Physica 6D (1983) 179–194
North-Holland Publishing Company

THE HAMILTONIAN STRUCTURE OF CLASSICAL CHROMOHYDRODYNAMICS

John GIBBONS*, Darryl D. HOLM and Boris KUPERSHMIDT†
Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA

Received 8 July 1982

Noncanonical Hamiltonian structures are presented both for Yang–Mills/Vlasov plasmas and for ideal fluids interacting with Yang–Mills fields. The Hamiltonian structure for the Yang–Mills/Vlasov system passes over to that for the Yang–Mills fluid in the "cold-plasma" limit. The resulting Hamiltonian structure is shown to correspond to a Lie algebra.

1. Introduction

The problem treated here is to find the Hamiltonian structure for an ideal fluid in interaction with self-consistent Yang–Mills fields (hence the name: chromohydrodynamics, i.e., the color fluid). The Abelian, electromagnetic case has been treated by Spencer and Kaufmann [1] and by two of the present authors [2].

We use two different methods to attack the problem. The first method is to derive a so-called "Clebsch" representation for the velocity of the fluid by means of a constrained variational principle. The variational principle involves additional variables: Lagrange multipliers (Clebsch variables) which are, by construction, canonical variables. This canonical structure restricts to a non-canonical structure in the original variables. The non-canonical structure obtained this way is not completely satisfactory, however, because the procedure forces a particular choice of gauge.

The second method is gauge invariant. It consists of a series of transformations from a particle system to a fluid description. Starting from a set of particles in a Yang–Mills field, we pass to the Vlasov description of the corresponding many-body system. Then we restrict to the case of a cold plasma, whose equations are equivalent to those of a barotropic fluid. The Hamiltonian structure–which survives these transformations–is gauge invariant, and readily provides the needed gauge invariant extension of the first Hamiltonian structure. The result is a Hamiltonian structure for Yang–Mills fluids which we associate with a certain Lie algebra.

The order of presentation is as follows. Section 2 sets notation for classical Yang–Mills fields.

Section 3 presents Hamilton's principle for the color fluid equations and recasts them as canonical Hamiltonian equations. In order to perform this step, auxiliary variables are introduced as Lagrange multipliers, which incorporate the subsidiary fluid equations into Hamilton's principle. The canonical description appears in section 4 where we (a) identify canonical variables, i.e., coordinate variables and conjugate momenta; (b) define a Hamiltonian in these variables; and (c) regain the color fluid equations as Hamiltonian equations with canonical Poisson structure. In section 5 we show that the canonical Poisson structure is

*Present address: Istituto di Fisica dell'Università, Università degli Studi-Roma, I-00185 Roma, Italy.
†Present address: Mathematics Department, The University of Michigan, Ann Arbor, Michigan 48109, U.S.A.

0167-2789/83/0000–0000/$03.00

compatible with a non-canonical one defined in terms of the original, reduced set of variables provided a certain gauge condition is satisfied. In order to obtain the gauge-invariant Hamiltonian structure of a color fluid, our approach after section 4 shifts to consideration of a barotropic fluid; one whose pressure depends only on its density. The equations of motion of such a fluid are derivable from a Vlasov equation in the "cold plasma" approximation, for which all the particles at a point have the same velocity and charge.

Sections 6 through 9 show how the Vlasov equation inherits a Hamiltonian structure from the particle equations and the barotropic fluid equations in turn inherit a Hamiltonian structure from the Vlasov equation. Gauge-invariance is kept at each step.

Section 6 reviews the Hamiltonian structure of the equations of motion of particles carrying a gauge charge in an external Yang–Mills field. We derive the canonical, nonrelativistic equations of motion, in which the Yang–Mills analog of the Lorentz force appears and 'electric' and 'magnetic' fields are introduced.

Section 7 describes the Hamiltonian structure of the gauge fields in a gauge-invariant manner. Here the scalar potential occurs in the Hamiltonian, but not in the Poisson bracket. On choosing the Coulomb gauge, the scalar potential vanishes, the Hamiltonian reduces to the usual one, and the Hamiltonian structure becomes canonical.

In section 8, the Vlasov equation is derived for the composite field–particle system by using the Klimontovich exact distribution function. In section 9, we consider the cold plasma case and derive the barotropic fluid equations with their associated Hamiltonian structure. Although the cold plasma equations lack a term in the pressure, this may be inserted by adding to the Hamiltonian a term dependent on the density alone. Finally, in section 10, we discuss the mathematical interpretation of the Hamiltonian structure for a color fluid.

2. Notation

The color fluid we treat is an ideal, classical fluid which carries a gauge-charge and moves in $\mathbb{R}^n$ under forces due to self-consistent Yang–Mills fields as well as ordinary pressure forces. Coordinates in $\mathbb{R}^n$ are x_i, $1 \le i \le n$, while $x_0 = t$ is the time component. The fluid variables are: velocity components v_i, $1 \le i \le n$; mass density ρ; specific entropy η; self-consistent Yang–Mills vector potentials A^a_μ, $0 \le \mu \le n$, $1 \le a \le N$; and gauge-charge density G_a. The vector potentials A^a_μ carry both a space–time index, μ, and an internal symmetry index, a. The gauge-charge densities, G_a, also carry an internal symmetry index $1 \le a \le N$, where N is the dimension of the symmetry algebra, $\mathfrak{G}$.

The component notation A^a_μ will be used interchangably with the matrix notation, e.g., $A_\mu = A^a_\mu \hat{e}_a$, where basis matrices $\hat{e}_a$ satisfy Lie-algebra commutation relations of $\mathfrak{G}$,

$$[\hat{e}_a, \hat{e}_b] = \hat{e}_c C^c_{ab}, \tag{1}$$

and one sums on repeated indices over their range as usual.

The Yang–Mills fields, denoted $F_{\mu\nu}$, are related to the potentials A_μ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]. \tag{2}$$

These fields are smooth functions of space and time which take their values in the Lie algebra $\mathfrak{G}$.

We need to consider multiplication in the Lie algebra $\mathfrak{G}$. For α and β in $\mathfrak{G}$ we write their product as $[\alpha, \beta]$, or, if we wish to consider Lie bracket multiplication by α as a linear map of the algebra to itself, we may write

$$[\alpha, \beta] = \mathrm{ad}_\alpha\, \beta = \mathrm{ad}(\alpha)\beta. \tag{3}$$

While the first notation is standard, the second (parenthetical) notation will be used below whenever the expression for α becomes cumbersome. The algebra $\mathfrak{G}$ has a dual, denoted $\mathfrak{G}^*$; the gauge charges of the particles belong to $\mathfrak{G}^*$. The pairing between $\mathfrak{G}^*$ and $\mathfrak{G}$ we denote by the brackets $\langle\, ,\rangle$; for γ in $\mathfrak{G}^*$, and α in $\mathfrak{G}$, their product is written as $\langle\gamma, \alpha\rangle$. To the linear operation ad_α on $\mathfrak{G}$, there corresponds another linear operation, ad^*_α, essentially the transpose, which acts on $\mathfrak{G}^*$ as defined by

$$\langle \mathrm{ad}^*_\alpha\gamma, \beta\rangle \equiv \langle\gamma, \mathrm{ad}_\alpha\beta\rangle. \tag{4}$$

We are now in a position to define the $(n+1)$ operators of covariant differentiation, which act on $\mathfrak{G}$-valued functions of space and time. They are

$$\boldsymbol{D} = \boldsymbol{\nabla} - \mathrm{ad}_{\boldsymbol{A}}, \tag{5}$$

with n components

$$D_i = \frac{\partial}{\partial x_i} - \mathrm{ad}_{A_i}, \tag{6}$$

and

$$D_t = \frac{\partial}{\partial t} - \mathrm{ad}_{A_0} \tag{7}$$

for the time component.

Similarly, one defines other operators, which act on $\mathfrak{G}^*$-valued functions

$$\boldsymbol{D}^* = \boldsymbol{\nabla} + \mathrm{ad}^*_{\boldsymbol{A}} \tag{8}$$

with components

$$D^*_i = \frac{\partial}{\partial x_i} + \mathrm{ad}^*_{A_i}, \tag{9}$$

and

$$D^*_t = \frac{\partial}{\partial t} + \mathrm{ad}^*_{A_0}. \tag{10}$$

If α and β are functions of space and time with values in $\mathfrak{G}^*$ and $\mathfrak{G}$, respectively, then

$$\frac{\partial}{\partial t}\langle\alpha, \beta\rangle = \langle D^*_t\alpha, \beta\rangle + \langle\alpha, D_t\beta\rangle. \tag{11}$$

From the covariant derivative operators we define the fields,

$$\begin{aligned}[D_i, D_t] &= \mathrm{ad}_{E_i},\\ [D_i, D_j] &= \mathrm{ad}_{B_{ij}}.\end{aligned} \tag{12}$$

This gives field components

$$\begin{aligned}E_i &= -\frac{\partial A_0}{\partial x_i} + \frac{\partial A_i}{\partial t} + [A_i, A_0] = F_{0i},\\ B_{ij} &= -\frac{\partial A_j}{\partial x_i} + \frac{\partial A_i}{\partial x_j} + [A_i, A_j] = -F_{ij}.\end{aligned} \tag{13}$$

We have chosen space to have dimension n; thus, the 1-form E has n components E_i, and the 2-form B has $n(n-1)/2$ independent components $B_{ij}(i<j)$ with the antisymmetry property $B_{ij} = -B_{ji}$.

Finally, to construct the Hamiltonian, we need an invertible linear map from the algebra to its dual. If the algebra has the finite dimension N, then α in $\mathfrak{G}$ may be represented by the operator ad_α, which in any basis on $\mathfrak{G}$ is a $N\times N$ matrix $\alpha_{\kappa\lambda}$, say. Then a map from $\mathfrak{G}\times\mathfrak{G}$ to $\mathbb{R}$ (the Killing form) is provided by the rule,

$$K(\alpha, \beta) = \sum_{\kappa,\lambda}\sum \alpha_{\kappa\lambda}\beta_{\lambda\kappa} = \mathrm{Tr}\,\mathrm{ad}_\alpha\,\mathrm{ad}_\beta \tag{14}$$

for α, $\beta\in\mathfrak{G}$. We then define $^*\alpha$, an element of $\mathfrak{G}^*$, for any α in $\mathfrak{G}$ by

$$\langle {}^*\alpha, \beta\rangle = K(\alpha, \beta). \tag{15}$$

Clearly, $^*\alpha$ is a linear function of α; also, if $\mathfrak{G}$ is semi-simple, this map is invertible. Note that $\langle\alpha^*, [\beta, \gamma]\rangle$ is antisymmetric in all its arguments α, β, γ. Hence, we may derive

$$^*(\boldsymbol{D}\alpha) = \boldsymbol{D}^{*}{}^*\alpha. \tag{16}$$

The Yang–Mills fields satisfy equations

$$*(D_\mu F_{\mu\nu}) = J_\nu, \tag{17}$$

where J_ν, with components $J_0 = G$, $J_i = v_i G$, is the gauge current density, which is conserved,

$$D^*_\nu J_\nu = 0, \tag{18}$$

by antisymmetry of $F_{\mu\nu}$.

Now we shall simply write down the equations of motion for a color fluid and derive their canonical and non-canonical Hamiltonian structure. Our procedure shall adhere to that of refs. 2 and 3.

2. Equations of motion

The classical equations for an ideal color fluid consist of (a) conservation laws for mass, entropy, and gauge charge, (b) dynamical Yang–Mills equations for the self-consistent fields, and (c) the fluid motion equation. In terms of the gauge current,

$$J_\mu = (G, G\boldsymbol{v}), \tag{19}$$

and fields,

$$E_j = F_{0j}, \quad B_{ij} = -F_{ij}, \tag{20}$$

the color fluid equations can be written as

$$\begin{aligned}
\dot\rho &= -\operatorname{div} \rho \boldsymbol{v},\\
\dot\eta &= -\boldsymbol{v}\cdot\nabla\eta,\\
\dot G &= -\operatorname{div} G\boldsymbol{v} - \mathrm{ad}^*(A_0 + \boldsymbol{v}\cdot\mathbf{A})G,\\
\dot{\mathbf{A}} &= \boldsymbol{E} + \boldsymbol{D}A_0,\\
{*\dot{\boldsymbol{E}}} &= G\boldsymbol{v} + {*(\boldsymbol{D}\times\boldsymbol{B})} - \mathrm{ad}^*(A_0)^*E,\\
\dot{\boldsymbol{v}} &= -(v\cdot\nabla)\boldsymbol{v} - \frac{1}{\rho}\nabla p - \frac{1}{\rho}\langle G, (\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\rangle,
\end{aligned} \tag{21}$$

The suggestive notation is

$$(\boldsymbol{D}\times\boldsymbol{B})_j = (D_k B_{kj}), \quad (\boldsymbol{v}\times\boldsymbol{B})_j = v_k B_{kj}, \tag{22}$$

and, for example,

$$\mathrm{ad}^*(A_0) = \mathrm{ad}^*_{A_0}, \tag{23}$$

in accordance with (3). The equations for $\dot G$, $\dot A$, $*\dot E$ re-express in vector notation the current conservation law (18), the field definition (13), and the spatial part of the source equations (17), respectively. The time component of the source equation (17) is the non-abelian version of Gauss's law,

$$-G = *(D_i E_i) = *E_{i,i} + *[E_i, A_i]. \tag{24}$$

This condition is preserved by the equations of motion, as one may readily show by direct computation with the dynamical equations (21).

4. Hamilton's principle

The color fluid equations (21) result from Hamilton's principle,

$$\delta \int \mathscr{L}\,\mathrm{d}t\,\mathrm{d}^n x = 0, \tag{25}$$

with constrained Lagrangian density

$$\begin{aligned}
\mathscr{L} = {} & \tfrac12\rho v^2 - \rho e(\rho, \eta) + \langle *E_i, \dot A_i\rangle - \tfrac12\langle *E_i, E_i\rangle\\
& - \tfrac14\langle *B_{ij}, B_{ij}\rangle\\
& + \langle G, \boldsymbol{v}\cdot\mathbf{A}\rangle + \langle *(D_i E_i) + G, A_0\rangle + \langle D^*_\nu J_\nu, \Gamma\rangle\\
& + \phi(\dot\rho + \operatorname{div}\rho\boldsymbol{v}) - \beta(\dot\eta + \boldsymbol{v}\cdot\nabla\eta)
\end{aligned} \tag{26}$$

defined in the space of dependent variables,

$$\{\boldsymbol{v}; \rho, \phi; \eta, \beta; G, \Gamma; *\boldsymbol{E}, \mathbf{A}; A_0\}. \tag{27}$$

Variations in Hamilton's principle produce the following equations:

$$\begin{aligned}
\delta v &: \tilde{M} = \rho v + \langle G, A\rangle + \langle \mathrm{ad}^*(A)G, \Gamma\rangle \\
&\qquad = \rho\nabla\phi + \beta\nabla\eta + \langle G, \nabla\Gamma\rangle, \\
\delta\rho &: \dot\phi = -v\cdot\nabla\phi + \tfrac{1}{2}v^2 - (e + p/\rho), \\
\delta\phi &: \dot\rho = -\operatorname{div}\rho v, \\
\delta\eta &: \dot\beta = -\operatorname{div}\beta v + \rho\partial e/\partial\eta, \\
\delta\beta &: \dot\eta = -v\cdot\nabla\eta, \\
\delta\Gamma &: \dot G = -\operatorname{div} G v - \mathrm{ad}^*(A_0 + v\cdot A)G, \\
\delta G &: \dot\Gamma = -v\cdot\nabla\Gamma + (A_0 + v\cdot A) + \mathrm{ad}(A_0 + v\cdot A)\Gamma, \\
\delta{*E_i} &: \dot A_i = E_i + D_i A_0, \\
\delta A_i &: {*\dot E_i} = G v_i + {*(D\times B)_i} - \mathrm{ad}^*(A_0){*E_i} \\
&\qquad + v_i\,\mathrm{ad}^*(\Gamma)G, \\
\delta A_0 &: {*(D_i E_i)} + G - \mathrm{ad}^*(\Gamma)G = 0.
\end{aligned} \tag{28}$$

The correct equations for the color fluid are obtained when $\Gamma = 0$, which forces the choice of the "hydrodynamic" gauge,

$$A_0 + v\cdot A = 0. \tag{29}$$

This is the Coulomb gauge in the frame co-moving with the fluid. With this choice of gauge one readily verifies that the variational equations (28) along with the expression for $\tilde{M}$ produce the correct motion equation for the color fluid.

5. Canonical and non-canonical Hamiltonian description

Since we have Hamilton's principle (25) for the color fluid, it is straightforward to verify in the hydrodynamic gauge (29) that eqs. (28) form a canonical Hamiltonian system:

$$\begin{aligned}
\dot p_A &= \{p_A, H\} = -\frac{\delta H}{\delta q_A}, \\
\dot q_A &= \{q_A, H\} = \frac{\delta H}{\delta p_A}.
\end{aligned} \tag{30}$$

These equations reproduce the color fluid equations (28) in the space of canonical variables, with Lie algebra indices written out,

$$\begin{aligned}
&p_A \in \{\phi, \eta, \Gamma^a, {*E_i^a}\}; \quad p_1 = \phi,\ p_2 = \eta,\ p_{a+2} = \Gamma^a, \\
&q_A \in \{\rho, \beta, G_a, A_i^a\}; \quad q_1 = \rho,\ q_2 = \beta,\ q_{a+2} = G_a.
\end{aligned} \tag{31}$$

The canonical pairs are (ϕ, ρ), etc. and the Hamiltonian density is given by

$$\begin{aligned}
\mathscr{H} &= \tfrac{1}{2}\rho v^2 + \rho e(\rho, \eta) + \tfrac{1}{2}\langle {*E_i}, E_i\rangle + \tfrac{1}{4}\langle {*B_{ij}}, B_{ij}\rangle \\
&\quad - \langle {*(D_i E_i)} + G, A_0\rangle, \quad H = \int \mathrm{d}^n x\, \mathscr{H}.
\end{aligned} \tag{32}$$

This Hamiltonian density corresponds to the total, conserved energy under imposition of the Gauss's Law condition, (24), which is preserved by the dynamics. In eq. (32) one uses the map (28a) to evaluate $v(p_A, q_A)$ in the space of canonical variables; and one specifies the gauge only after all variational derivatives have been performed.

So far, canonical variables for the color fluid have been identified in an extended space of the original variables plus auxiliary variables, introduced as Lagrange multipliers. An energy Hamiltonian has also been defined in the extended space of variables, whose canonical Hamiltonian equations reproduce the color fluid equations via the map (28a).

Now our task is to restrict the canonical Poisson structure (30), defined in the extended space, back to the original set of variables. We perform this restriction again via the map (28a). Thereby we derive a non-canonical Poisson structure, which is compatible with the canonical one and reproduces the color fluid equations as a Hamiltonian system in the original variables.

The canonical Poisson bracket in the extended space is given by

$$\{F, H\} = \frac{\delta F}{\delta p_\alpha}\frac{\delta H}{\delta q_\alpha} - \frac{\delta H}{\delta p_\alpha}\frac{\delta F}{\delta q_\alpha} - \frac{\delta F}{\delta {}^*E_j^a}\frac{\delta H}{\delta A_j^a} + \frac{\delta H}{\delta {}^*E_j^a}\frac{\delta F}{\delta A_j^a}, \quad \alpha = 1, 2, \ldots, N+2. \tag{33}$$

We leave the second piece of the canonical bracket unchanged, and consider a transformation from variables (p_α, q_α) into new variables $(\tilde{M}_j, \rho, \sigma, G_a)$

$$\tilde{M}_i = p_\alpha q_{\alpha,i}; \quad \rho = p_1, \sigma = p_1 q_2. \tag{34}$$

As we know from earlier work, [2] and [3], the resulting bracket in $(\tilde{M}_i, \rho, \sigma, G_a, A_k^a, {}^*E_k^a)$ space is given by

$$\begin{aligned} -\{F, H\} = {} & \frac{\delta F}{\delta \rho}\partial_k \rho \frac{\delta H}{\delta \tilde{M}_k} + \frac{\delta F}{\delta \sigma}\partial_k \sigma \frac{\delta H}{\delta \tilde{M}_k} \\ & + \frac{\delta F}{\delta G_a}\partial_k G_a \frac{\delta H}{\delta \tilde{M}_k} \\ & + \frac{\delta F}{\delta \tilde{M}_j}\Big[\rho \partial_j \frac{\delta H}{\delta \rho} + \sigma \partial_j \frac{\delta H}{\delta \sigma} + G_a \partial_j \frac{\delta H}{\delta G_a} \\ & + (\tilde{M}_k \partial_j + \partial_j \tilde{M}_j)\Big]\frac{\delta H}{\delta \tilde{M}_k} \\ & - \frac{\delta F}{\delta A_k^a}\frac{\delta H}{\delta {}^*E_k^a} + \frac{\delta F}{\delta {}^*E_k^a}\frac{\delta H}{\delta A_k^a}. \end{aligned} \tag{35}$$

Up to a minus sign, this is the direct sum of the canonical $(\boldsymbol{A}, \boldsymbol{E})$ bracket with the bracket on the dual to

$$\mathfrak{D}(\mathbb{R}^n) \sim (C^\infty(\mathbb{R}^n) \oplus C^\infty(\mathbb{R}^n) \bigoplus_a C_a^\infty(\mathbb{R}^n)). \tag{36}$$

In the hydrodynamic gauge (29) the following variational derivative vanishes for the Hamiltonian (32),

$$\frac{\delta H}{\delta G_a} = 0. \tag{37}$$

Thus, gauge dependence could be removed at this point simply by observing that the correct gauge-independent equations for the color fluid do result when the bracket (35) is modified as

$$\{F, H\} \rightarrow \{F, H\}' = \{F, H\} - \frac{\delta F}{\delta G_a} G_c C_{ab}^c \frac{\delta H}{\delta G_b} \tag{38}$$

to account for non-commutation of the gauge charges. This heuristic modification, while correct, still requires corroboration by a systematic derivation. This corroboration is provided in the next section by starting from a particle description.

6. The particles

To obtain a gauge-invariant Hamiltonian structure for a color fluid, we reconsider the problem from the viewpoint of a particle description.

Let us consider the Hamiltonian structure of a system of particles, each with unit mass, interacting with an external Yang–Mills field. If the lth particle has position x_l and momentum p_l, each in $\mathbb{R}^n$, and charge g_l in $\mathfrak{G}^*$, then Hamilton's equations are

$$\dot{\boldsymbol{x}}_l = \frac{\partial H}{\partial \boldsymbol{p}_l}, \quad \dot{\boldsymbol{p}}_l = -\frac{\partial H}{\partial \boldsymbol{x}_l}, \tag{39}$$

and the equation of motion for the charge g_l is

$$\dot{g}_l = \mathrm{ad}^*\left(\frac{\delta H}{\delta g_l}\right) g_l. \tag{40}$$

These equations define a Poisson bracket between functions of the $\boldsymbol{x}_l$, $\boldsymbol{p}_l$ and g_l,

$$\{J, K\} = \sum_l \frac{\partial J}{\partial \boldsymbol{p}_l}\cdot\frac{\partial K}{\partial \boldsymbol{x}_l} - \frac{\partial J}{\partial \boldsymbol{x}_l}\cdot\frac{\partial K}{\partial \boldsymbol{p}_l} + \left\langle g_l, \left[\frac{\partial J}{\partial g_l}, \frac{\partial K}{\partial g_l}\right]\right\rangle. \tag{41}$$

To avoid a profusion of indices, we use vector or dyadic notation as much as possible. For instance,

$$\frac{\partial a}{\partial \boldsymbol{x}}\cdot\frac{\partial b}{\partial \boldsymbol{p}}=\sum_{i=1}^{n}\frac{\partial a}{\partial x_i}\frac{\partial b}{\partial p_i},$$

$$[(\nabla \mathbf{A})\cdot \dot{\boldsymbol{x}}]_i = A_{j,i}\dot{x}_j. \tag{42}$$

Expression (41) is a Poisson bracket since it is antisymmetric and satisfies the Jacobi identity. The only non-trivial terms which occur when one checks the Jacobi identity come from the explicit dependence of $\{J, K\}$ on the charges g_l. However, if the Lie bracket $[\,,\,]$ satisfies the Jacobi identity, as is certainly the case, these terms sum to zero.

To describe non-relativistic motion of particles in a gauge field, with vector and scalar potentials $\mathbf{A}$, A_0, which take values in $\mathfrak{G}$, one chooses the Hamiltonian

$$H=\sum_l \tfrac{1}{2}|\boldsymbol{p}_l-\langle g_l, \mathbf{A}(\boldsymbol{x}_l, t)\rangle|^2-\langle g_l, A_0(\boldsymbol{x}_l, t)\rangle. \tag{43}$$

Then the equations of motion for the particles are, in dyadic notation

$$\begin{aligned}\dot{\boldsymbol{x}}_l &= \boldsymbol{p}_l-\langle g_l, \mathbf{A}(\boldsymbol{x}_l, t)\rangle,\\ \dot{\boldsymbol{p}}_l &= \langle g_l, \nabla\mathbf{A}(\boldsymbol{x}_l, t)\rangle\cdot(\boldsymbol{p}_l-\langle g_l, \mathbf{A}(\boldsymbol{x}_l, t)\rangle)\\ &\quad+\langle g_l, \nabla A_0(\boldsymbol{x}_l, t)\rangle,\\ \dot{g}_l &= -\operatorname{ad}^*(A^0(\boldsymbol{x}_l, t)\\ &\quad+(\boldsymbol{p}_l-\langle g_l, \mathbf{A}(\boldsymbol{x}_l, t)\rangle)\cdot\mathbf{A}(\boldsymbol{x}_l, t))g_l.\end{aligned} \tag{44}$$

The third of these may be simplified by using the covariant derivative operators D_t^*, $\boldsymbol{D}^*$, defined in section 2. It becomes

$$\begin{aligned}D_t^* g_l &= -\dot{\boldsymbol{x}}_l\cdot\operatorname{ad}^*\mathbf{A}(\boldsymbol{x}_l, t)g_l\\ &= -\dot{\boldsymbol{x}}_l\cdot\boldsymbol{D}^* g_l,\end{aligned} \tag{45}$$

where in the second step we observe that g_l has no explicit space dependence. The interpretation of (45) is that the covariant derivative of g_l along the trajectory of the lth particle is zero. The analog of this result in electrodynamics is, of course, that the charge is constant.

Let us now differentiate the $\dot{\boldsymbol{x}}_l$ equation with respect to time t, to obtain

$$\begin{aligned}\ddot{\boldsymbol{x}}_l &= \dot{\boldsymbol{p}}_l-\frac{\partial}{\partial t}\langle g_l, \mathbf{A}(\boldsymbol{x}_l, t)\rangle\\ &= \langle g_l, \nabla\mathbf{A}(\boldsymbol{x}_l, t)\rangle\cdot\dot{\boldsymbol{x}}_l+\langle g_l, \nabla A_0(\boldsymbol{x}_l, t)\rangle\\ &\quad-\frac{\partial}{\partial t}\langle g_l, \mathbf{A}(\boldsymbol{x}_l, t)\rangle.\end{aligned} \tag{46}$$

In differentiating $\langle g_l, \mathbf{A}(\boldsymbol{x}_l, t)\rangle$ we should bear in mind not only the explicit dependence of g_l and $\mathbf{A}$ upon t, but also the implicit dependence of $\mathbf{A}(\boldsymbol{x}_l, t)$ upon t through $\boldsymbol{x}_l(t)$:

$$\begin{aligned}\frac{\partial}{\partial t}\langle g_l, \mathbf{A}(\boldsymbol{x}_l, t)\rangle &= \langle \dot{g}_l, \mathbf{A}(\boldsymbol{x}_l, t)\rangle\\ &\quad+\langle g_l, \frac{\partial\mathbf{A}}{\partial t}+(\dot{\boldsymbol{x}}_l\cdot\nabla)\mathbf{A}(\boldsymbol{x}_l, t)\rangle\\ &= \langle -\operatorname{ad}^*(A_0+\dot{\boldsymbol{x}}_l\cdot\mathbf{A})g_l, \mathbf{A}(\boldsymbol{x}_l, t)\rangle\\ &\quad+\left\langle g_l, \frac{\partial\mathbf{A}}{\partial t}+(\dot{\boldsymbol{x}}_l\cdot\nabla)\mathbf{A}(\boldsymbol{x}_l, t)\right\rangle\\ &= -\langle g_l, [(A_0(\boldsymbol{x}_l, t)+\dot{\boldsymbol{x}}_l\cdot\mathbf{A}(\boldsymbol{x}_l, t)), \mathbf{A}(\boldsymbol{x}_l, t)]\rangle\\ &\quad+\left\langle g_l, \frac{\partial\mathbf{A}}{\partial t}+(\dot{\boldsymbol{x}}_l\cdot\nabla)\mathbf{A}(\boldsymbol{x}_l, t)\right\rangle.\end{aligned} \tag{47}$$

Substituting this result into (46) produces

$$\begin{aligned}\ddot{\boldsymbol{x}}_l &= \langle g_l, \nabla\mathbf{A}(\boldsymbol{x}_l, t)\rangle\cdot\dot{\boldsymbol{x}}_l+\langle g_l, \nabla A_0(\boldsymbol{x}_l, t)\rangle\\ &\quad+\langle g_l, [A_0(\boldsymbol{x}_l, t)+\dot{\boldsymbol{x}}_l\cdot\mathbf{A}(\boldsymbol{x}_l, t), \mathbf{A}(\boldsymbol{x}_l, t)]\rangle\\ &\quad-\left\langle g_l, \frac{\partial\mathbf{A}}{\partial t}+(\dot{\boldsymbol{x}}_l\cdot\nabla)\mathbf{A}(\boldsymbol{x}_l, t)\right\rangle\\ &= \left\langle g_l, \nabla A_0(\boldsymbol{x}_l, t), +[A_0(\boldsymbol{x}_l, t), \mathbf{A}(\boldsymbol{x}_l, t)]-\frac{\partial\mathbf{A}}{\partial t}(\boldsymbol{x}_l, t)\right\rangle\\ &\quad+\langle g_l, (\nabla\mathbf{A}(\boldsymbol{x}_l, t)\cdot\dot{\boldsymbol{x}}_l-(\dot{\boldsymbol{x}}_l\cdot\nabla)\mathbf{A}(\boldsymbol{x}_l, t)\\ &\quad+[\dot{\boldsymbol{x}}_l\cdot\mathbf{A}(\boldsymbol{x}_l, t), \mathbf{A}(\boldsymbol{x}_l, t)])\rangle\\ &= -\langle g_l, \boldsymbol{E}(\boldsymbol{x}_l, t)\rangle-\langle g_l, \dot{\boldsymbol{x}}_l\times\boldsymbol{B}(\boldsymbol{x}_l, t)\rangle,\end{aligned} \tag{48}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are as defined in (22). In components, expression (48) can be written as

$$\ddot{x}_l^i = -\langle g_l, E_i(\boldsymbol{x}_l, t)\rangle-\langle g_l, \dot{x}_l^j B_{ji}(\boldsymbol{x}_l, t)\rangle. \tag{49}$$

In 3-dimensional space, and with an Abelian gauge group, this result reduces to the usual Lorentz force law,

$$\ddot{x}_l^i = -g_l(E(x_l, t) + \dot{x}_l \times B(x_l, t)). \qquad (50)$$

Thus, (49) may be regarded as a nonabelian generalization of the equation of motion for particles in an electromagnetic field.

6. The fields

To obtain a more complete description of the system, we must construct the equations of motion for the fields. These follow from a Lagrangian. The Lagrangian density is

$$\mathscr{L} = \tfrac{1}{2}\langle {}^*E_i, E_i\rangle - \tfrac{1}{4}\langle {}^*B_{ij}, B_{ij}\rangle. \qquad (51)$$

Here E and B depend upon A_i, A_0; $\mathscr{L}$ can be written as

$$\mathscr{L} = \tfrac{1}{2}\Big\langle {}^*\Big(\frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x_i} + [A_i, A_0]\Big), \frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x_i} + [A_i, A_0]\Big\rangle + \tfrac{1}{4}\Big\langle {}^*\Big(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_j, A_i]\Big), \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_j, A_i]\Big\rangle. \qquad (52)$$

Since $\mathscr{L}$ is independent of $\partial A_0/\partial t$, variation with respect to A_0 gives

$$\delta\mathscr{L} = \sum_i \langle {}^*E_i, -D_i\delta A_0\rangle,$$

$$\frac{\delta\mathscr{L}}{\delta A_0} = {}^*(\boldsymbol{D}\cdot\boldsymbol{E}), \qquad (53)$$

which must vanish. Consequently, the covariant divergence of $\boldsymbol{E}$ is zero. Variation of A_i gives

$$\delta\mathscr{L} = \langle {}^*E_i, D_t\delta A_i\rangle + \langle {}^*B_{ij}, -D_j\delta A_i\rangle. \qquad (54)$$

Thus, we obtain the equation of motion

$$D_t^{*}{}^{*}E_t = D_j^{*}{}^{*}B_{ij}. \qquad (55)$$

Eqs. (53) and (55) must be supplemented by the Bianchi identities, which relate other derivatives of the fields. For example, consider the Jacobi identity

$$[D_t, [D_i, D_j]] = [[D_t, D_i], D_j] + [D_i, [D_t, D_j]]. \qquad (56)$$

This is the same as

$$D_tB_{ij} = D_jE_i - D_iE_j. \qquad (57)$$

Similarly, the Jacobi identity for D_i, D_j and D_k gives

$$D_iB_{jk} + D_jB_{ki} + D_kB_{ij} = 0. \qquad (58)$$

To pass to a Hamiltonian formulation, we calculate the conjugate momentum to A_i, namely,

$$\frac{\partial\mathscr{L}}{\partial\dot{A}_i} = {}^*E_i, \qquad (59)$$

For the variable A_0, we are unable to pass to a Hamiltonian formulation, since $\mathscr{L}$ does not depend on the time derivative $\dot{A}_0$. However, the function

$$R = \langle {}^*E_i, \dot{A}_i\rangle - \mathscr{L} \qquad (60)$$

expressed in terms of $^*\boldsymbol{E}$, $\boldsymbol{A}$ and A_0 may be considered as a Hamiltonian in the canonical pairs of variables $({}^*E_i, A_i)$ and a Lagrangian in A_0. Such a function is called a Routhian in mechanics textbooks. Here

$$R = \tfrac{1}{2}\langle E_i^*, E_i\rangle + \langle E_i^*, D_iA_0\rangle + \tfrac{1}{4}\Big\langle {}^*\Big(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_j, A_i]\Big), \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_j, A_i]\Big\rangle \qquad (61)$$

and the equations of motion come out in the form

$$\dot{A}_i = \frac{\delta R}{\delta {}^*E_i} = E_i + D_i A_0,$$

$$^*\dot{E}_i = -\frac{\delta R}{\delta A_i} = -D_j^{**}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_j, A_i]\right) - \mathrm{ad}^*_{A_0}{}^*E_i, \tag{62}$$

with the condition

$$0 = \frac{\delta R}{\delta A_0} = -\sum_i D_i^{**}E_i. \tag{63}$$

There is no equation of motion for A_0; this reflects the freedom of choice of gauge. For instance, we may set

$$A_0 = 0 \tag{64}$$

in the local neighborhood of any point. The Routhian then collapses to a function of $^*E_i$ and A_i alone and can be treated (locally) as a Hamiltonian. One may then observe that (63) is preserved by the equations of motion.

8. The Vlasov equation

We now consider the composite system of particles and fields together; the Hamiltonian for this system is just the sum of the particle Hamiltonian (43) and the Routhian for the fields (61), namely,

$$\begin{aligned}\mathcal{H} &= H + R \\ &= \sum_l \left(\tfrac{1}{2}|p_l - \langle g_l, A(x_l, t)\rangle|^2 - \langle g_l, A_0(x_l, t)\rangle\right) \\ &\quad + \int \mathrm{d}^N x (\tfrac{1}{2}\langle {}^*E, \cdot E\rangle + \langle {}^*E, \cdot DA^0\rangle + \tfrac{1}{4}\langle {}^*B_{ij}, B_{ij}\rangle).\end{aligned} \tag{65}$$

The Poisson bracket for this system is the direct sum of those for the particle and field systems separately,

$$\dot{x}_l = \frac{\partial \mathcal{H}}{\partial p_l}, \quad \dot{p}_l = -\frac{\partial \mathcal{H}}{\partial x_l}, \quad \dot{g}_l = \mathrm{ad}^* \frac{\partial \mathcal{H}}{\partial g_l} g_l,$$

$$\dot{A} = \frac{\delta \mathcal{H}}{\delta {}^*E}, \quad {}^*\dot{E} = -\frac{\delta \mathcal{H}}{\delta A}, \quad 0 = \frac{\delta \mathcal{H}}{\delta A_0}. \tag{66}$$

The particle equations are the same as before, in (44),

$$\begin{aligned}\dot{x}_l &= p_l - \langle g_l, A(x_l, t)\rangle, \\ \dot{p}_l &= \langle g_l, \nabla A(x_l, t)\rangle \cdot (p_l - \langle g_l, A(x_l, t)\rangle) \\ &\quad + \langle g_l, \nabla A_0(x_l, t)\rangle, \\ \dot{g}_l &= -\mathrm{ad}^*(A^0(x_l, t) \\ &\quad + (p_l - \langle g_l, A(x_l, t)\rangle) \cdot A(x_l, t)) g_l\end{aligned} \tag{67}$$

The field equations, however, differ from (62) and (63),

$$\begin{aligned}\dot{A} &= E + DA_0, \\ {}^*\dot{E}_i &= -\mathrm{ad}^*(A_0){}^*E_i + {}^*(D_j B_{ij}) \\ &\quad - \sum_{l=1}^{M} g_l(p_l^i - \langle g_l, A^i(x_l, t)\rangle)\delta(x - x_l), \\ D \cdot {}^*E &= -\sum_{l=1}^{M} g_l \delta(x - x_l).\end{aligned} \tag{68}$$

We now consider the case of many particles. To do this, we introduce the exact distribution function, f, on the single-particle phase space whose coordinates are x, p, and g. This space has dimension $2n + N$, where n is the dimension of space, and N the dimension of the Lie algebra, $\mathfrak{G}$. The function f is defined as a sum of δ-functions,

$$f(x, p, g; t) = \sum_l \delta(x - x_l)\delta(p - p_l)\delta(g - g_l). \tag{69}$$

We see that f satisfies

$$\frac{\partial f}{\partial t} = \sum_l \dot{x}_l \cdot \frac{\partial f}{\partial x_l} + \dot{p}_l \cdot \frac{\partial f}{\partial p_l} + \left\langle \dot{g}_l, \frac{\partial f}{\partial g_l}\right\rangle. \tag{70}$$

That is,

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + F \cdot \frac{\partial f}{\partial p} + \left\langle \gamma, \frac{\partial f}{\partial g}\right\rangle = 0, \tag{71}$$

where

$$v = \frac{\partial h}{\partial \boldsymbol{p}},$$

$$\boldsymbol{F} = -\frac{\partial h}{\partial \boldsymbol{x}}, \quad h = \tfrac{1}{2}|\boldsymbol{p} - \langle g, \boldsymbol{A}(\boldsymbol{x}, t)\rangle|^2 - \langle g, A_0(\boldsymbol{x}, t)\rangle, \tag{72}$$

$$\gamma = \mathrm{ad}^*\left(\frac{\partial h}{\partial g}\right)g.$$

In the notation of section 8, (eq. (42)) we may write

$$\frac{\partial f}{\partial t} + \{h, f\}_1 = 0, \tag{73}$$

where we use the single-particle Poisson bracket. This equation is itself Hamiltonian. For if we rewrite the Hamiltonian of the system as,

$$\begin{aligned}\mathcal{H} = \int &(\tfrac{1}{2}|\boldsymbol{p} - \langle g, \boldsymbol{A}(\boldsymbol{x}, t)\rangle|^2 \\ &- \langle g, A_0(\boldsymbol{x}, t)\rangle) f \, \mathrm{d}^n\boldsymbol{x}\mathrm{d}^n\boldsymbol{p}\,\mathrm{d}^N g \\ &+ \int \tfrac{1}{2}\langle \boldsymbol{E}^*, \cdot \boldsymbol{E}\rangle + \langle {}^*\boldsymbol{E}, \cdot \boldsymbol{D}A_0\rangle + \tfrac{1}{4}\langle {}^*B_{ij}, B_{ij}\rangle \,\mathrm{d}^n\boldsymbol{x},\end{aligned} \tag{74}$$

then we obtain trivially

$$h = \frac{\delta \mathcal{H}}{\delta f}. \tag{75}$$

Consequently, eq. (73) becomes

$$\frac{\partial f}{\partial t} + \left\{\frac{\delta \mathcal{H}}{\delta f}, f\right\}_1 = 0. \tag{76}$$

This equation is Hamiltonian when associated with the Poisson bracket

$$\{H, J\}_f = \int f\left\{\frac{\delta H}{\delta f}, \frac{\delta J}{\delta f}\right\}_1 \mathrm{d}^n\boldsymbol{x}\,\mathrm{d}^n\boldsymbol{p}\,\mathrm{d}^N g \tag{77}$$

which satisfies the Jacobi identity for functionals H, J, on the single-particle phase space; the only troublesome terms here vanish because of the Jacobi identity for the single-particle bracket $\{\,,\}_1$.

The Hamiltonian nature of (76) does not depend upon the form of f, so we may safely generalize it to the case when f is smooth. Such an equation, known as a Vlasov equation, may be derived from the physical system under the assumption that two-particle correlations are negligible. The smooth function f is then the average of the exact distribution taken over some ensemble of states.

The equations of motion are thus

$$\begin{aligned}&\frac{\partial f}{\partial t} + \left\{\frac{\delta \mathcal{H}}{\delta f}, f\right\}_1 = 0, \\ &\frac{\partial \boldsymbol{A}}{\partial t} = \frac{\delta \mathcal{H}}{\delta {}^*\boldsymbol{E}}, \\ &\frac{\partial {}^*\boldsymbol{E}}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \boldsymbol{A}}, \\ &\frac{\delta \mathcal{H}}{\delta A_0} = 0.\end{aligned} \tag{78}$$

The Hamiltonian structure (78) is the extension to nonabelian gauge fields of the Marsden–Weinstein structure [5] for the Maxwell–Vlasov equations. The field part of the bracket is the same here as for Maxwell's equations. The difference lies in the particle bracket, from which the f-equation is constructed, for now the charge of a particle is not independent of time. If we restrict this bracket to an Abelian theory, then γ, as defined in (72c) to be the time derivative of the charge of a particle, will vanish. Then the equations reduce to the Maxwell–Vlasov equations; for each charge g, we may consider the single particle phase space of $2n$ dimensions with coordinates $(\boldsymbol{x}, \boldsymbol{p})$, instead of the $(2n + N)$ dimensional phase space we have here. This Poisson bracket then collapses to the direct sum of the Kirillov bracket for the Vlasov equation [6] and the canonical bracket for the field equations, which is the Marsden–Weinstein structure.

The equation for f reads, in full,

$$\frac{\partial f}{\partial t}+(p-\langle g,\mathbf{A}\rangle)\cdot\frac{\partial f}{\partial x}$$
$$+\left[\left\langle g,\frac{\partial}{\partial x}\mathbf{A}\right\rangle\cdot(p-\langle g,\mathbf{A}\rangle)+\frac{\partial A_0}{\partial x}\right]\cdot\frac{\partial f}{\partial p}$$
$$-\left\langle g,\left[(p-\langle g,\mathbf{A}\rangle)\cdot\mathbf{A}+A_0,\frac{\partial f}{\partial g}\right]\right\rangle=0. \qquad (79)$$

From this, the particle density,

$$\rho=\int f\,\mathrm{d}^n p\,\mathrm{d}^N g, \qquad (80)$$

must satisfy

$$\dot\rho+\int(p-\langle g,\mathbf{A}\rangle)\cdot\frac{\partial f}{\partial x}\mathrm{d}^n p\,\mathrm{d}^N g$$
$$+\int\left[\left\langle g,\frac{\partial}{\partial x}\mathbf{A}\right\rangle\cdot(p-\langle g,\mathbf{A}\rangle)\right]\cdot\frac{\partial f}{\partial p}\mathrm{d}^n p\,\mathrm{d}^N g=0. \qquad (81)$$

The term in $\partial A_0/\partial x\cdot\partial f/\partial p$ in (79) is an exact p derivative and has dropped out accordingly. So has the term in $\partial f/\partial g$, an exact g-derivative. Representing g, $\mathbf{A}$, and A_0 in some basis yields

$$\left\langle g,\left[(p-\langle g,\mathbf{A}\rangle)\cdot\mathbf{A}+A_0,\frac{\partial f}{\partial g}\right]\right\rangle$$
$$=C^a_{bc}g_a[(p-\langle g,\mathbf{A}\rangle)\cdot\mathbf{A}^b+A_0^b]\frac{\partial f}{\partial g_c}$$
$$\approx -C^a_{bc}\delta_{ac}[(p-\langle g,\mathbf{A}\rangle)\cdot\mathbf{A}^b+A_0^b]f$$
$$+C^a_{bc}g_a\mathbf{A}^c\cdot\mathbf{A}^b$$
$$=-C^a_{ba}[(p-\langle g,\mathbf{A}\rangle)\cdot\mathbf{A}^b+A_0^b]=0, \qquad (82)$$

which vanishes since the sum C^a_{ba} vanishes. Therefore,

$$\dot\rho+\nabla\cdot\int(p-\langle g,\mathbf{A}\rangle)f\,\mathrm{d}^n p\,\mathrm{d}^N g=0. \qquad (83)$$

Similarly, we may calculate the charge conservation equation. We take

$$G=\int gf\,\mathrm{d}^n p\,\mathrm{d}^N g \qquad (84)$$

and obtain

$$\frac{\partial G}{\partial t}+\int g(p-\langle g\mathbf{A}\rangle)\cdot\frac{\partial f}{\partial x}\mathrm{d}^n p\,\mathrm{d}^N g+$$
$$+\int\left[g\left\langle g,\frac{\partial}{\partial x}\mathbf{A}\right\rangle\cdot(p-\langle g,\mathbf{A}\rangle\right]\cdot\frac{\partial f}{\partial p}\mathrm{d}^n p\,\mathrm{d}^N g$$
$$-\int g\frac{\partial}{\partial g_c}C^a_{bc}g_a[(p-\langle g,\mathbf{A}\rangle)\cdot\mathbf{A}^b+A_0^b]\,\mathrm{d}^n p\,\mathrm{d}^N g$$
$$=0. \qquad (85)$$

which simplifies to

$$\frac{\partial G_c}{\partial t}+\nabla\cdot\int g_c(p-\langle g,\mathbf{A}\rangle)f\,\mathrm{d}^n p\,\mathrm{d}^N g$$
$$+\int C^a_{bc}g_a[(p-\langle g,\mathbf{A}\rangle)\cdot\mathbf{A}^b+A_0^b]\,\mathrm{d}^n p\,\mathrm{d}^N g=0. \qquad (86)$$

This is the same as

$$D_t^* G+\boldsymbol{D}^*\cdot\int g(p-\langle g,\mathbf{A}\rangle)f\,\mathrm{d}^n p\,\mathrm{d}^N g=0. \qquad (87)$$

Similar equations may be calculated for higher moments of the momentum and charge. These, however, carry no more information than the Vlasov equation (79) itself and are much more cumbersome. If we go to the "cold plasma" approximation by taking

$$f=\rho(x,t)\delta(p-\bar p(x,t))\delta(g-\bar g(x,t)) \qquad (88)$$

(an Ansatz which is preserved by the Vlasov equation), then f and all functionals of f depend only upon ρ, $\bar p$, $\bar g$. Consequently, the Hamiltonian structure simplifies and we may derive equations for the functions ρ, $\bar p$ and $\bar g$. This is the subject of the next section.

9. The cold plasma

The state of a cold plasma is uniquely characterized by the following moments;

$$\rho = \int f \, \mathrm{d}^n \boldsymbol{p} \, \mathrm{d}^N g,$$
$$\boldsymbol{M} = \int \boldsymbol{p} f \, \mathrm{d}^n \boldsymbol{p} \, \mathrm{d}^N g = \rho \bar{\boldsymbol{p}}, \tag{89}$$
$$G = \int g f \, \mathrm{d}^n \boldsymbol{p} \, \mathrm{d}^N g = \rho \bar{g}.$$

On the right we have expressed $\boldsymbol{M}$ and G, the momentum density and charge density, in terms of $\bar{\boldsymbol{p}}$ and $\bar{g}$, which are introduced in (88). Suppose the Hamiltonian functional $\mathscr{H}$ depends on f through moments alone. By the chain rule,

$$\frac{\delta \mathscr{H}}{\delta f} = \frac{\delta \mathscr{H}}{\delta \rho} + \boldsymbol{p} \cdot \frac{\delta \mathscr{H}}{\delta \boldsymbol{M}} + \left\langle g, \frac{\delta \mathscr{H}}{\delta G} \right\rangle, \tag{90}$$

f must satisfy

$$\frac{\partial f}{\partial t} + \left\{ \frac{\delta \mathscr{H}}{\delta f}, f \right\}_1 = 0. \tag{76}$$

That is,

$$\frac{\partial f}{\partial t} + \frac{\delta \mathscr{H}}{\delta \boldsymbol{M}} \cdot \frac{\partial f}{\partial \boldsymbol{x}} - \frac{\partial}{\partial \boldsymbol{x}} \left(\frac{\delta \mathscr{H}}{\partial \rho} + \boldsymbol{p} \cdot \frac{\delta \mathscr{H}}{\delta \boldsymbol{M}} + \left\langle g, \frac{\delta \mathscr{H}}{\delta G} \right\rangle \right) \cdot \frac{\partial f}{\partial \boldsymbol{p}} + \left\langle g, \left[\frac{\delta \mathscr{H}}{\delta G}, \frac{\partial f}{\partial g} \right] \right\rangle = 0. \tag{91}$$

One now calculates $\partial \rho / \partial t$;

$$\frac{\partial \rho}{\partial t} = - \int \frac{\delta \mathscr{H}}{\delta \boldsymbol{M}} \cdot \frac{\partial f}{\partial \boldsymbol{x}} - \frac{\partial}{\partial \boldsymbol{x}} \left(\boldsymbol{p} \cdot \frac{\delta \mathscr{H}}{\delta \boldsymbol{M}} \right) \cdot \frac{\partial f}{\partial \boldsymbol{p}} \, \mathrm{d}^n \boldsymbol{p} \, \mathrm{d}^N g = - \nabla \cdot \left(\frac{\delta \mathscr{H}}{\delta \boldsymbol{M}} \rho \right). \tag{92}$$

Similarly, one calculates the equation for charge density,

$$\frac{\partial G}{\partial t} = - \int g \left(\frac{\delta \mathscr{H}}{\delta \boldsymbol{M}} \cdot \frac{\partial f}{\partial x} - \left[\frac{\partial}{\partial \boldsymbol{x}} \left(\boldsymbol{p} \cdot \frac{\partial \mathscr{H}}{\partial \boldsymbol{M}} \right) \right] \cdot \frac{\partial f}{\partial \boldsymbol{p}} + \left\langle g, \left[\frac{\delta \mathscr{H}}{\delta G}, \frac{\partial f}{\partial g} \right] \right\rangle \right) \mathrm{d}^n \boldsymbol{p} \, \mathrm{d}^N g$$
$$= - \nabla \cdot \left(\frac{\delta \mathscr{H}}{\delta \boldsymbol{M}} G \right) + \mathrm{ad}^* \left(\frac{\delta \mathscr{H}}{\delta G} \right) G. \tag{93}$$

The equation for momentum density is found to be

$$\frac{\partial \boldsymbol{M}}{\partial t} = - \int \boldsymbol{p} \left(\frac{\delta \mathscr{H}}{\delta \boldsymbol{M}} \cdot \frac{\partial f}{\partial x} - \frac{\partial}{\partial \boldsymbol{x}} \left(\frac{\delta \mathscr{H}}{\delta \rho} + \boldsymbol{p} \cdot \frac{\delta \mathscr{H}}{\delta \boldsymbol{M}} + \left\langle g, \frac{\delta \mathscr{H}}{\delta G} \right\rangle \right) \cdot \frac{\partial f}{\partial \boldsymbol{p}} \right) \mathrm{d}^n \boldsymbol{p} \, \mathrm{d}^N g$$
$$= - \left(\frac{\delta \mathscr{H}}{\delta \boldsymbol{M}} \cdot \nabla \right) \boldsymbol{M} - \left(\nabla \frac{\delta \mathscr{H}}{\delta \boldsymbol{M}} \right) \cdot \boldsymbol{M} - \left(\mathrm{div} \frac{\delta \mathscr{H}}{\delta \boldsymbol{M}} \right) \boldsymbol{M} - \rho \nabla \frac{\delta \mathscr{H}}{\delta \rho} - \left\langle G, \nabla \frac{\delta \mathscr{H}}{\delta G} \right\rangle$$
$$= - \nabla \cdot \left(\frac{\delta \mathscr{H}}{\delta \boldsymbol{M}} \boldsymbol{M} \right) - \left(\nabla \frac{\delta \mathscr{H}}{\delta \boldsymbol{M}} \right) \cdot \boldsymbol{M} - \rho \nabla \frac{\delta \mathscr{H}}{\delta \rho} - \left\langle G, \nabla \frac{\delta \mathscr{H}}{\delta G} \right\rangle, \tag{94}$$

Thus we can express the Hamiltonian structure in terms of ρ, $\boldsymbol{M}$, and G as

$$\frac{\partial}{\partial t}\begin{vmatrix}\rho\\ G\\ M_i\end{vmatrix} = -\begin{vmatrix}0 & 0 & \nabla_j\rho\\ 0 & -\mathrm{ad}^*(\ )G & \nabla_j G\\ \rho\nabla_i & G\nabla_i & \nabla_j M_i + M_j\nabla_i\end{vmatrix}\begin{vmatrix}\dfrac{\delta\mathcal{H}}{\delta\rho}\\ \dfrac{\delta\mathcal{H}}{\delta G}\\ \dfrac{\delta\mathcal{H}}{\delta M_j}\end{vmatrix}, \tag{95}$$

where the G–G term in the middle is to be read $-\mathrm{ad}^*(\delta\mathcal{H}/\delta G)G$, when expanded. To describe a barotropic fluid, we take as the Hamiltonian

$$\mathcal{H} = \int (|\boldsymbol{M} - \langle G, \boldsymbol{A}\rangle|^2/2\rho - \langle G, A^0\rangle + U(\rho) + \tfrac{1}{2}\langle {}^*E_i, E_i\rangle + \langle {}^*E_i, D_iA_0\rangle + \tfrac{1}{4}\langle {}^*B_{ij}, B_{ij}\rangle)\, \mathrm{d}^n x, \tag{96}$$

which is, apart from the term $U(\rho)$, the restriction of the Hamiltonian (74) to a cold plasma. The Hamiltonian (96) is also the same as (32), in the isentropic case. The term $U(\rho)$ is the internal energy density, which has been introduced so that the fluid will have a pressure. The fluid equations are

$$\begin{aligned}
&\frac{\partial\rho}{\partial t} + \nabla\cdot(\boldsymbol{M} - \langle G, \boldsymbol{A}\rangle) = 0,\\
&\frac{\partial G}{\partial t} + \nabla\cdot(\rho^{-1}G(\boldsymbol{M} - \langle G, \boldsymbol{A}\rangle)) + \mathrm{ad}^*(A^0 + \rho^{-1}(\boldsymbol{M} - \langle G, \boldsymbol{A}\rangle)\cdot\boldsymbol{A})G = 0,\\
&\frac{\partial \boldsymbol{M}}{\partial t} + \rho\nabla\Big(\frac{\partial U}{\partial\rho} - \frac{|\boldsymbol{M} - \langle G, \boldsymbol{A}\rangle|^2}{2\rho^2}\Big) - \langle G, \nabla(A^0 + \rho^{-1}(\boldsymbol{M} - \langle G, \boldsymbol{A}\rangle)\cdot \boldsymbol{A}\rangle + \nabla\cdot\Big(\frac{(\boldsymbol{M} - \langle G, \boldsymbol{A}\rangle)}{\rho}\boldsymbol{M}\Big)\\
&\quad + \Big(\nabla\frac{\boldsymbol{M} - \langle G, \boldsymbol{A}\rangle}{\rho}\Big)\cdot(\boldsymbol{M}) = 0,
\end{aligned} \tag{97}$$

which are to be taken together with the field equations

$$\boldsymbol{D}^*\cdot{}^*\boldsymbol{E} + G = 0, \quad D_i^{*\,*}E_j - \rho^{-1}G(M_j - \langle G, A_j\rangle) = D_i^{*\,*}B_{ij}, \quad \frac{\partial\boldsymbol{A}}{\partial t} = \boldsymbol{E} + \boldsymbol{D}A^0. \tag{98}$$

The fluid equations simplify considerably upon rewriting them in terms of the fluid velocity,

$$\boldsymbol{v} = \frac{1}{\rho}(\boldsymbol{M} - \langle G, \boldsymbol{A}\rangle). \tag{99}$$

Namely,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0, \quad D_t^*G + \boldsymbol{D}^*\cdot(G\boldsymbol{v}) = 0, \tag{100}$$

and

$$\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \rho^{-1}\langle G, \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\rangle + \nabla\frac{\partial U}{\partial\rho} = 0, \tag{101}$$

where $(\boldsymbol{v}+\boldsymbol{B})_i = v_j B_{ji}$. Eq. (101) is the Yang–Mills analog of the equation of motion for a charged barotropic fluid. The first two terms compose the comoving derivative of $\boldsymbol{v}$, the third is the generalization of the Lorentz force, and the last is the pressure term when $\partial U/\partial\rho$ is taken to be the specific enthalpy of the fluid.

The gauge-invariant Hamiltonian structure for the full color fluid equations (21) may be written immediately now, by analogy with (95), as

$$\frac{\partial}{\partial t}\left|\begin{matrix} \rho \\ \sigma \\ G \\ M_i \\ {}^*E_i \\ A_i \end{matrix}\right| = -\left|\begin{matrix} 0 & 0 & 0 & \nabla_j\rho & 0 & 0 \\ 0 & 0 & 0 & \nabla_j\sigma & 0 & 0 \\ 0 & 0 & -\mathrm{ad}^*(\ )G & \nabla_j G & 0 & 0 \\ \rho\nabla_i & \sigma\nabla_i & G\nabla_i & (\nabla_j M_i + M_j\nabla_i) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_{ij} \\ 0 & 0 & 0 & 0 & -\delta_{ij} & 0 \end{matrix}\right| \left|\begin{matrix} \dfrac{\delta\mathscr{H}}{\delta\rho} \\ \dfrac{\delta\mathscr{H}}{\delta\sigma} \\ \dfrac{\delta\mathscr{H}}{\delta G} \\ \dfrac{\delta\mathscr{H}}{\delta M_j} \\ \dfrac{\delta\mathscr{H}}{\delta {}^*E_j} \\ \dfrac{\delta\mathscr{H}}{\delta A_j} \end{matrix}\right|, \tag{102}$$

where $\boldsymbol{M} = \rho\boldsymbol{v} + \langle G, \boldsymbol{A}\rangle$ is the total momentum density and $\mathscr{H}$ is the Hamiltonian given in eq. (32). The corresponding bracket is eq. (38), with $\tilde{\boldsymbol{M}}$ in (35) replaced by $\boldsymbol{M}$.

10. Interpretation of the Poisson bracket

Every Hamiltonian matrix such as (102) linearly dependent upon the field variables must come from a certain Lie algebra [7, 8, 9]. In this section we determine the Lie algebra which is responsible for the linear part of our matrix (102). The bracket (102) is a direct sum of two parts: the field part, $(\boldsymbol{E}, \boldsymbol{A})$, which is canonical; and the fluid part, (M_i, G_a, ρ, σ), which is not canonical. We restrict ourselves to the non-canonical fluid part. The procedure we follow is explained in detail in [2].

Consider a free $C^\infty(\mathbb{R}^n)$-module L of dimension $n+N+2$. For any two elements in L, $P = (X_i, Y^a, Z^\alpha)$, and $Q = (\bar{X}_j, \bar{Y}^b, \bar{Z}^\beta)$, $1 \le i, j \le n$, $1 \le a, b \le N = \dim \mathfrak{G}$, $1 \le \alpha, \beta \le 2$, there is a bracket between them defined by formula (32) of [2],

$$\langle u, [P, Q]\rangle \equiv P^t B Q \pmod{\mathrm{Im} D}, \tag{103}$$

where: u stands for variables M_i, G_a, ρ and σ; $\langle u, R\rangle$ denotes $u_k R_k$, $k = 1, \ldots, n+N+2$; $\mathrm{Im}\, D = \Sigma_{l=1}^n \mathrm{Im}\, \partial/\partial x_l$. Using our matrix B from eq. (102) one finds components of $[P, Q]$:

$$[P, Q]_i = X_j \bar{X}_{i,j} - \bar{X}_j X_{i,j}; \tag{104}$$

$$[P, Q]^c = Y^a \bar{Y}^b C^c_{ab} + X_i \bar{Y}^c_{,i} - \bar{X}_i Y^c_{,i}, \quad c = 1, \ldots, N; \tag{105}$$

$$[P, Q]^\alpha = X_i \bar{Z}^\alpha_{,i} - \bar{X}_i Z^\alpha_{,i}. \tag{106}$$

We see from (104) that the $\boldsymbol{X}$-parts commute as vector fields $\{X_i \partial/\partial x_i\}$; while (106) shows that $\{Z^\alpha\}$ serve as coordinates in $C^\infty(\mathbb{R}^n)(\alpha)$ ((106) is formula (43) of [2]).

Formula (105) is a new feature, not encountered in refs. 2 and 3. This part of the bracket can be interpreted as follows.

Denote by $\hat{\mathfrak{G}}$ the set of smooth functions on $\mathbb{R}^n$ with values in a Lie algebra $\mathfrak{G}$. Pointwise multiplication naturally makes $\mathfrak{G}$ into a Lie algebra. Now the Lie algebra $\mathscr{D}(\mathbb{R}^n)$ of vector fields on $\mathbb{R}^n$ acts on $\mathfrak{G}$ by derivations,

$$X(f^a(x)\otimes e_a) = X(f^a(x))\otimes e_a, \tag{107}$$

for any basis e_a in $\mathfrak{G}$. Thus we can form the new Lie algebra $\mathscr{D}(\mathbb{R}^n)\ominus\mathfrak{G}$ (semidirect product; see [2], section 4). Formula (105) is the component representation of the bracket in $\mathscr{D}(\mathbb{R}^n)\ominus\mathfrak{G}$. Indeed, let us take two elements from $\mathscr{D}(\mathbb{R}^n)\ominus\mathfrak{G}$: $(X; Y) = (X_i\partial/\partial x_i; Y^a\otimes e_a)$ and similarly for $(\bar{X}; \bar{Y})$. Then the bracket of these elements is

$$[(X; Y), (\bar{X}; \bar{Y})] = ([X, \bar{X}]; X(\bar{Y}) - \bar{X}(Y) + [Y, \bar{Y}]). \tag{108}$$

The $[X, \bar{X}]$ part of (108) is, of course, just (104). The rest is

$$X(\bar{Y}) - \bar{X}(Y) + [Y, \bar{Y}] = \{X_i\bar{Y}^c_{,i} - \bar{X}_iY^c_{,i} + Y^a Y^b C^c_{ab}\}\otimes e_c, \tag{109}$$

which is exactly (105).

This reasoning results in the following

Proposition. The natural Poisson bracket on the dual space of the Lie algebra

$$\mathscr{D}(\mathbb{R}^n)\ominus\mathfrak{G}\oplus C^\infty(\mathbb{R}^n)\oplus C^\infty(\mathbb{R}^n)) \tag{110}$$

is the noncanonical part of the Yang–Mills hydrodynamical Poisson bracket (102).

We comment briefly on the case when we have an arbitrary n-dimensional manifold M instead of $\mathbb{R}^n$.

Let π: $T(M)\to M$ be the tangent bundle of M. Let us add to π the trivial bundle $(\mathbb{R}^2\oplus\mathfrak{G})\times M\to M$, and denote the resulting bundle by Π: $E\to M$. Let $\Gamma(\Pi)$ be the set of sections of Π, which can be thought of as the geometric version of the Lie algebra (110). Since Π is a vector bundle, the cotangent bundle $\bar{\Pi}$: $T^*(\Pi)\to M$ (see [10], ch. III) can be reduced to have the same dimension of fibers as that of Π itself. The sections of $\bar{\Pi}$ are analogs of the elements of the dual space of the Lie algebra (110).

For any two Hamiltonians H and G, which are n-dimensional horizontal forms on the jet bundle $\bar{\Pi}_\infty : J^\infty\bar{\Pi}\to M$ ([10], ch. III), the "functional derivatives" have a natural interpretation in terms of the Lie algebra $\Gamma(\Pi)$. Applying then the standard procedure which makes a new Lie algebra out of functions on the dual space to a Lie algebra (see, e.g. [9]) we arrive at a Poisson bracket, which is exactly the non-canonical part of our bracket (102) when written in local coordinates. Thus we may construct globally a Poisson bracket over the manifold M which when written locally is identical to that constructed for $\mathbb{R}^n$.

11. Conclusion

We have derived the Hamiltonian structure (102) for a classical fluid which carries a Yang–Mills field. In physical variables, this Hamiltonian structure is expressible as a direct sum,

$$*(\boldsymbol{M}; \sigma, \rho, G)\oplus(\boldsymbol{E}; \boldsymbol{A}). \tag{111}$$

The physical picture of the dynamics follows accordingly. The quantities σ, ρ, G are dragged along in the fluid motion by Lie-derivation with respect to the "minimal-coupling" momentum density, $\boldsymbol{M} = \rho\boldsymbol{v} + \langle G, \boldsymbol{A}\rangle$. This fluid motion is a source for the Yang–Mills field, which in turn exerts a Lorentz force on the fluid. Hence, the Lie algebraic description of the Yang–Mills fluid equations, considered as a Hamiltonian system, places fluid dynamics and field equations into a geometrical framework which mirrors the underlying physics. The abelian case has applications in electromagnetic plasma theory (see, e.g., [2], [3], and [5]). Two areas in physics where the non-abelian description could be useful are in (a) condensed-matter theories, e.g., the theory of spin-glasses, which is based upon the O(3) Yang–Mills algebra, and (6) self-gravitating fluids in general relativity, which can be regarded as a Yang–Mills theory based upon the SL(2, ℂ) algebra.

Acknowledgments

We are grateful to the Center for Nonlinear Studies at Los Alamos National Laboratory, whose facilities and support made this work possible. We thank Roy Axford and Craig Nohl for helpful comments on the manuscript. A brief description of the present work is given in [4].

References

[1] R. G. Spencer and A. N. Kaufman, Hamiltonian Structure of Two-Fluid Plasma Dynamics, Lawrence Berkeley Laboratory preprint LBL-13720.

[2] Darryl D. Holm and Boris Kupershmidt, Poisson Brackets and Clebsch Representations for Magnetohydrodynamics, Multifluid Plasmas, and Elasticity. LA-UR-82-1103, Submitted to Physica D.

[3] Darryl D. Holm and Boris Kupershmidt, Canonical and Noncanonical Hamiltonian Formulations of Ideal Magnetohydrodynamics. LA-UR-82-124. Submitted to J. Math. Phys.

[4] John Gibbons, Darryl D. Holm and Boris Kupershmidt, Gauge-Invariant Poisson Brackets for Chromohydrodynamics, Phys. Lett. 90A (1982) 281.

[5] J. E. Marsden and A. Weinstein, The Hamiltonian Structure of the Maxwell–Vlasov Equations, Physica 4D (1982) 394.

[6] J. Gibbons, Physica 3D (1981) 503.

[7] D. R. Lebedev and Yu. I. Manin, Hamiltonian operator of Gel'fand-Dikii and Coadjoint Representation of the Volterra Group, Funk. Anal. Appl. 13: 4 (1979) 40–46.

[8] D. R. Lebedev, Benney's Long Waves Equations: Hamiltonian Formalism, Lett. Math. Phys. 3 (1979) 481–488.

[9] B. A. Kupershmidt, On Dual Spaces of Differential Lie Algebras, *Lie Algebras and Related Topics*, Rutgers 1981 Proceedings, Lecture Notes in Mathematics (to appear).

[10] B. A. Kupershmidt, Geometry of Jet Bundles and the Structure of Lagrangian and Hamiltonian Formalisms, in Lect. Notes Math. #775 (Springer, Berlin, 1980) pp. 162–218.