Wavelet and Other Multi-resolution Methods for Time Series Analysis

Jeffrey D. Scargle

ABSTRACT Wavelets are the central idea of a broad framework for thinking about, displaying, and analyzing data. These localized basis functions provide flexible and efficient methods for analysis of time series and image data – including noise removal, characterization of stochastic behavior, depiction of power spectrum time-variations, and data compression.

1 Wavelet Analysis of Digital Signals

Wavelets comprise a complete set of basis functions which are localized in both time and frequency. As astronomers understand their great utility in problems of noise reduction, signal characterization and modeling, data compression, and so on, the applications [Sc96] will continue to grow. Within a few years multiresolution techniques will be familiar fare, and wavelets will be more used than Fourier techniques. This section summarizes the special properties which make wavelets so useful.

1.1 Basic Properties

Wavelets are *localized.* That is, each member of the basis is concentrated on a subinterval of T, the total sampled time interval. (In contrast, the Fourier basis functions are all spread over T.) This feature makes them especially useful for signals containing sharply defined, localized features like jumps, jerks, and bumps. But their frequency (or, equivalently, time-scale) properties make them useful as well for representing $\frac{1}{f}$ noise, chirps, fractals, and other multi-scale or self-similar structures.

Another important feature of wavelets is that there is a close relationship between the magnitudes of the wavelet coefficients and the smoothness of the corresponding function – a relationship not shared by Fourier analysis. In simple terms, an attempt to smooth a function by diminishing the amplitudes of some of its wavelet components is guaranteed to succeed – no such guarantee applies to Fourier components. Technically, wavelets form *unconditional bases*, which are in turn optimal bases for data compression and for statistical estimation [Me92a, Me92b, Do93c]. The practical importance is that smoothing algorithms with very nice properties can be implemented using wavelets: decreasing the size (or eliminating entirely) some of the wavelet coefficients results in a relatively noise-free version of the sampled data (see §1.4).

Wavelets elucidate not just fluctuations on a specific scale, but also relationships between those on different scales. One invents display techniques for highlighting behaviors on different scales and visualizing their relationships. One analyzes data (especially images) into the sum of a smoothed and a detailed version, and then treats them separately. One decomposes data into components on a hierarchy of scales, treats them individually, and then re-synthesizes the data; this theme pervades much of wavelet data analysis.

One thinks in terms of not just the harmonic and time-scale content of fluctuation spectra, but of the time-evolution of such spectra, and the detection of sharp ("localized") changes in them. One seeks methods to remove random observational noise from data, without rounding off the sharp edges present in the signal – "denoising without smoothing." And one seeks data analysis and visualization techniques that are suited to the multi-scale behaviors characteristic of $\frac{1}{f}$-noise and chaotic dynamical systems.

This paper is not meant to introduce the reader to technical aspects of wavelets, or to their actual use in data analysis. There is now a large literature on the theory and applications of wavelets; good sources include [Da92, Me92a, Me92b, Me93, Ch92a, Ch92b]. See also [Gr95] and the World Wide Web sites given in §3. Much of the published research is oriented toward abstract functional analysis, but the monograph by Meyer [Me93] discusses the goals of wavelets in data analysis and provides a historical overview, a treatment of time-frequency methods, plus four chapters on applications (computer vision, fractals, turbulence, and the study of distant galaxies). The book [Wi94] is a comprehensive overview of wavelet theory with C source code for many practical algorithms. A recent review [BDG] compares over thirty commercial and public domain wavelet analysis software packages.

Here I will outline the basic definitions for wavelets in which both the data and the scale variable in the transform are discrete – to set the terminology and notation. For definiteness, assume that we have $N = 2^M$ (M an integer) evenly spaced samples of a function X; the data are then

$$X_n, n = 1, 2, 3, \ldots N. \tag{1.1}$$

The *mother wavelet* (also called the *analyzing wavelet*) ψ, can be thought of as a localized shape function, that in turn generates the set of basis functions. For example, the simplest mother wavelet is the *Haar wavelet*, which in discrete time is:

$$\psi(n) = \begin{cases} \frac{1}{2}, & n = 1; \\ -\frac{1}{2}, & n = 2; \\ 0, & \text{all other } n \end{cases} \tag{1.2}$$

The wavelets making up the complete, orthonormal basis are shifted and time-scaled copies of the mother wavelet:

$$\psi_{s,l}(n) = 2^{-\frac{s}{2}}\psi(\frac{n}{2^s} - l). \tag{1.3}$$

The shift and scale factors are defined by the location index l scale index s, respectively. The index ranges

$$s = 0, 1, \ldots, M - 1 \tag{1.4}$$

$$l = 0, 2^{s+1}, \ldots, 2^M - 2^{s+1} \tag{1.5}$$

generate $N - 1$ copies. (For Haar, the remaining degree of freedom is $\overline{X}$.) Since this set is complete and orthogonal, the data can be represented as:

$$X_n = \sum_s \sum_l W_{s,l}\psi_{s,l}(n), \tag{1.6}$$

the *wavelet coefficients* $W_{s,l}$ being defined as the inner product of the corresponding wavelet with the data:

$$W_{s,l} = \sum_n \psi(n)_{s,l} X_n. \tag{1.7}$$

In practice, however, this *wavelet transform* is computed with a fast recursive algorithm.

1.2 Analysis and Visualization of Temporal Structure: The Scalogram

The wavelet transform is a kind of quick-and-dirty time-scale distribution (see §1.5). It is therefore useful to display the log of the absolute value of the wavelet coefficients as a function of scale and location – this is called the *scalogram* [Ri92], or sometimes the *wavelet modulus*. For example, [Go90, Go91] contain pictures of such three-dimensional displays.

Szatmary and co-workers have used scalograms to study changes in the harmonic content of regular and semiregular variable stars [Sz92a, Sz92b]. They carried out simulations on a suite of synthetic signals [Sz93, Sz94], providing useful information on how the wavelet transform of periodic signals responds to things like additive noise, uneven sampling, data gaps, amplitude and phase modulation, phase jumps, period changes, and mode switching.

1.3 Detection/Characterization of 1/f noise: The Scalegram

To study a signal as a function of scale, ignoring location, wavelets offer an alternative to the power spectrum, namely the *scalegram*, defined as the mean square wavelet coefficient (averaged over the location index):

$$V(s) = \frac{2^s}{N} \sum_l (W_{s,l})^2, \tag{1.8}$$

where the l-sum is over the $2^{-s}N$ location index values allowed at scale s [Eq. (1.5)]. To avoid confusion with the scalogram, the scalegram is sometimes called the *wavelet spectrum*. For data with non-normal errors, a modification of the scalegram using the absolute value of the wavelet transform in place of its square is more robust [No94, Cl73]. [CM96] compares the scalegram with other wavelet approaches related to power spectra.

Randomly and/or chaoticly fluctuating systems are of great importance in astronomy [Sc92]. The power spectrum is perforce continuous and usually steeply declining with frequency. The famous "$\frac{1}{f}$" noise and the related *fractal Brownian motion* are examples. Flandrin and co-workers have applied wavelet methods, including the scalegram, to the study of such processes [Fl92a, Fl92b, Ab95, Ab96]. A recent monograph on self-similar processes, with a wavelet flavor is [Wo96].

It is instructive to evaluate the scalegram of a noisy signal and to relate it to the scalegram of the unknown intensity X^{true}. Assume that the observed data consist of a series of photon counts in N evenly spaced time bins. Then X_n is a Poisson-distributed random variable with expected value $E(X_n) = X_n^{true}$, and has the probability distribution

$$Prob\{X_n = k\} = \frac{(X_n^{true})^k e^{-X_n^{true}}}{k!}. \tag{1.9}$$

Let this relationship between the measured and true signals be written

$$\boxed{X_n = Pois(X_n^{true}).} \tag{1.10}$$

It should be stressed that this relation is:

- Very different in form from the additive noise model that most statisticians assume is relevant to scientific data.
- Quite accurate; in practice deviations are small correlations produced by detector dead-time effects, *etc.*, not departures from this formula.

The structure of this non-additive, non-Gaussian noise is determined by the precisely Poisson nature of photon counting.

Assuming that any other errors are in the form of an additive, stationary, normally distributed process R_n, we have for the observed data:

$$X_n = Pois(X_n^{true}) + R_n. \tag{1.11}$$

We assume that both R and the Poisson photon noise are uncorrelated (white) and uncorrelated with each other. A straightforward computation gives

$$E[V^{X_n}(s)] = V^{X_n^{true}}(s) + \sigma_R^2 + \frac{1}{N}\sum_n X_n^{true} \tag{1.12}$$

In short, the net effect of both noise processes is to add a constant to the scalegram. An identical result holds for the power spectrum.

If there is no counting noise, the last term on the right-hand side vanishes, and one needs an estimate of σ_R, *e.g.* from instrumental considerations. If, as is often the case in X-ray or γ-ray data, there is no additive noise, the constant $\frac{1}{N}\sum_n X_n^{true}$ in Eq. (1.12) can be estimated from the total counts, $\frac{1}{N}\sum_n X_n^n$. This procedure seems to work very well in practice [Sc93]. Indeed, using the corrected scalegram we identified self-similar variability in Sco X-1; this led to a model – the "dripping handrail" [Yo96] – for chaotic accretion in this low mass X-ray binary star/disk system. Fritz [FR95] has used the scalegram in a wavelet-based study of flickering in cataclysmic variables.

1.4 Wavelet Denoising Methods

In a marvelous series of papers, Donoho and co-coworkers [Do93a, Do93b, Do93c, Do93d, Do93e, Do93f, Do93g, DJ93a, DJ93b, DJ93c, DJ93d, DJKP93a, DJKP93b] have proposed wavelet procedures for dealing with a number of estimation and time series analysis problems.

Of great potential application in astronomy is *denoising*. These algorithms yield accurate estimates of signals embedded in noise, without loosing sharp features such as steps, bumps, or spikes. This methodology should not be confused with the usual approach to noise reduction by *smoothing*, which invariably blurs such features.

For a practical introduction to these methods, I recommend consulting [Do93f] or the documentation in the **WaveLab** system (see §3) – even if the reader does not use the software itself. For details on estimating unknown functions embedded in noise see [Do93a, Do93b, Do93d]; more technical matters are discussed in [DJ93c]. The use of wavelets to estimate probability distributions is discussed in [DJKP93a]. For an introduction to the theory of wavelet and wavelet packet based de-noising, inverse methods, segmented multiresolutions, and nonlinear multi-resolutions, consult [Do93g, DJKP93b].

Among the mathematical results in these papers are proofs that the basic thresholding technique is amazingly good both in a practical sense (as depicted with synthetic examples) and in a theoretical sense (in that the rate at which the variance of the estimator improves with sample size is greater than for conventional methods – nearly as great as for an ideal estimator [Do93a]).

Kolaczyk [Ko96] has developed denoising theory for the case of Poisson noise. His simulation studies and applications to γ-ray bursts are promising.

Bendjoya, Petit and Spahn [Be93] have developed a procedure that attempts to recognize significant patterns in the wavelet coefficients of noisy data. Their approach begins with the isolation of significant data structures by retaining only wavelet coefficients above a relatively high threshold. On a second pass smaller coefficients are added if they relate in a specified way to those retained in the first pass. For example, coefficients can be added if they exceed a second, lower, threshold and are contiguous in scale-location space to the first-pass coefficients.

1.5 Wavelet Time-Frequency Methods

The topic of time-frequency [Coh89] (and time-scale) representations is wide-ranging and complicated, primarily because its goal (the portrayal of the time-evolution of a quantity that is defined as a time-integral) is somewhat self contradictory. See [Da90] for a wavelet-flavored discussion.

From their definition as localized basis functions, oscillating about zero, wavelets clearly have a potential role in the representation of time-varying spectra [Ri92, Fl92c]. Donoho and co-workers have developed and included in **WaveLab** (see §3) a number of algorithms for carrying out a generalized multi-scale analysis of time series, with the goal of producing time-frequency distributions. The basic idea involves displaying the sum of the time-frequency distributions of the atoms in an atomic representation [see Eq. (1.13) below].

Wavelet methods appear to be able to avoid some unpleasant side-effects of the classical approach to time-frequency distributions, such as the infamous interference terms between multiple periodic components [Coh89, Ka92]. Donoho and von Sachs are developing wavelet based tools for time-frequency and time-scale displays [VS96].

Maurice Priestly, in his contribution to this volume, discusses some problems with the use of wavelets to study time-dependent spectra.

1.6 Data Compression

Most of the "action" in many signals arising in the real world is contained in a few wavelet coefficients. That is to say, only a very few of the N coefficients are significantly different from zero. So if the many small coefficients are discarded, one obtains a very parsimonious but accurate representation of the signal – mage or time series. Indeed, §1.4 above indicates that such a procedure will often discard more noise than signal. With wavelet compresseion one gets denoising free. Donoho [Do93c] describes the further compression obtained by quantizing wavelet coefficients.

1.7 Detection/Characterization of Discontinuities: the CWT

The *continuous wavelet transform* is evaluated at a continuum of scales, not just discrete scales related to each other by factors of 2. It is useful in locating and measuring the order of discontinuities in time series. On a 3D map of the wavelet coefficients, the ridge lines (loci of local maxima) locate discontinuities, and the logarithmic slopes of the ridge lines give the orders [Al95]. This procedure works surprisingly well even in the presence of substantial noise.

1.8 Spectral Estimation

It is perhaps natural that wavelet denoising be used to estimate the notoriously noisy power spectrum. The distribution of the power spectrum is so pathological that the statisticians [Ga93a, Ga93b, Mo92, Mo93, Mo94] have attcked the better-behaved logarithm of the spectrum.

A variant of the STF transform, called the Gabor transform, is being used by a group at NASA-Goddard and elsewhere [Bo94a, Bo94b] to investigate astronomical systems, both observational and theoretical, containing harmonic signals with time-varying frequencies. The goal of their techniques is to provide insight into non-stationary evolution of chaotic and other non-linear physical systems. Boyd *et al.* [Bo94b] studied numerical data from computations of a single star scattering gravitationally off of a binary system. The Gabor transform was used to generate time-frequency plots exhibiting the dual-frequency behavior of this 3-body system, and to show the evolution of the frequencies over time. In addition, they analyzed Hubble telescope photometric data of the star HD 605435 and found evidence that changes in the frequencies are too abrupt to be due to a beat phenomenon, as proposed by other workers.

Stark (personal communication) is experimenting with wavelet denoising of spectra from the complicated multiply periodic power spectra of solar oscillations obtained by the Global Oscillation Network Group (GONG).

1.9 Fixing Up the Wavelet Transform

Here are three practical problems, and their fixes, with the use of the standard waveelt transform.

Edge Effects

The simplest implementations of the wavelet transform assume that the data are periodically replicated outside of the sampled interval. As with Fourier methods, it is desirable to avoid this "wraparound" by zero-padding the data or otherwise. This problem of "wavelet bases for the interval" has been solved by Cohen, Daubechies, Jawerth and Vial [CDJV], by constructing special sets of wavelets such that the inner product in Eq. (1.7) never extends beyond the edges of the sampled data. The **WaveLab** implementation of these left-edge, middle, and right-edge wavelets is called the *CDJV Boundary-Corrected Transform.*

Wavelet Transform of Unevenly Spaced Data

For well-known reasons, astronomical time series are often unevenly spaced. Techniques have been developed to compute – for data with arbitrary sampling in time – Fourier transforms [Sc89], power spectra [Sc82], and correlation functions [Sc89]. The goals are computational techniques for estimating these quantities, and ways to correct for the uneveness of the sampling to the extent possible.

Since the Haar wavelet is piecewise constant, wavelet coefficients can be computed by simply counting the number of points in the corresponding intervals – no matter what the sampling. But it is not obvious which levels in the scale hierarchy to include, since the *minimum scale length* is not well defined. Taking it to be somewhat less than the mean sampling interval is probably reasonable in practice. The only other problem is the treatment of any dyadic intervals that contain no samples – as will sometimes be the case due to pure chance if the sampling is random. In the computation of the scalegram, one should average the squared wavelet coefficients corresponding only to intervals which contain samples.

Lehto [Le96a, Le96b] has used this approach to the scalegram to study unevenly sampled optical observations of OJ 287. He introduced the interesting idea of studying the effects of the sampling by computing the scalegram of white noise sampled at the same times as the real data (the *noisegram*).

Translation-invariant Wavelet Transforms

A third practical problem with wavelets results from their localized nature. Suppose the signal has one or more localized features. Consider those wavelets whose characteristic scale is on the same order as that of the feature – and whose location overlaps the feature. The coefficients of such wavelets will depend strongly on the relative locations of the wavelet and the feature. This renders the analysis sensitive to the whims of how the wavelets fall.

Coifman and Donoho [Co95] have solved this problem with a special *translation invariant wavelet transform* that, in essence but not directly, considers all possible rotations of the data by one sample (with wraparound) – thus in effect averaging over all phases. The **WaveLab** software contains tools for implementing this extension of the ordinary wavelet transform.

2 Beyond Wavelets

There is a class of signal processing methods in which overcomplete sets of functions, some of which are wavelets or wavelet-like, are used in place of complete, orthogonal bases. The general idea is that we want to represent the observed signal as a linear superposition of the form

$$X_n = \sum_k c_k \psi_k(n), \quad n = 1, 2, \ldots, N. \tag{1.13}$$

The c_k are constant coefficients, and the ψ_k belong to the set

$$D = \{\psi_k(n)\}, k = 1, 2, 3, \ldots, \tag{1.14}$$

chosen so that any function of interest can be accurately, but not necessarily uniquely, so represented.

The relation in (1.13) is called an *atomic representation* of the data, the functions ψ_k are called *atoms*, and D is a *dictionary*. Behind these colorful terms is the idea that the data are like a body of information that can be described by a sequence of words (atoms) listed in a dictionary. The same information can be translated to another language, using words from its dictionary. Even more generally, one can construct multi-lingual libraries, the shelves of which house many complete dictionaries. A expression (read "signal") can be represented in many ways in any of these languages. It is nevertheless understandable that the data may find its most efficient representation in one language. Thus to express a given idea, in *matching pursuit* we seek the best wording, in *basis pursuit* the best language. As with human languages, the best choice depends on the data – French is the language of philosophy, Italian the language of love, *etc.*

Atoms are intended to be elementary building blocks, well suited to compose the structures known or likely to be present in the signal of interest. Often the atoms are simple in form; atomic representation then amounts to *assembling a complicated structure out of simple building blocks.* An example is classical Fourier analysis, where the building blocks – sine and cosine functions – comprise a complete, orthonormal set which provides a unique expansion in the form in Equation (1.13) for any sufficiently well-behaved function.

The Fourier basis functions – sines and cosines – are precisely localized in frequency, but completely unlocalized in time. Atoms that, on the contrary, are well localized in both time and frequency are important in theory and applications, and are called *time-frequency atoms.* An example is the windowed or short-time Fourier transform (STF), in which the dictionary consists of sines and cosines modulated by a (sliding) localizing window function (see [Da92] for the relation of STF to wavelets, and [Coh89] for its shortcomings).

One general way to make a dictionary is to pick a function $g(t)$, and then construct copies of it shifted, $g(t-l)$, and scaled, $g(\frac{t}{S})$, in time. If g is localized in time, and its Fourier transform is localized in frequency, then these copies – $g(\frac{t-l}{S})$, for all allowed values of S and l – comprise a dictionary of *time-frequency atoms*. Wavelets are an example of this construction. In some applications it makes sense to also frequency-modulate the function.

But in the most general setting atoms are arbitrary functions, considered in some way to be suitable for piecing together functions of interest.

If the dictionary D is a complete, orthogonal set (*i.e.*, a basis), then each f has a unique atomic representation, and we have an invertible transformation between the data and the coefficients c_k. Fourier and wavelet transforms are examples. But we allow D to be *overcomplete*. Since each f can be constructed in a number of ways, one wants to find the linear combination of basis functions that best represents f. There are many choices for what "best" means in this context – for example, the one with the fewest number of atoms.

Further, one can define a set of dictionaries, called a *library*; then the goal is to find the best dictionary, and in turn its best representation of the data. Wavelet packets and cosine packets [Co92] are examples of such libraries.

An important case is that in which the functions $\psi_k(t)$ can be grouped into subsets S_k of D that correspond to different "scales of variation." Start with a basic function $\psi(t)$ confined to the unit interval $0 \leq t \leq 1$. Define S_0 to be the set consisting of this function and all copies of it translated in time by integer multiples of one unit – *i.e.*, $\psi(t-l), \quad l = 0, \pm 1, \pm 2, \ldots$.

Then consider the functions $\psi(\frac{t}{2^s})$ – stretched out or compressed versions of $\psi(t)$, according as s is > 0 or < 0. Define D_s as the set of this scaled function translated by all integer multiples of 2^s. This generates a *hierarchy of scales*. The set of all linear combinations of functions in D_s is completely contained in the set of all linear combinations of functions in D_{s-1}. Any linear combination of functions in D_s is said to *vary on the time scale* 2^s. This abstraction, known as *multiresolution analysis* (MRA), was invented and rigorously developed in 1986 by Mallat and Meyer, as described in [Da92, Me92b]. Wavelets are an example.

3 WaveLab – Wavelet and MRA software tools

There is a tremendous amount of information about wavelets available on the Internet and the World Wide Web (WWW). TeX/LaTeX and other versions of most of the papers by Donoho and co-workers are readily available on the Internet. Do an ftp to the node **playfair.stanford.edu** as "anonymous," enter your internet address in lieu of a password, change to the directory ./pub/donoho, (type "cd pub/donoho"), and then "get" followed by the name of the file you want. This can be done for any of the papers for which a file name in the form ——.tex appears in the bibliography below;

the WWW address is `http://playfair.stanford.edu/~donoho/`. In addition, a software package called **WaveLab**, implementing all of the wavelet operations discussed here and many more, can be obtained at no cost from this WWW site: `http://playfair.stanford.EDU:80/~wavelab/`. *Wavelet Digest* is a good general source, with pointers to papers and meetings, plus a Q & A forum: `http://www/math.scarolina.edu/~wavelet/`. Fionn Murtagh has an extensive bibliography on astronomical applications of wavelets, multiresolution, noise suppression, filtering, image restoration and compression at `http://http.hq.eso.org/~fmurtagh/wavelets.html`.

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