Time-Reversal Symmetry and Concurrence Dynamics

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Outline

I. Multi-partite Entanglement
II. Concurrence monotone & time-reversal
III. Matrix decompositions
IV. Kramers' non-degeneracy
V. Perturbations (entanglement phase-transition)

Entangled States

- Multi-partite Hilbert space: Kronecker product $\mathcal{H} = \otimes_{j=1}^{n} \mathcal{H}_j$

- Local (mixed) state: $\rho = \sum_{k=1}^{\ell} p_k \bigotimes_{j=1}^{n} \rho_k^j, \ 0 \leq p_k \leq 1, \ \sum p_k = 1$

- Entangled state: non-local states (not of this form)

- Remark: recent extensions focus on constructible projectors rather than particle (Kronecker product) projectors

Applications of Entangled States

- Quantum Communication
  - Teleportation & Super-dense coding: (2q) Bell states
  - Quantum networks, parallel architecture: many-partite states
- Quantum circuit emulation using cluster state
  - Evolve full superposition by Ising interaction
  - Subsequently: only one-qubit operations, one-qubit measurement
- Precision measurement: **GHZ**-state shot noise achieves quantum limit

Multi-Partite Entanglement Types

- Bipartite state: $|\psi\rangle_{AB}$ from $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with $\rho_{AB} = |\psi\rangle_{AB}\langle\psi|_{AB}$
  - All ent. measures: fxns. of entropy $S(\rho_A) = -\text{Tr}(\rho_A \log \rho_A)$
  - Unique entanglement type

- $n \geq 3$ qubits: multiple entanglement measures, types
  - Entanglement type: states not convertible by stochastic LOCC
  - $3q$, two tri-partite types of **GHZ** & $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$
  - 4 qubits, nine types

Quantifying Multi-partite Entanglement

- Entanglement monotones:
  - Does not increase on average under LOCC
  - Vanishes on local mixed states
  - Convex on density matrices
- Examples: Concurrence, concurrence^2, 3-tangle ($3q$), generalized Schmidt measure, geometric distance to local states, etc.
- Mixed states: $\mu(\rho) = \min\left\{ \sum_{j=1}^{\ell} \langle x_j | x_j \rangle \, \mu(|x_j\rangle) \; ; \; \rho = \sum_{j=1}^{\ell} |x_j\rangle\langle x_j| \right\}$

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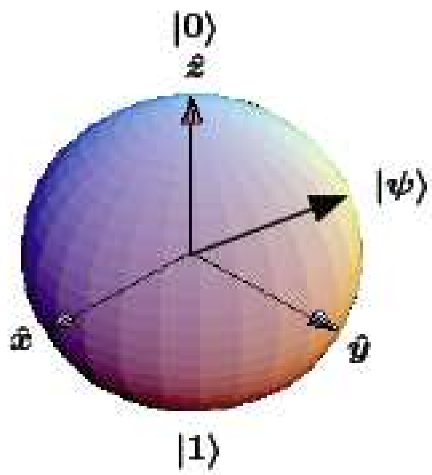
Time-reversal Symmetry Operator Θ

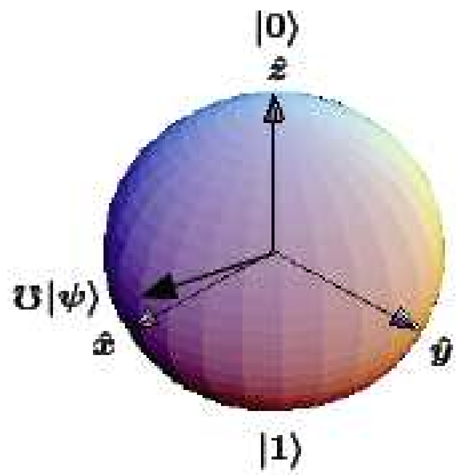
- Time-reversal symmetry operator $\Theta : \mathcal{H} \to \mathcal{H}$ definition (Wigner):
  - $\mathbb{C}$-antilinear & orthogonal (i.e. anti-unitary)
  - projectively involutive: $\Theta^2 = e^{i\varphi} I$
- Physical intuition: $+1$ eigenspace position, -1 eigenspace momentum
- Bosonic or fermionic as $\Theta^2 = \pm I$

Quantum Bit-Flip $\mho$: One Qubit

- Picture: reverse Bloch sphere vector

- Example of Time-reversal symmetry operator





Quantum Bit-Flip $\mho$ & Concurrence

- One qubit formula: $|\psi\rangle \overset{\mho}{\mapsto} (-i\sigma^y)\overline{|\psi\rangle}$

- $\mathbb{C}$-antilinear: not unitary evolution (aphysical)

- n-qubit formula: $|\psi\rangle \overset{\mho}{\mapsto} (-i\sigma^y)^{\otimes n}\overline{|\psi\rangle} = \overline{(-i\sigma^y)^{\otimes n}|\psi\rangle}$

- Physical interpretation: time-reversal symmetry operator

- Concurrence Entanglement Monotone: $C_{2p}(|\psi\rangle) = |\langle\psi|\mho|\psi\rangle|/\langle\psi|\psi\rangle$

Concurrence Entanglement Monotone: Examples

- $C_{2p-1}(|\psi\rangle) = 0$, for any $|\psi\rangle$ since $[(-i\sigma^y)^{\otimes n}]^T = -(-i\sigma^y)^{\otimes n}$

- $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(\,|0000\rangle + |1111\rangle)$, then $C_4(|\text{GHZ}\rangle) = 1$

- $|\text{W}\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$, then $C_4(|\text{W}\rangle) = 0$

- $C_{2p}(|\psi_{2p-1}\rangle \otimes |\psi_1\rangle) = 0$ since $\langle\psi_1|\mho_1|\psi_1\rangle = 0$

- (Scott, Caves) $\langle\, C_{2p}(|\psi\rangle)^2 \,\rangle \in O(1/N)$

Concurrence of Mixed States is a Monotone (Brennen)

- Def: $C_{2p}(\rho) \; = \;$ min$\left\{ \sum_{j=1}^{\ell} \langle x_j | x_j \rangle \, C_{2p}(|x_j\rangle) \; ; \; \rho = \sum_{j=1}^{\ell} |x_j\rangle\langle x_j| \; \right\}$

- Proof of monotone (Brennen, following Wong,Christensen for C_{2p}^2)

- Properties to check for monotonicity
  - $C_{2p}(\sum_{k=1}^{\ell} p_k \otimes_1^n \rho_j) = 0$
  - Convex: $C_{2p}(p\rho_1 + q\rho_2) \leq pC_{2p}(\rho_1) + qC_{2p}(\rho_2)$, any ρ_1, ρ_2, $p + q = 1$
  - Non-increasing on average under LOCC

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Matrix Decompositions

- Goal: Study entangling properties of $u : \mathcal{H}_n \to \mathcal{H}_n$ (entire state space)
- Technique: Matrix decompositions (factorization of matrix class)
- Examples: Singular Value Decomposition (SVD), Cosine-Sine Decomposition (CSD), QR-Decomposition
- Producing Decompositions:
  - Invoke existence theorem for factorization (implicit)
  - Provide factorization algorithm (explicit)

$G = KAK$ Theorem for Decompositions

- $G = KAK$ theorem outputs matrix decomposition
  - Lie group G is one of three inputs
  - $K \subset G$, $A \subset G$ always Lie subgroups

- Many examples, several useful to quantum computing
  - SVD: $Gl(n,\mathbb{C}) \;=\; U(n)\Delta U(n) \;=\; KAK$
  - CSD: $SU(2n) \;=\; [SU(n) \oplus SU(n)]T[SU(n) \oplus SU(n)]$, appropriate T
  - Bloch sphere rotations: $SU(2) = \{R_z(\alpha)\}\{R_y(\theta)\}\{R_z(\alpha)\}$

New $G = KAK$ example: Concurrence Canonical Decomposition

- Generalizes canonical dec.: $SU(4) = [SU(2) \otimes SU(2)]\Delta[SU(2) \otimes SU(2)]$
  - $SU(2) \otimes SU(2)$: two-qubit local unitary (LU) group
  - Δ: relative phase computations on Bell basis
- Sample applications: 2q control theory, 2q quantum logic circuits, 2q computation times, 2q entanglement theory
- n-qubit CCD: extend theorem inputs for 2q-CD to n qubits

Concurrence CD Cont.: A & K

- What K subgroup does this produce?
  - Concurrence form: $\mathcal{C}_{2p}(|\phi\rangle, |\psi\rangle) \;=\; \overline{\langle\phi|\mho|\psi\rangle}$, ($\mathbb{C}$-bilinear)
  - K: symmetry subgroup
  - $(k \in K) \iff \mathcal{C}_{2p}(\, k|\phi\rangle, k|\psi\rangle \,) = \mathcal{C}_{2p}(\, |\phi\rangle, |\psi\rangle \,)$
- What A subgroup does this produce?
  - Complicated: degrees of $G = KAK$ freedom in choice of A
  - One choice: relative phasing of basis of Greenberger-Horne-Zeilinger states

Two-Qubit Entanglement Capacities

- Jun Zhang et al (Berkeley): two-qubit entanglement capacities
  - Def: $\mathcal{E}_2(v) \;=\;$ max$\{\, C_2(v|\psi\rangle)\,;\, C_2(|\psi\rangle) = 0,\ \langle\psi|\psi\rangle = 1\,\}$
  - Strategy: study changes in entanglement induced by a factor of two-qubit canonical decomp, $v = [b\otimes c]a[d\otimes f]$
  - Thm: $\mathcal{E}_2(v) = 1$ iff the convex hull (polygonal span) of spec$[(\sigma^y)^{\otimes 2}\, v\, (\sigma^y)^{\otimes 2}\, v^T]$ holds $0 \in \mathbb{C}$
- Application: B gate; two-qubit computation with minimal possible (2) applications needed to build $SU(4)$ from $\{B, LU\}$

Even Qubit Concurrence Capacities

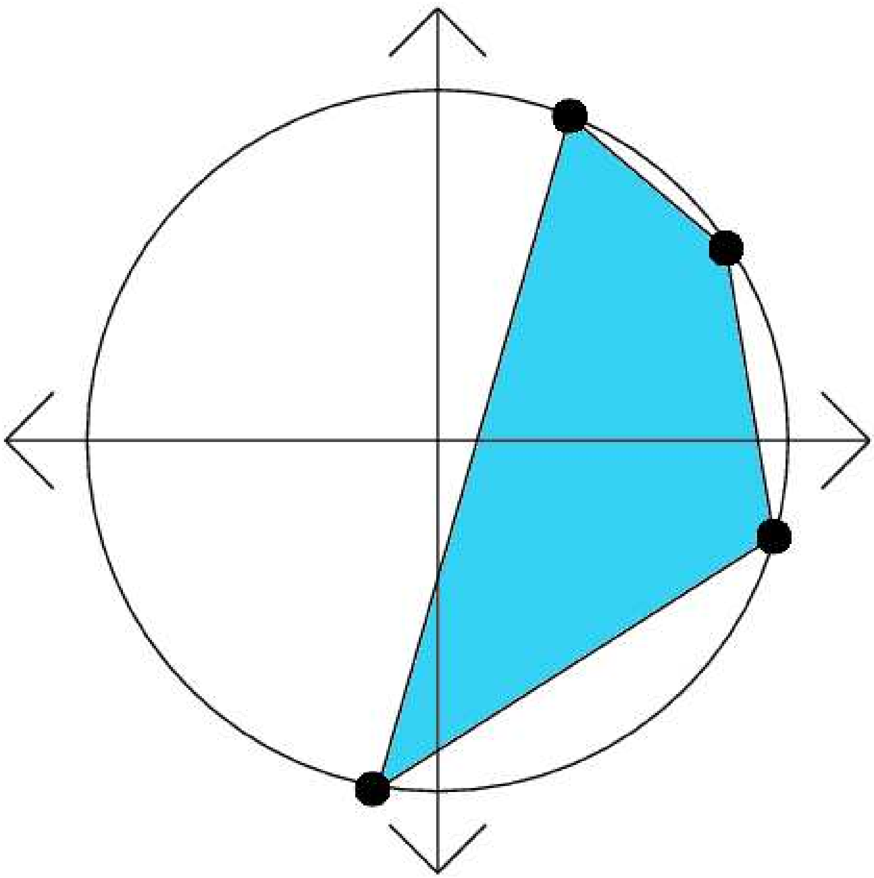
- Fix $n = 2p$, integral spin qubit system
  - Def: Concurrence capacity

$$\kappa_{2p}(v) \;=\; \max\{\, C_{2p}(v|\psi\rangle)\;;\; C_{2p}(|\psi\rangle) = 0,\; \langle\psi|\psi\rangle = 1\,\}$$

  - Prop:(BB) $\kappa_{2p}(k_1 a k_2) = \kappa_{2p}(a)$
  - Thm:(BB) $\kappa_{2p}(v) = 1$ iff the convex hull (polygonal span) of $\lambda_c(v) = \mathrm{spec}[(-i\sigma^y)^{\otimes 2p}\, v\, (-i\sigma^y)^{\otimes 2p}\, v^T] = \mathrm{spec}(a^2)$ holds $0 \in \mathbb{C}$

- Application: Choose an element a at random for $2p$ large $\Longrightarrow$ the concurrence capacity of $v = k_1 a k_2$ is probably one

Picture: Sample Convex Hull

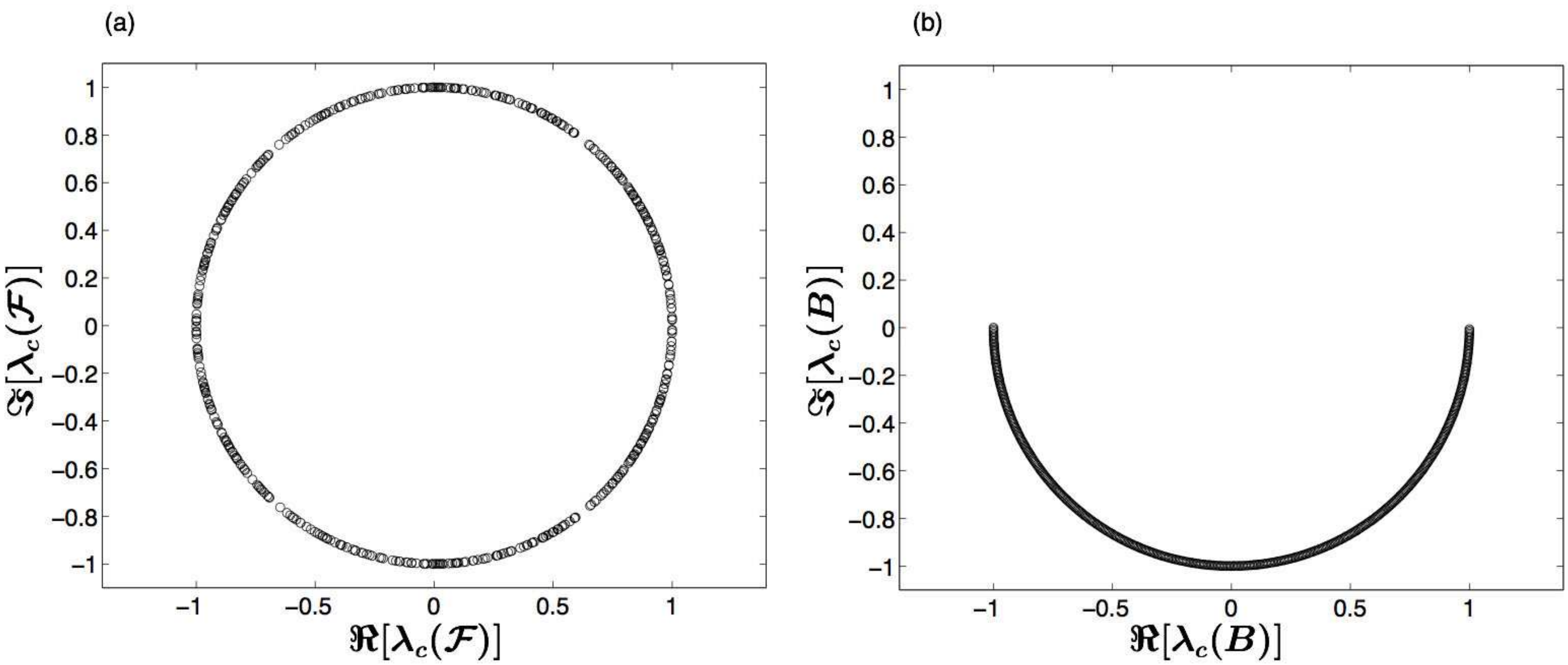


Concurrence Spectra of Other Computations

- Quantum Fourier transform: Fix $N = 2^n$, $\omega = \mathrm{e}^{2\pi i/N}$
  - $\mathcal{F} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \omega^{jk} |k\rangle\langle j|$
  - Admits $\Theta(\log N)$-sized quantum logic circuit
  - Component of other quantum algorithms, e.g. Shor's factoring

- Quantum baker's map: $B_n = \mathcal{F}_n (I_2 \otimes \mathcal{F}_{n-1}^{-1})$
  - Used in quantum chaos theory; quantum mixing
  - Concurrence of $B_n^k|00\cdots 0\rangle$ studied by Scott, Caves

$\lambda_c(v) = \textbf{spec}(a^2)$ of $\mathcal{F}_{10}$ (left) & B_{10} (right)



Algorithms Computing the CCD

- Parity dependece: inner automorphism class of $K \subset SU(2^n)$
  - $n = 2p$: $K = E\ SO(2^n)E^\dagger$, some matrix E
  - $n = 2p-1$: $K = F\ Sp(2^{n-1})\ F^\dagger$, some matrix F
- $n = 2p$: Algorithm very similar to two-qubit case
- $n = 2p-1$: requires symplectic diagonalization argument
  - Dongarra, Gabriel, Koelling, Wilkinson *Linear Algebra & App.*, '84
  - Similar to work w/ quaternions of Dyson, **JMP**

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Second $G = KAK$ input: $\theta : \mathfrak{g} \to \mathfrak{g}$

- (linear) Lie group G yields Lie algebra $\mathfrak{g}$: set of matrix logarithms

- Lie algebra: carries Lie bracket $[-,-]$

- Cartan involution intuition: polar-like decomposition of $\mathfrak{g}$

- Cartan involution: $\theta : \mathfrak{g} \to \mathfrak{g}$, $\theta^2 = I$ & $[\theta X, \theta Y] = \theta[X,Y]$

Second $G = KAK$ input: $\theta : \mathfrak{g} \to \mathfrak{g}$ Cont.

- Example: for Singular value decomposition
  - $\mathfrak{gl}(n,\mathbb{C}) = \mathbb{C}^{n\times n} = \log Gl(n,\mathbb{C})$
  - $\theta(X) = -X^{\dagger}$, w/ -1 eigenspace Hermitian matrices and $+1$ eigenspace antiHermitian
  - Exponentials of eigenspaces: Hermitian & unitary matrices
- $\mathfrak{su}(2^n) \ = \ \{iH\ ;\ \mathrm{tr}(H) = 0, H = H^{\dagger}\ \}$
- CCD involution: $\theta(\ iH\ ) \ = \ [(-i\sigma^y)^{\otimes n}]^{\dagger}(iH)(-i\sigma^y)^{\otimes n} \ = \ \mho^{-1}(iH)\mho$

CCD at Hamiltonian Level

- Observed slightly earlier: Bremner-Dodd-Nielsen-Bacon

- Notation: $i\sigma^{\otimes J}$, multiindex $J = j_1 j_2 \cdots j_k \cdots j_n$
  - $\sigma^0 = I,\ j \in \{0, x, y, z\}$
  - $i\sigma^{\otimes J} = i \otimes_{j=1}^{n} \sigma^j \in \mathfrak{u}(N)$
  - Traceless if some $j_k \neq 0$; $\#J = \#\{k\,;\ j_k \neq 0, 1 \leq k \leq n\}$

- $\mathfrak{su}(N) \ = \{\ iH\ ;\ H = H^\dagger, \text{tr}(H) = 0\ \} \ = \ \bigoplus_{\#J \neq 0} \mathbb{R}\ i\sigma^{\otimes J}$

CCD at Hamiltonian Level Cont.

- $\mathbb{F}_2$ grading: $\mathfrak{su}(N) = \left(\bigoplus_{\#J \equiv 0 \text{ mod2}, \, \#J \neq 0} \mathbb{R}\{i\sigma^{\otimes J}\} \right) \oplus \left(\bigoplus_{\#J \equiv 1 \text{ mod2}} \mathbb{R}\{i\sigma^{\otimes J}\} \right)$

- time-reversal symmetry w.r.t. $\mho$: $\mathfrak{p} = \bigoplus_{\#J \equiv 0, \text{ mod2}, \#J \neq 0} \mathbb{R} \, i\sigma^{\otimes J}$

- time-reversal anti-symmetry w.r.t. $\mho$: $\mathfrak{k} = \bigoplus_{\#J \equiv 1 \text{ mod2}} \mathbb{R} \, i\sigma^{\otimes J}$

- Preview: Demonstrate eigenstates of all $iH \in \mathfrak{p}$ are degenerate or else have concurrence one

Offhand Remarks

- Recall K earlier as symmetries of concurrence form
  - So K is a group
  - $K = \{ \exp(iH) \; ; \; H$ time-reversal anti-symmetric $\}$
  - Anti-symmetric evolutions form a group
- Last $G = KAK$ input is $\mathfrak{a} = \log A$; must be in -1 e-space
- Time-symmetric evolutions do not form a group; cf. Hermitian matrices

Kramers' Degeneracy & $G = KAK$ Classification

- Kramers' degeneracy: Half integral spin system, time-symmetric energy Hamiltonian $\Longrightarrow$ degenerate eigenstates

- Time-symmetric Hamiltonian: CCD simplifies
  - $v = \exp(iH)$, $v = kak^\dagger$ and $H = kH_{\mathfrak{a}}k^\dagger$, $H_{\mathfrak{a}} \in \mathfrak{a}$, $k \in K$

- Structure of $\mathfrak{a}$ algebra (depends on parity of n)
  - Cartan classication of all θ up to Lie isomorphism
  - $K \cong Sp(2^{n-1}) \Longrightarrow$ repeat eigenvalues

Eigenstates for Time-Symmetric H (Kramers' proof)

Prop (Kramers): Let H be a traceless Hamiltonian which is time-reversal symmetric (i.e. $iH \in \mathfrak{su}(N)$, $iH \in \mathfrak{p}$.) Say $|\psi\rangle \in \mathcal{H}_n$ is a λ-eigenstate. Then the bit-flip $\mho|\psi\rangle$ is also a λ-eigenstate.

Proof: Since $H = H^\dagger$, note that λ is real. Thus antilinearity causes $\mho\lambda|\psi\rangle = \lambda\mho|\psi\rangle$. By symmetry, $\mho H \mho^{-1} = H$, i.e. $\mho H = H\mho$. Thus given $H|\psi\rangle = \lambda|\psi\rangle$,

$$H\mho|\psi\rangle \;=\; \mho H|\psi\rangle \;=\; \mho\lambda|\psi\rangle \;=\; \lambda\mho|\psi\rangle$$

Thus $\mho|\psi\rangle$ is a λ-eigenstate. $\square$

Kramers' Non-degeneracy (B B O'L)

- Fix $n = 2p$, H traceless, time-symmetric, nondegenerate

- $H|\psi\rangle = \lambda|\psi\rangle \Longrightarrow \mho|\psi\rangle$ also λ-eigenstate

- Nondegenerate: must have $\mho|\psi\rangle = \mathrm{e}^{i\varphi}|\psi\rangle$

- Consequence: $C_{2p}(|\psi\rangle) = |\langle\psi|\mho|\psi\rangle| = 1$;
  any nondegenerate eigenstate of H is maximally concurrent
  ($\Longrightarrow$ entangled)

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Examples of Time-Symmetric Hamiltonians

- E.g. XY Hamiltonian: $H_{XY} = J\sum_{j=0}^{2p-1}\left(\frac{1+g}{4}\sigma_j^x\sigma_{(j+1)\mathtt{mod}2p}^x + \frac{1-g}{4}\sigma_j^y\sigma_{(j+1)\mathtt{mod}2p}^y\right)$

- Time symmetric: Each summand has even # of Pauli operators.

- Literature (Lieb): nondegenerate (no repeat eigenvalues)

- Consequence: Very entangled eigenstates, in particular ground state

- Could one produce maximally concurrent states by cooling?

Perturbations of Time Symmetry (GKB)

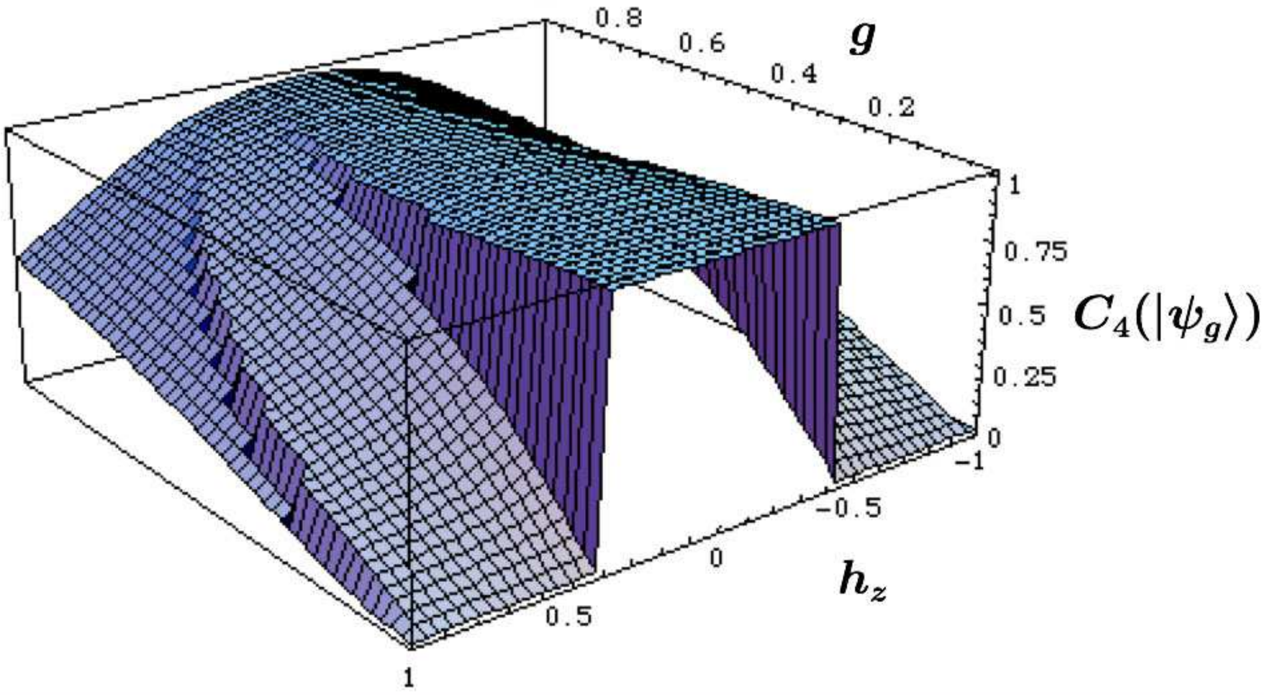
- Proof fails if perturbative individual one-qubit spins are added:

$$H = J \sum_{j=0}^{2p-1} \left(\frac{1+g}{4} \sigma_j^x \sigma_{(j+1)\mathtt{mod}2p}^x + \frac{1-g}{4} \sigma_j^y \sigma_{(j+1)\mathtt{mod}2p}^y \right) + \frac{h_z}{2} \sum_{j=0}^{2p-1} \sigma_j^z$$

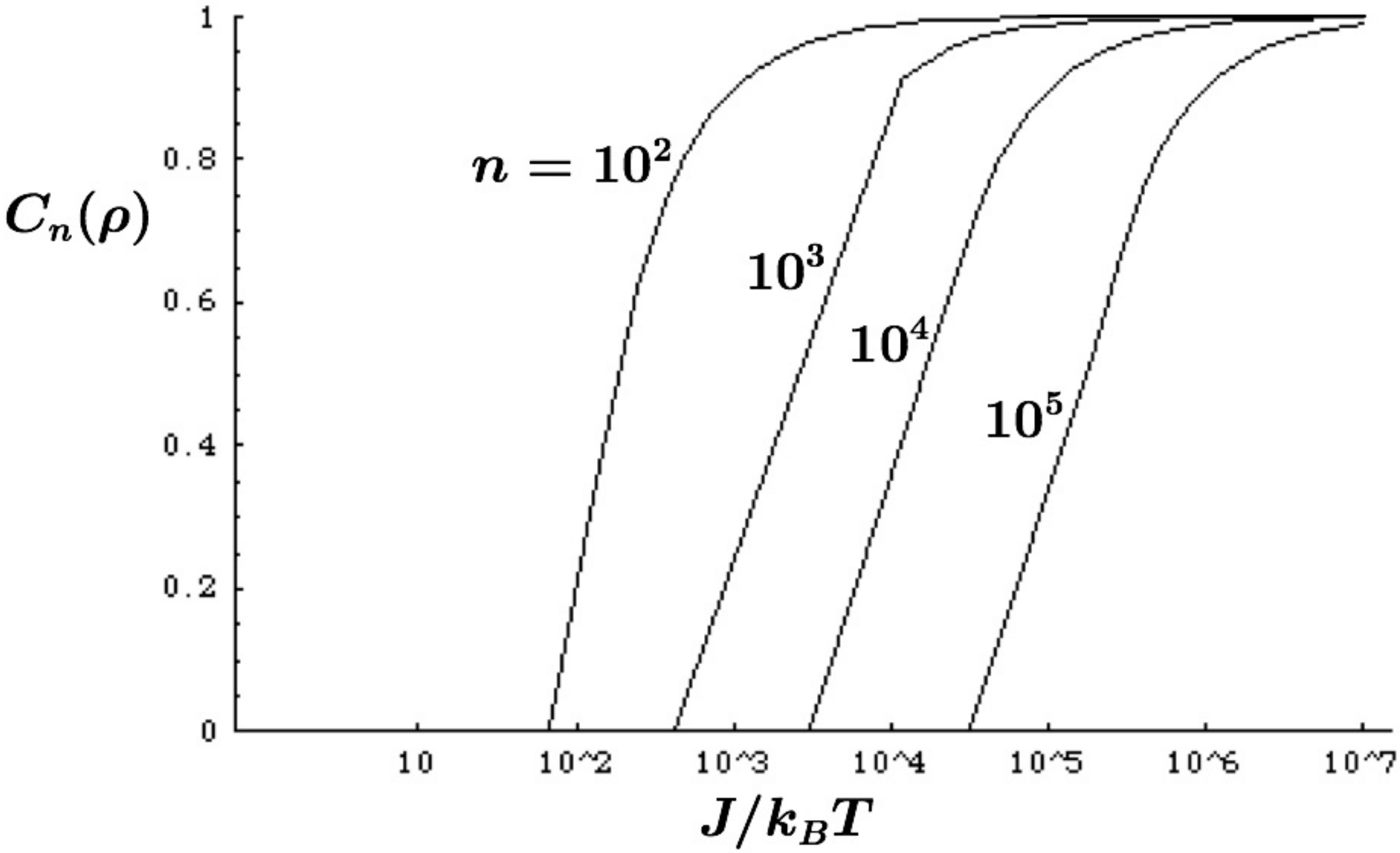
- Question (GKB): How does concurrence change in h_z and in balance of $\sigma^x \otimes \sigma^x$ vs. $\sigma^y \otimes \sigma^y$ given by g?

- Four qubit answer: over

Perturbations of Time Symmetry (GKB) Cont.



Concurrence at Finite Temp, $g = h_z = 0$ (GKB)



Conclusions

- Concurrence Canonical Decomposition:
  - Loosely: Singular Value Decomp. for time-reversal symmetry
  - Generalizes 2q-entanglement dynamics to n-qubit concurrence dynamics

- Structures visible at $n > 2$ qubits, exclusive
  - Kramers' degeneracy, as total spin integral, half-integral
  - Many-partite concurrence of eigenstates

Ongoing Work

- Scaling of sides of concurrence phase-transition as h_z varies
  - Seems to scale as $1/n$, $n = $ #qubits
  - Break less marked if $(\sigma^x)^{\otimes 2}$, $(\sigma^y)^{\otimes 2}$ weights differ
- Bang-bang correction of real time-reversal symmetric Hamiltonians, & Bang-gnab for K (anti-symmetric)
- Concurrence of finite-temperature (mixed) states of H_{XY}
- Measurement of concurrence of n-qubit system, n large