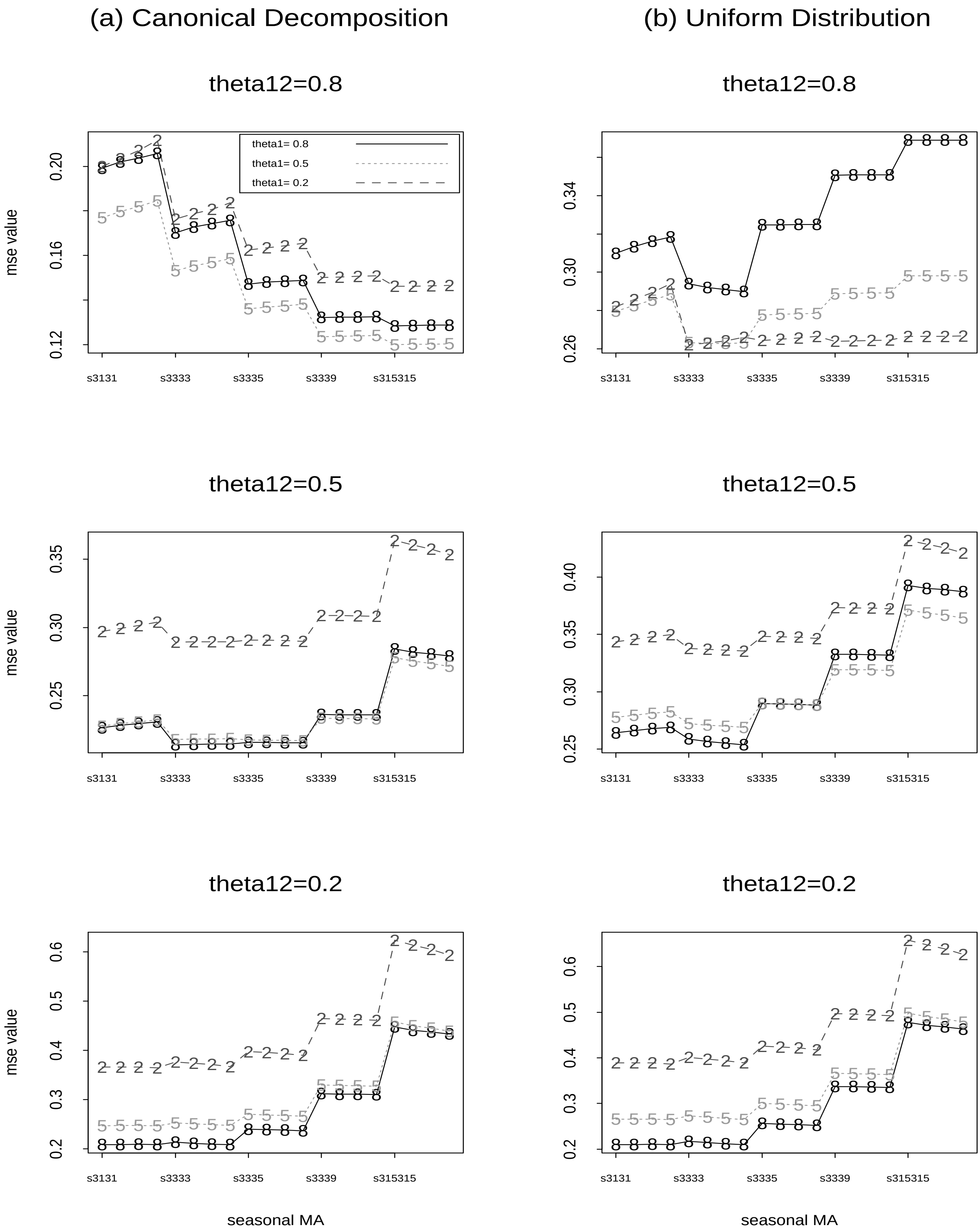

Figure 2: The MSEs when using various X-12 concurrent seasonal filters to estimate the seasonal component of the airline model with various parameter values ((a) canonical decomposition, (b) uniform prior on γ)

6 Conclusions and Topics for Future Research

In this paper we examined the performance of X-12 symmetric and concurrent filters for estimating the seasonal component of a model-based decomposition of a time series following the airline model. The performance was assessed in terms of average MSE for estimating the seasonal component, with these average MSEs compared to those of the optimal model-based filters. The average MSE was computed over a prior distribution for the parameter γ that allocates white noise between the seasonal and nonseasonal components of the model-based decomposition. We considered two priors for γ: the canonical decomposition (a degenerate prior on the minimum value of 0 for γ), and a uniform prior over the admissible range $[0, \gamma_{\max}]$ of γ. A Lemma showed that the average MSE over the prior for γ equals the MSE for estimating the seasonal component, $S_t^{\mu_\gamma}$, corresponding to setting γ equal to its prior mean. For the canonical decomposition the prior mean is just the minimum value 0; for the uniform prior the mean is $\mu_\gamma = \gamma_{\max}/2$.

As a criterion for picking an X-12 filter from among the various possible options, we suggested picking the X-12 filter that minimizes the average MSE for estimating S_t^{γ}, i.e., the X-12 filter that minimizes the MSE for estimating $S_t^{\mu_\gamma}$. Results showed that increases in MSE from using the best X-12 filter rather than the best model-based filter are generally small for the canonical decomposition, especially for concurrent filters. Table 2 provided results relating the best choices of seasonal MAs to values of the seasonal moving average parameter θ_{12} for the airline model. For the uniform prior on γ the MSE increases from using the best X-12 filter are much larger, especially for the symmetric filters.

MSE results for X-12 filters other than the best choices were mixed. Choice of the Henderson trend MA rarely had an appreciable effect on the MSE. Typically, choice of one of the seasonal MAs "close to" the best did not appreciably increase the MSE, but choice of other seasonal MAs could lead to more substantial MSE increases. We also noted that the best choices of X-12 symmetric filters were generally the same as or close to the best choices of X-12 concurrent filters, leading to the conclusion that MSE would not necessarily increase much if one made the same choice of X-12 filter for both symmetric and concurrent seasonal adjustment. (Here the "concurrent X-12 filters" were those resulting from full forecast extension of the time series followed by application of the symmetric X-12 filters. "Same choice" of symmetric and concurrent filters refers to choosing a single symmetric filter as the basis for both symmetric and concurrent adjustments.)

An experiment with time series simulated from the airline model with various parameter values revealed that automatic filter choices made by the X-12-ARIMA program tended to yield the best or close to the best choices of X-12 filters for estimating the canonical seasonal from models with values of $\theta_{12} \leq .5$. For $\theta_{12} > .5$ the X-12 automatic filter choices tended to use shorter seasonal MAs than were best for estimating the canonical seasonal. Under the uniform prior even the best X-12 filters don't do very well, so it appears that X-12-ARIMA with its automatic filter choices fares best for estimating the canonical seasonal when $\theta_{12} \leq .5$.

Several questions remain for future research. One concerns whether similar results to those shown here would be obtained with different models than the airline model? The results here suggest that for other seasonal ARIMA models the value of the seasonal moving average parameter θ_{12} would be an important determining factor in the results. Another question concerns the accuracy of X-12 trend estimates. The approach presented here extends in a straightforward fashion to estimation of the trend component. Results of DP suggest that MSEs for X-12 trend filters would depend on the choices of Henderson trend MAs as well as on the seasonal MAs.

A final question concerns the relative performance of X-12 filters with finite data. The results obtained here assumed series sufficiently long for the symmetric filters to apply without forecast or backcast extension of the series, and for application of the concurrent filters to require only forecast (not backcast) extension. For shorter time series and models with large values of θ_{12}, MSEs will increase both for X-12 and model-based filters. The increase in MSEs should tend to be greater for the model-based than for the best X-12 filters (since the model-based filters tend to be longer), making the relative accuracy losses from using the best X-12 filters in finite series even less than in the results shown here. We have some preliminary results that appear to confirm this.

Notes:

1. We use the term "X-12 filter" to refer to the filters available in the X-12-ARIMA program, though we could equally well use the term "X-11 filter," as is done by some authors, such as DP. The basic filtering approach of X-12-ARIMA (and also of X-11-ARIMA) is that of the original X-11 program, and, in fact, the seasonal adjustment procedure in the X-12 program is referred to as X11. Also, most of the filters used in X-12 were available in the X-11 program, though X-12 does provide some additional choices based on a few seasonal and trend moving averages not available in X-11.

2. There are two minor differences in X-12-ARIMA between automatic selection of a given seasonal MA and user specification of the same MA. First, automatic selection applies only to the second seasonal MA in the X-11 filtering; the first seasonal MA under automatic selection is always the 3×3. User specification, in contrast, applies to both seasonal MAs. Thus, automatic selection of the 3×5 seasonal MA implies, in our notation, the S3335 filter, whereas user specification of the 3×5 seasonal MA implies S3535. As noted in Section 4, these two filters are quite close. The second minor difference is that automatic selection affects only the final iteration (D) of the X-11 procedure, while user specification also determines the filters used in iterations B and C (Ladiray and Quenneville 2001). The B and C iterations are for preliminary and final estimation (by the X-11 procedure, not via the modeling capabilities in X-12) of calendar effects and extreme values. Our focus here is on the final seasonal filtering at iteration D.

References

Bell, W. R. (1984), "Signal Extraction for Nonstationary Time Series," *The Annals of Statistics*, **12**, 646–684.

Bell, W. R. (1992), "On Some Properties of X-11 Symmetric Linear Filters," paper presented at the Seasonal Adjustment Workshop, June 2-3, 1992, Bureau of the Census, U.S. Department of Commerce, Washington, DC.

Bell, W. R. and Hillmer, S. C. (1984), "Issues involved with the seasonal adjustment of economic time series," *Journal of Business & Economic Statistics*, **2**, 291–320.

Bell, W. R. and Martin, D. E. (2004), " Computation of Asymmetric Signal Extraction Filters and Mean Squared Error for ARIMA Component Models," *Journal of Time Series Analysis*, forthcoming.

Bell, W. R. and Monsell, B. C. (1992), "X–11 Symmetric Linear Filters and Their Transfer Functions," Research Report Series No. RR92/15, Statistical Research Division, Bureau of the Census, U.S. Department of Commerce, Washington, D.C.

Bell, W. R. and Otto, M. C. (1992), "Bayesian Assessment of Uncertainty in Seasonal Adjustment with Sampling Error Present," Research Report 92/12, Statistical Research Division, Bureau of the Census, U.S. Department of Commerce, Washington, D.C.

Box, G.E.P. and Jenkins, G. M. (1976), *Time Series Analysis: Forecasting and Control*, 2nd edition, San Francisco: Holden Day.

Burman, J. P. (1980), "Seasonal Adjustment by Signal Extraction," *Journal of the Royal Statistical Society, A*, **143**, 321–337.

Burridge, P. and Wallis, K. F. (1984), "Unobserved Components Models for Seasonal Adjustment Filters," *Journal of Business and Economic Statistics*, **2**, 350-359.

Cleveland, W. P. and Tiao, G. C. (1976), "Decomposition of Seasonal Time Series: A Model for the X-11 Program," *Journal of the American Statistical Association*, **71**, 581-587.

Chu, Y. J., Tiao, G. C. and Bell, W. R. (2002), "Model-Based Interpretation for the Census X-12 Procedure," working paper.

Dagum, E. B. (1975), "Seasonal Factor Forecasts from ARIMA Models," Proceedings of the 40th Session of the International Statistical Institute, Warsaw, 206-219.

Dagum, E. B. (1985), "Moving Averages," in *Encyclopedia of Statistical Sciences*, *Vol. 5*, ed. S. Kotz and N. L. Johnson, New York: John Wiley, 630-634.

Depoutot, R. and Planas, C. (1998), "Comparing Seasonal Adjustment and Trend Extraction Filters with Application to a Model-Based Selection of X11 Linear Filters," Eurostat Working Papers, No. 9/1998/A/9, Eurostat, Luxembourg.

Findley, D. F., Monsell, B. C., Bell, W. R., Otto, M. C., and Chen, B. C. (1998), "New capabilities and methods of the X-12-ARIMA seasonal-adjustment program," *Journal of Business & Economic Statistics*, **16**, 127–176.

Geweke, J. (1978), "Revision of Seasonally Adjusted Time Series," SSRI Report No. 7822, Department of Economics, University of Wisconsin.

Harvey, A. C. (1989), *Forecasting, Structural Time Series Models, and the Kalman Filter*, Cambridge: Cambridge University Press.

Hillmer, S. C. and Tiao, G. C. (1982), "An ARIMA-model-based approach to seasonal adjustment," *Journal of the American Statistical Association*, **77**, 63–70.

Ladiray, D. and Quenneville, B. (2001), *Seasonal Adjustment with the X-11 Method*, New York: Springer-Verlag.

Pierce, David A. (1979), "Signal Extraction Error in Nonstationary Time Series," *The Annals of Statistics*, **7**, 1303–1326.

Pierce, David A. (1980), "Data Revisions With Moving Average Seasonal Adjustment Procedures," *Journal of Econometrics*, **14**, 95-114.

Planas, C. and Depoutot, R. (2002), "Controlling Revisions in ARIMA-Model-Based Seasonal Adjustment," *Journal of Time Series Analysis*, **23**, 194–213.

Shiskin, J., Young, A. H., and Musgrave, J. C. (1967), "The X-11 Variant of the Census Method II Seasonal Adjustment Program," Technical Paper No. 15, U.S. Department of Commerce, Bureau of Economic Analysis.

U.S. Census Bureau (2002), "X-12-ARIMA Reference Manual, Version 0.2.10," Time Series Staff, Statistical Research Division, Washington, DC, available online at http://www.census.gov/srd/www/x12a/.

Watson, M. W. (1987), "Uncertainty in Model-Based Seasonal Adjustment Procedures and Construction of Minimax filters," *Journal of the American Statistical Association*, **82**, 395–408.