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On Acyclic Orientations and Sequential Dynamical Systems

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We study a class of discrete dynamical systems that consists of the following data: (a) a finite (labeled) graph Y with vertex set $\{1, \dots, n\}$, where each vertex has a binary state, (b) a vertex labeled multi-set of functions $(F_{i,Y}: \mathbb{F}_2^n \to \mathbb{F}_2^n)_i$, and (c) a permutation $\pi \in S_n$. The function $F_{i,Y}$ updates the binary state of vertex i as a function of the states of vertex i and its Y-neighbors and leaves the states of all other vertices fixed. The permutation π represents a Y-vertex ordering according to which the functions $F_{i,Y}$ are applied. By composing the functions $F_{i,Y}$ in the order given by π we obtain the sequential dynamical system (SDS):

$$[\mathfrak{F}_Y, \pi] = F_{\pi(n),Y} \circ \cdots \circ F_{\pi(1),Y}: \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n.$$

In this paper we first establish a sharp, combinatorial upper bound on the number of non-equivalent SDSs for fixed graph Y and multi-set of functions $(F_{i,Y})$. Second, we analyze the structure of a certain class of fixed-point-free SDSs.

Key Words: acyclic orientations; sequential dynamical system; orderings; graph automorphisms.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let Y be a loop-free, labeled, undirected graph with vertex set $\mathrm{v}[Y] = \{1, \dots, n\}$ and edge set $\mathrm{e}[Y]$. In particular, let Line_n be the graph with edge set $\{\{i, i+1\} \mid i = 1, \dots, n-1\}$, Circ_n the graph with edge set $\{\{1, n\}\} \cup \{\{i, i+1\} \mid i = 1, \dots, n-1\}$, Wheel_n the vertex join of Circ_n and 0, and finally Star_n the graph with vertex set $\{1, \dots, n\}$ and edge set $\{\{1, i\} \mid i = 2, \dots n\}$. We denote the set of Y-vertices adjacent to vertex i by $S_1(i)$, $B_1(i) = S_1(i) \cup \{i\}$ and set $\delta_i = |S_1(i)|$, $d(Y) = \max_{1 \le i \le n} \delta_i$. To emphasize the underlying base graph we will sometimes refer to $S_1(i)$, $B_1(i)$ as $S_{1,Y}(i)$, $B_{1,Y}(i)$. The increasing sequence of elements of the sets $S_1(i)$

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and $B_1(i)$ is referred to as

$$\widetilde{S}_1(i) = (j_1, \dots, j_{\delta_i}), \qquad \widetilde{B}_1(i) = (j_1, \dots, i, \dots, j_{\delta_i}). \tag{1.1}$$

Each vertex i has associated a state $x_i \in \mathbb{F}_2$, and for each $k = 1, \dots, d+1$ we have a symmetric function $f_{(k)}\colon \mathbb{F}_2^k \to \mathbb{F}_2$. In view of (1.1) we introduce the map

$$\mathrm{proj}[i]\colon \mathbb{F}_2^n \to \mathbb{F}_2^{\delta_i+1}, \qquad (x_1, \dots, x_n) \mapsto (x_{j_1}, \dots, x_i, \dots, x_{j_{\delta_i}}),$$

and denote the permutation group over k letters by S_k. For each i there exists a (Y-local) map $F_{i,Y}$ given by

$$y_i(x) = f_{(\delta_i+1)} \circ \mathrm{proj}\,[i](x)$$

$$F_{i,Y}(x) = (x_1, \dots, x_{i-1}, y_i(x), x_{i+1}, \dots, x_n)$$

and we refer to the multi-set $(F_{i,Y})_i$ as $\mathfrak{F}_Y$. Clearly, for each $Y < K_n$ the multi-set $(f_{(k)})_{1 \le k \le n}$ induces a multi-set $\mathfrak{F}_Y$.

DEFINITION 1. Let $[\mathfrak{F}_Y,]$ be the mapping

$$[\mathfrak{F}_Y,]\colon S_n \to \mathbb{F}_2^{n\mathbb{F}_2^n}, \quad [\mathfrak{F}_Y, \pi] = \prod_{i=1}^{n} F_{\pi(i),Y} = F_{\pi(n),Y} \circ \cdots \circ F_{\pi(2),Y} \circ F_{\pi(1),Y}. \tag{1.2}$$

We call $[\mathfrak{F}_Y, \pi]$ the *sequential dynamical system* (SDS) over Y with respect to the ordering π.

In the following we will study SDSs that are induced by the multi-sets $(\mathrm{nor}_{(k)})$ and $(\mathrm{nand}_{(k)})$, where

$$\mathrm{nor}_{(k)}(x_1, \dots, x_k) = \begin{cases} 1 & \text{if } (x_1, \dots, x_k) = (0, \dots, 0) \\ 0 & \text{else} \end{cases} \tag{1.3}$$

$$\mathrm{nand}_{(k)}(x_1, \dots, x_k) = \begin{cases} 0 & \text{if } (x_1, \dots, x_k) = (1, \dots, 1) \\ 1 & \text{else.} \end{cases} \tag{1.4}$$

We will refer to these SDSs as $[\mathrm{Nor}_{\mathrm{Y}}, \pi]$ and $[\mathrm{Nand}_Y, \pi]$, respectively.

Sequential dynamical systems have been studied in [1, 3] in the context of foundations of a theory of computer simulations and in [5] as dynamical systems.

Let the graph Y and the multi-set $\mathfrak{F}_Y$ be fixed. Obviously, an SDS $[\mathfrak{F}_Y, \pi]$ induces the labeled digraph, $\mathbb{G}[\mathfrak{F}_Y, \pi]$, with vertex set $\mathbb{F}_2^n$ and edge set $\{(x, [\mathfrak{F}_Y, \pi](x)) \mid x \in \mathbb{F}_2^n\}$. We will call $\mathbb{G}[\mathfrak{F}_Y, \pi]$ the phase space of $[\mathfrak{F}_Y, \pi]$, denote its set of vertices contained in cycles by $\mathrm{Per}[\mathfrak{F}_Y, \pi]$, and call $\mathbb{G}[\mathfrak{F}_Y, \pi]$-cycles periodic orbits. A periodic orbit of size 1 is called a fixed-point. One central question in SDS analysis is that of two SDSs $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Y, \sigma]$ being *equivalent*. Equivalence of SDS is defined with respect to a category $\mathfrak{C}[Y, \mathfrak{F}_Y]$ whose objects are the digraphs $\mathbb{G}[\mathfrak{F}_Y, \pi]$. Here, we consider the category $\mathfrak{C}_{\mathrm{di}}[Y, \mathfrak{F}_Y]$ having all digraph-morphisms as morphisms and therefore considering two SDSs $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}'_Y, \pi']$ to be equivalent if and only if $\mathbb{G}[\mathfrak{F}_Y, \pi] \cong \mathbb{G}[\mathfrak{F}_Y, \pi']$ holds. In the following we will analyze the set of non-equivalent SDSs for fixed Y and $\mathfrak{F}_Y$ which we denote by $\mathbf{E}[Y, \mathfrak{F}_Y]$. SDSs with different Boolean functions can be equivalent, too: let $[\mathfrak{F}_Y, \pi]$ be an arbitrary SDS and let inv: $\mathbb{F}_2^n \to \mathbb{F}_2^n$, $\mathrm{inv}(x_i) = (\overline{x_i})$, and $\mathfrak{F}_Y^{\mathrm{inv}} = (\mathrm{inv} \circ F_{i,Y} \circ \mathrm{inv})$. Then $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Y^{\mathrm{inv}}, \pi]$ are equivalent SDSs. In particular, $[\mathrm{Nor}_Y, \pi]$ and $[\mathrm{Nand}_Y, \pi]$ are equivalent.

To state our first result we introduce some basic terminology. Let G be a group and let Y be an undirected graph with automorphism group $\mathrm{Aut}(Y)$. Then G acts on Y if there exists a group homomorphism $u: G \longrightarrow \mathrm{Aut}(Y)$. If G acts on the graph Y, then its action induces (i) the graph $G \setminus Y$, where

$$\mathrm{v}[G\setminus Y]=\{G(i)\,|\,i\in \mathrm{v}[Y]\} \quad \text{and} \quad \mathrm{e}[G\setminus Y]=\{G(\{i,k\})\,|\,\{i,k\}\in \mathrm{e}[Y]\},$$

and (ii) the surjective graph morphism π_G given by

$$\pi_G\colon Y \longrightarrow G \setminus Y, \qquad i \mapsto G(i).$$

In our first result we give a combinatorial upper bound on the number of non-equivalent SDSs which is sharp for certain classes of SDS. Let $\mathrm{Acyc}(Y)$ denote the set of acyclic orientations of Y and set $a(Y) = |\mathrm{Acyc}(Y)|$.

THEOREM 1. *Let Y be an arbitrary graph, let $\pi \in S_n$, and let $[\mathfrak{F}_Y, \pi]$ be an SDS over Y. Then we have*

$$|\mathbf{E}[Y, \mathfrak{F}_Y]| \leq \frac{1}{|\mathrm{Aut}(Y)|} \sum_{\gamma \in \mathrm{Aut}(Y)} |a(\langle\gamma\rangle \setminus Y)| \tag{1.5}$$

$$|\mathbf{E}[\mathrm{Star}_n, \mathrm{Nor}_{\mathrm{Star}_n}]| = \frac{1}{|\mathrm{Aut}(\mathrm{Star}_n)|} \sum_{\gamma \in \mathrm{Aut}(\mathrm{Star}_n)} |a(\langle\gamma\rangle \setminus \mathrm{Star}_n)| = n. \tag{1.6}$$

In [2] one can find further analysis on the sharpness of the bound in (1.5), which can be computed for the graphs Circ_n and Wheel_n:

PROPOSITION 1. *Let $n > 2$, $\pi \in S_n$, and let ϕ be the Euler ϕ-function. Then the following assertions hold*:

$$|\mathbf{E}[\mathrm{Circ}_n, \mathfrak{F}_{\mathrm{Circ}_n}]| \leq \begin{cases} \dfrac{1}{2n} \displaystyle\sum_{d|n} \phi(d)\big(2^{n/d} - 2\big) + 2^{n/2}/4 & \text{iff } n \equiv 0 \bmod 2 \\ \dfrac{1}{2n} \displaystyle\sum_{d|n} \phi(d)\big(2^{n/d} - 2\big) & \text{iff } n \equiv 1 \bmod 2 \end{cases} \tag{1.7}$$

$$(1.8)\qquad |\mathbf{E}[\mathrm{Wheel}_n, \mathfrak{F}_{\mathrm{Wheel}_n}]| \leq \begin{cases} \dfrac{1}{2n}\sum_{d|n}\phi(d)\big(3^{n/d}-3\big)+3^{n/2}/2 & \text{iff } n\equiv 0 \bmod 2\\ \dfrac{1}{2n}\sum_{d|n}\phi(d)\big(3^{n/d}-3\big) & \text{iff } n\equiv 1 \bmod 2. \end{cases}$$

A permutation $\pi = (i_1, \dots, i_n)$ induces an orientation $\mathfrak{O}(Y)_\pi$ of Y by setting for $\{i_k, i_r\} \in \mathrm{e}[Y]$ and $k < r$, $o(\{i_k, i_r\}) = i_k$, and $t(\{i_k, i_r\}) = i_r$. By construction $\mathfrak{O}(Y)_\pi$ is acyclic and we have a mapping w: $S_n \to \mathrm{Acyc}(Y)$, $\pi \mapsto \mathfrak{O}(Y)_\pi$. w is surjective and for any $\pi, \sigma \in S_n$, $\mathfrak{O}_\pi = \mathfrak{O}_\sigma$ implies $[\mathfrak{F}_Y, \pi] = [\mathfrak{F}_Y, \sigma]$. Accordingly, we obtain that

$$(1.9)\qquad h\colon \mathrm{Acyc}(Y) \longrightarrow \{[\mathfrak{F}_Y, \pi] \mid \pi \in S_n\}, \qquad \mathfrak{O}_\pi \mapsto [\mathfrak{F}_Y, \pi]$$

is well defined. Let $\mathcal{J}(Y)$ be the set of Y-independence sets. We will next analyze the structure of SDSs that are induced by a multi-set $(f_{(k)})_k$ such that they are fixed-point-free for any graph Y:

THEOREM 2. *Let $(f_{(m)})_m$ be a family of Boolean, symmetric functions inducing for an arbitrary graph Y the fixed-point-free SDS $[\mathfrak{F}_Y, \pi]$. Then $[\mathfrak{F}_Y, \pi]$ is equivalent to $[\mathrm{Nor}_Y, \pi]$.*

Suppose $[\mathfrak{F}_Y, \pi]$ is equivalent to $[\mathrm{Nor}_Y, \pi]$, then we have:

(a) *Each periodic point of $[\mathfrak{F}_Y, \pi]$ corresponds uniquely to a Y-independence set; i.e., there exists a bijective mapping ι: $\mathrm{Per}[\mathfrak{F}_Y, \pi] \longrightarrow \mathcal{J}(Y)$.*

(b) *Each $\mathbb{G}[\mathfrak{F}_Y, \pi]$-vertex is either periodic or has in-degree 0. Furthermore, (0) has maximal in-degree in $\mathbb{G}[\mathfrak{F}_Y, \pi]$.*

(c) *Let $Y = \mathrm{Line}_n$ or $Y = \mathrm{Circ}_n$. Then $\mathbb{G}[\mathfrak{F}_Y, \pi] \cong_\lambda \mathbb{G}[\mathfrak{F}_Y, \sigma]$ implies $\lambda((0)_i) = (0)_i$. In particular, the corresponding orbits containing (0) are isomorphic.*

(d) *Suppose $\mathrm{Aut}(Y)$ is transitive and there exist $\rho, \sigma, \pi \in S_n$ such that $[\mathfrak{F}_{\rho(Y)}, \sigma] = [\mathfrak{F}_Y, \pi]$ holds. Then we have $\rho \in \mathrm{Aut}(Y)$ and $\mathfrak{O}(Y)_{\rho^{-1}\sigma} = \mathfrak{O}(Y)_\pi$.*

2. SOME GROUP ACTIONS ON SDS

S_n acts on the set of Y-vertices by permutation and thereby induces the natural group action on the set of all mappings t: $\{1, \dots, n\} \longrightarrow \mathbb{F}_2$ given by $\{\rho \cdot t\}(i) = t(\rho^{-1}(i))$. In particular, we may view t as an n-tuple, $(x_1, \dots, x_n)$ and accordingly obtain the S_n-action on $\mathbb{F}_2^n$:

$$(2.1)\qquad \cdot : S_n \times \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n, \qquad (\rho, (x_j)) \mapsto \rho \cdot (x_j) = (x_{\rho^{-1}(j)}).$$

Clearly, we have $hg \cdot (x_j) = (x_{g^{-1}h^{-1}(j)}) = h \cdot (g \cdot (x_j))$. The action $\cdot : S_n \times \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n$ induces an S_n-action on mappings $\Phi \colon \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n$ given by

$$\{\rho \bullet \Phi\}(x_j) = \rho \cdot (\Phi(\rho^{-1} \cdot (x_j)). \tag{2.2}$$

PROPOSITION 2. *Let Y be an arbitrary graph with vertex set $\{1, \dots, n\}$ acted upon by the group G. Then we have the group-action*

$$\bullet : S_n \times \{[\mathfrak{F}_{\pi(Y)}, \sigma] \mid \pi, \sigma \in S_n\} \to \{[\mathfrak{F}_{\pi(Y)}, \sigma] \mid \pi, \sigma \in S_n\} \tag{2.3}$$

$$(\rho, [\mathfrak{F}_{\pi(Y)}, \sigma]) \mapsto \rho \bullet [\mathfrak{F}_{\pi(Y)}, \sigma] = [\mathfrak{F}_{\pi(Y)}, \rho\sigma] \tag{2.4}$$

and $\bullet$ induces by restriction the action

$$\bullet : G \times [\mathfrak{F}_Y, S_n] \longrightarrow [\mathfrak{F}_Y, S_n] \tag{2.5}$$

$$(g, [\mathfrak{F}_Y, \sigma]) \ \mapsto \ g \bullet [\mathfrak{F}_Y, \sigma] = [\mathfrak{F}_Y, g\sigma]. \tag{2.6}$$

Furthermore, G acts naturally on $\mathrm{Acyc}(Y)$ *via $g\mathfrak{O}(\{i, k\}) = \mathfrak{O}(\{g^{-1}(i), g^{-1}(k)\})$ and h:* $\mathrm{Acyc}(Y) \longrightarrow [\mathfrak{F}_Y, S_n]$ *is a G-map.*

Proof. We first show

$$\forall \rho \in S_n, i = 1, \dots, n, \qquad \rho \cdot F_{i,Y}(\rho^{-1} \cdot (x_j)) = F_{\rho(i), \rho(Y)}(x_j). \tag{2.7}$$

To prove (2.7) we first note that, for arbitrary $\rho \in S_n$, we have $\rho(B_{1,Y}(i)) = B_{1,\rho(Y)}(\rho(i))$. In view of $(\rho^{-1} \cdot (x_j))_i = x_{\rho(i)}$ and $(\rho \cdot (y_j))_{\rho(i)} = y_i$ we derive

$$\begin{aligned} &\rho \cdot F_{i,Y}(\rho^{-1} \cdot (x_j)) \\ &\qquad = \big(x_1, \dots, y_{\rho(i)} = f_{(|B_{1,Y}(i)|)}((x_{\rho(k)})_{k \in B_{1,Y}(i)}), \dots, x_n\big) \end{aligned} \tag{2.8}$$

$$\begin{aligned} &F_{\rho(i), \rho(Y)}(x_j) \\ &\qquad = \big(x_1, \dots, y_{\rho(i)} = f_{(|B_{1,\rho(Y)}(\rho(i))|)}((x_k)_{k \in B_{1,\rho(Y)}(\rho(i))}), \dots, x_n\big). \end{aligned} \tag{2.9}$$

Now (2.7) follows in view of

$$\{x_{\rho(s)} \mid \rho(s) \in B_{1,\rho(Y)}(\rho(i))\} = \{x_{\rho(s)} \mid s \in B_{1,Y}(i))\}. \tag{2.10}$$

Obviously, (2.4) is implied by composing the corresponding local maps and it remains to prove (2.6). Since G acts on Y we have, for all $\rho \in G$, $B_{1,\rho(Y)}(i) = B_{1,Y}(i)$ and since $F_{i,Y}$ is a symmetric function we have

$$\forall \rho \in G, \qquad F_{i,\rho(Y)} = F_{i,Y}. \tag{2.11}$$

Assertion (2.6) follows immediately from (2.11) and it remains to show that h is a G-map. In view of $\mathfrak{O}_{g\pi} = g\mathfrak{O}_\pi$ and (2.6) we derive

$$h(g\mathfrak{O}_\pi) = [\mathfrak{F}_Y, g\pi] = g \bullet [\mathfrak{F}_Y, \pi] = g \bullet h(\mathfrak{O}_\pi)$$

completing the proof of the proposition. ∎

3. PROOF OF THEOREM 1

Let $\mathfrak{O}(Y)$ be an acyclic orientation of Y and let $\mathrm{P}(\mathfrak{O}(Y))$ be the set of all directed $\mathfrak{O}(Y)$-paths, π. Further let $\omega(\pi)$, $\tau(\pi)$, and $\ell(\pi)$ be its start-vertex, end-vertex, and length of the directed $\mathfrak{O}(Y)$-path π, respectively. We consider the mapping

$$\mathrm{rk}\colon \mathrm{v}[Y] \longrightarrow \mathbb{N}, \qquad \mathrm{rk}(i) = \max\{\ell(\pi) \mid \pi \in \mathrm{P}(\mathfrak{O}(Y)); \\ \omega(\pi) \text{ is an } \mathfrak{O}\text{-origin and } \tau(\pi) = i\}.$$

An acyclic orientation $\mathfrak{O}$ induces a partial ordering $<_{\mathfrak{O}}$, by setting $i <_{\mathfrak{O}} k$ if and only if $\mathrm{rk}(i) < \mathrm{rk}(k)$. Since $\mathrm{v}[Y] = \{1, \dots, n\}$ we can consider an acyclic orientation $\mathfrak{O}$ as a mapping $\mathfrak{O}\colon \mathrm{e}[Y] \longrightarrow \mathbb{F}_2$, where

$$\mathfrak{O}(\{i, k\}) = \begin{cases} 1 & \text{if either } \{i >_{\mathfrak{O}} k \text{ and } i > k\} \text{ or } \{k >_{\mathfrak{O}} i \text{ and } k > i\} \\ 0 & \text{otherwise.} \end{cases}$$

According to Proposition 2 the G-action on Y induces a G-action on $\mathrm{Acyc}(Y)$ given by

$$g\mathfrak{O}(\{i, k\}) = \mathfrak{O}(\{g^{-1}(i), g^{-1}(k)\}).$$

We set $\mathrm{Acyc}(Y)^G = \{\mathfrak{O} \in \mathrm{Acyc}(Y) \mid \forall g \in G;\ g\mathfrak{O} = \mathfrak{O}\}$ and $\mathrm{Fix}(g) = \mathrm{Acyc}(Y)^{\langle g \rangle}$. Moreover, $\pi_G\colon Y \longrightarrow G \setminus Y$ induces the mapping

$$\omega'_G\colon \mathrm{Acyc}(G \setminus Y) \longrightarrow \mathrm{Acyc}(Y), \qquad \overline{\mathfrak{O}} \mapsto \mathfrak{O}, \tag{3.1}$$

where $\mathfrak{O}(\{i, k\}) = \overline{\mathfrak{O}}(\{G(i), G(j)\})$. It is immediately clear that $\omega'_G(\mathrm{Acyc}(G \setminus Y)) \subset \mathrm{Acyc}(Y)^G$ holds. Next we prove that $\omega_G : \mathrm{Acyc}(G \setminus Y) \longrightarrow \mathrm{Acyc}(Y)^G$ is bijective having the inverse

$$\psi_G : \mathrm{Acyc}(Y)^G \longrightarrow \mathrm{Acyc}(G \setminus Y), \qquad \mathfrak{O} \mapsto \mathfrak{O}_G, \tag{3.2}$$

where $\mathfrak{O}_G(\{G(i), G(k)\}) = \mathfrak{O}(\{i, k\})$.

PROPOSITION 3. *Let Y be an undirected graph being acted upon by the group G. Then ψ_G is bijective and we have $\psi_G \circ \omega_G = \mathrm{id}$ and $\omega_G \circ \psi_G = \mathrm{id}$. In particular, $\mathrm{Acyc}(Y)^G \neq \varnothing$ if and only if all G-vertex orbits are contained in Y-independence sets.*

Proof. Let $\mathfrak{O} \in \mathrm{Acyc}(Y)^G$. By construction we have, for $g \in G$, $\mathfrak{O}(\{g^{-1}(i), g^{-1}(k)\}) = \mathfrak{O}(\{i, k\})$, whence $\mathfrak{O}\colon \mathrm{e}[Y] \longrightarrow \mathbb{F}_2$ is constant on G-edge orbits.

To define $\mathfrak{O}_G$, let $\{G(i), G(k)\}$ be a $G \setminus Y$-edge. We select $\{j, h\} \in \pi_G^{-1}(\{G(i), G(k)\})$ and set $\mathfrak{O}_G(\{G(i), G(k)\}) = \mathfrak{O}(\{j, h\})$. Since $\mathfrak{O}(\{g^{-1}(i), g^{-1}(k)\}) = \mathfrak{O}(\{i, k\})$ the mapping $\mathfrak{O}_G\colon \mathrm{e}[G \setminus Y] \longrightarrow \mathbb{F}_2$ is well defined and for $\mathfrak{O} \in \mathrm{Acyc}(Y)^G$ the mapping $\mathfrak{O} \mapsto \mathfrak{O}_G$ is bijective. It remains to prove that $\mathfrak{O}_G \in \mathrm{Acyc}(G \setminus Y)$. To prove this let L be a directed

$G \setminus Y$-loop w.r.t. $\mathfrak{O}_G$ over the vertices $G(i_1), \ldots, G(i_s)$ and the edges $G(y_1), \ldots, G(y_s)$. Restricting $\mathfrak{O}$ to the subgraph $Y' = \pi_G^{-1}(L)$ we obtain the acyclic orientation $\mathfrak{O}'$.

Claim. Each vertex-orbit $G(i_j)$, $j = 1, \ldots, s$, contains only Y' vertices which are not $\mathfrak{O}'$-origins.

Suppose $G(i_j)$ contains a Y' vertex, k, that is an $\mathfrak{O}'$-origin. Since L is an $\mathfrak{O}_G$-directed loop there exists a $G \setminus Y$-vertex $G(h)$ that precedes $G(k)$ in $\mathfrak{O}_G$. Since π_G is locally surjective there exists a Y-edge of the form $\{k', k\} \in \pi_G^{-1}(\{G(h), G(k)\})$ and we obtain $\mathfrak{O}'(\{k', k\}) = \mathfrak{O}(\{k', k\}) = \mathfrak{O}(\{G(h), G(k)\})$ contradicting the fact that k is an $\mathfrak{O}'$-origin. Consequently, there exists no Y'-vertex in a $G(i_j)$-orbit that is an $\mathfrak{O}'$-origin, proving the claim.

Obviously, the acyclicity of $\mathfrak{O}'$ implies that there exists at least one Y'-vertex i_j that is an $\mathfrak{O}'$-origin, which is impossible. Therefore, $\mathfrak{O} \in \mathrm{Acyc}(Y)^G$ implies $\mathfrak{O}_G \in \mathrm{Acyc}(G \setminus Y)$, whence ψ_G: $\mathrm{Acyc}(Y)^G \longrightarrow \mathrm{Acyc}(G \setminus Y)$ is a well-defined bijection and $\psi_G \circ \omega_G = \mathrm{id}$ and $\omega_G \circ \psi_G = \mathrm{id}$ follow immediately. It is straightforward to show that $\mathrm{Acyc}(Y)^G \neq \varnothing$ holds if and only if $G \setminus Y$ contains no loop of size 1. Obviously, the non-existence of a $G \setminus Y$-loop of size 1 is equivalent to the statement that all G-vertex orbits are contained in Y-independence sets, completing the proof of the proposition. ∎

In [4] one can find a generalization of Proposition 3 for locally surjective graph morphisms.

An immediate consequence of Propositions 2 and 3 reads

Corollary 1. *Let Y be an undirected graph with automorphism group G. Then we have*

$$|\mathbf{E}[Y, \mathfrak{F}_Y]| \leq \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)| = \frac{1}{|G|} \sum_{g \in G} a(\langle g \rangle \setminus Y). \tag{3.3}$$

Proof. Any $g \in G$ induces the bijective mapping $\lambda_g : \mathbb{F}_2^n \to \mathbb{F}_2^n$, $\lambda_g(x_j) = g \cdot (x_j)$ (see (2.1)), and in view of Proposition 2 we have

$$\begin{tikzcd}
g^{-1} \cdot (x_j) \arrow[r, "{[\mathfrak{F}_Y, \pi]}"] & {[\mathfrak{F}_Y, \pi](g^{-1} \cdot (x_j))} \arrow[d, "\lambda_g"] \\
(x_j) \arrow[u, "\lambda_g^{-1}"] \arrow[r, "{g \bullet [\mathfrak{F}_Y, \pi]}"] & g \bullet [\mathfrak{F}_Y, \pi](x_j) = g \cdot [\mathfrak{F}_Y, \pi](g^{-1} \cdot (x_j)).
\end{tikzcd}$$

Accordingly, λ_g: $\mathbb{G}[\mathfrak{F}_Y, \pi] \to \mathbb{G}[\mathfrak{F}_Y, g\pi]$ is a digraph-isomorphism. Using Burnside's lemma and Proposition 3 we derive

$$|\mathbf{E}[Y, \mathfrak{F}_Y]| \leq \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)| = \frac{1}{|G|} \sum_{g \in G} a(\langle g \rangle \setminus Y),$$

which proves the corollary. ∎

The second statement of Theorem 1 consists of the following

PROPOSITION 4.

$$|\mathbf{E}[\mathrm{Star}_n, \mathrm{Nor}_{\mathrm{Star}_n}]| = \frac{1}{|\mathrm{Aut}(\mathrm{Star}_n)|} \sum_{\gamma \in \mathrm{Aut}(\mathrm{Star}_n)} |a(\langle\gamma\rangle \setminus \mathrm{Star}_n)| = n.$$

The proof can be found in [5].

In fact, the RHS of (3.3) can be calculated efficiently for several classes of graphs. As an illustration we give a new proof of the formulas for the graphs Circ_n and Wheel_n [5] which were originally proved by a somewhat tedious computation.

Proof of Proposition 1. In the following we prove

$$\frac{1}{|G|} \sum_{\gamma \in G} a(\langle\gamma\rangle \setminus \mathrm{Circ}_n) = \begin{cases} \frac{1}{2n} \sum_{d|n} \phi(d) \left(2^{n/d} - 2\right) + 2^{n/2}/4 & \text{iff } \; n \equiv 0 \bmod 2 \\ \frac{1}{2n} \sum_{d|n} \phi(d) \left(2^{n/d} - 2\right) & \text{iff } \; n \equiv 1 \bmod 2 \end{cases} \tag{3.4}$$

$$\frac{1}{|G|} \sum_{\gamma \in G} a(\langle\gamma\rangle \setminus \mathrm{Wheel}_n) = \begin{cases} \frac{1}{2n} \sum_{d|n} \phi(d) \left(3^{n/d} - 3\right) + 3^{n/2}/2 & \text{iff } \; n \equiv 0 \bmod 2 \\ \frac{1}{2n} \sum_{d|n} \phi(d) \left(3^{n/d} - 3\right) & \text{iff } \; n \equiv 1 \bmod 2. \end{cases} \tag{3.5}$$

In view of Proposition 3, we have to compute the set $\mathrm{Acyc}(\mathrm{Circ}_n)^{\langle\gamma\rangle}$ for $\gamma \in \mathrm{Aut}(\mathrm{Circ}_n)$. First we observe that $\mathrm{Aut}(\mathrm{Circ}_n) = \langle\sigma\rangle \rtimes \langle\tau\rangle$, where $\sigma = (2, 3, \dots, n, 1)$ and $\tau = \prod_{i=2}^{\lceil n/2 \rceil} (i, n - i + 2)$. Furthermore we have $a(\mathrm{Circ}_n) = 2^n - 2$ and $a(\mathrm{Wheel}_n) = 3^n - 3$. Second, let $(0 \otimes Y)$ be the vertex-join of Y and 0, then π_G has the property

$$\forall\, Y, d(Y) < |\mathrm{v}[Y]|, \qquad G \setminus (0 \otimes Y) \cong 0 \otimes (G \setminus Y). \tag{3.6}$$

Accordingly, the formula for (3.5) follows by taking the vertex-joins of the graphs $\langle\gamma\rangle \setminus \mathrm{Circ}_n$. Thus it remains to compute $\langle\gamma\rangle \setminus \mathrm{Circ}_n$. Since $\mathrm{Aut}(\mathrm{Circ}_n)$ is a dihedral group we have either $\gamma = \sigma^k$ or $\gamma = \tau\sigma^k$. Suppose $d|n$ then $\langle\sigma^{n/d}\rangle \setminus \mathrm{Circ}_n \cong \mathrm{Circ}_{n/d}$ and the automorphisms of the form σ^k contribute $\sum_{d|n} \phi(d)(2^{n/d} - 2)$. For $n \equiv 1 \bmod 2$ we immediately observe that $\langle\tau\sigma^k\rangle$ contains at least one loop of size 1 and we are done. In case of $n \equiv 0$

mod 2, $\langle \tau\sigma^k \rangle$ has for $k \equiv 1$ mod 2 a vertex that corresponds to a $\langle \tau\sigma^k \rangle$-orbit which contains two adjacent vertices, whence $\mathrm{Acyc}(Y)^{\langle \tau\sigma^k \rangle} = \varnothing$. For $k \equiv 0$ mod 2 we conclude that $\langle \tau\sigma^k \rangle \setminus \mathrm{Circ}_n \cong \mathrm{Line}_{n/2}$, which has $2^{n/2}$ acyclic orientations and (3.4) follows.

In view of (3.6) it remains to take the vertex-joins of the graphs $\langle \gamma \rangle \setminus \mathrm{Circ}_n$ that have no loops of size 1 and the second formula follows in view of $0 \otimes \mathrm{Circ}_{n/d} \cong \mathrm{Wheel}_{n/d}$ and $a(0 \otimes \mathrm{Line}_{n/2}) = 2 \cdot 3^{n/2}$, whence Proposition 1. ∎

4. PROOF OF THEOREM 2

Let us begin by showing

LEMMA 1. *Let $(f_{(m)})_m$ be a family of Boolean symmetric functions that induces a fixed-point-free SDS $[\mathfrak{F}_Y, \pi]$ for arbitrary graphs Y. Then $[\mathfrak{F}_Y, \pi]$ and $[\mathrm{Nor}_Y, \pi]$ are equivalent.*

Proof. *Claim* 1. For any $m \in \mathbb{N}$ we have either $f_{(m)} = \mathrm{nor}_{(m)}$ or $f_{(m)} = \mathrm{nand}_{(m)}$.

Let us first consider the case $m = 2$. It is clear that a fixed-point-free symmetric function $f_{(2)}\colon \mathbb{F}_2^2 \to \mathbb{F}_2$ has the properties $f_{(2)}(0,0) = 1$, $f_{(2)}(1,1) = 0$. We have either $f_{(2)}(0,1) = f_{(2)}(1,0) = 1$ in which case $f_{(2)} = \mathrm{nand}_{(2)}$ or $f_{(2)}(0,1) = f_{(2)}(1,0) = 0$, that is, $f_{(2)} = \mathrm{nor}_{(2)}$. Let now $m > 2$. Suppose $f_{(m)} \neq \mathrm{nor}_{(m)}$ and $f_{(m)} \neq \mathrm{nand}_{(m)}$; then there exist two m-tuples $a = (a_1, \dots, a_m)$, $b = (b_1, \dots, b_m)$ with $|\{i \mid a_i = 1\}| = \ell$ and $|\{i \mid b_i = 1\}| = \ell'$ such that $0 < \ell, \ell' < m$ and $f_{(m)}(a) = 1$, $f_{(m)}(b) = 0$. We consider the graph K_2. Accordingly, we have either (i) $f_{(2)}(0,1) = 0$ or (ii) $f_{(2)}(0,1) = 1$.

In case (i) we take $Y(\ell, m-1)$ to be the graph over $\ell(m-\ell)$ vertices and $\binom{\ell}{2} + \ell(m-\ell)$ edges having K_ℓ as a subgraph such that each K_ℓ-vertex has degree $m-1$ and 1 otherwise. In view of $f_{(2)}(0,1) = 0$ and $f_{(m)}(a) = 1$ we obtain a fixed-point by assigning to any $Y(\ell, m-1)$-vertex with degree $m-1$ the state 1 and state 0 otherwise.

In case (ii), we consider $Y(m-\ell', m-1)$ defined as above. We assign to each $Y(m-\ell', m-1)$-vertex with degree $m-1$ the state 0 and state 1 otherwise and obtain, in view of $f_{(2)}(0,1) = f_{(2)}(1,0) = 1$ and $f_{(m)}(b) = 0$, a fixed-point, and the claim follows.

Claim 2. We have either, for all $m \in \mathbb{N}$, $f_{(m)} = \mathrm{nor}_{(m)}$ or, for all $m \in \mathbb{N}$, $f_{(m)} = \mathrm{nand}_{(m)}$ holds.

Suppose there exist $\ell, \ell' \in \mathbb{N}$ such that $f_{(\ell)} = \mathrm{nor}_{(\ell)}$ and $f_{(\ell')} = \mathrm{nand}_{(\ell')}$. We consider the bipartite graph $K_{\ell-1,\,\ell'-1}$ having the vertex set $A \cup B$, where each $a \in A$ has degree $\ell - 1$ and each $b \in B$ degree $\ell' - 1$. We assign to each $a \in A$ the state 0 and to each $b \in B$ the state 1 and obtain a fixed-point. This proves Claim 2.

In view of $[\mathrm{Nor}_Y, \pi] = \mathrm{inv} \circ [\mathrm{Nand}_Y, \pi] \circ \mathrm{inv}$ and Observation 1 of the Introduction, $[\mathrm{Nor}_Y, \pi]$ and $[\mathrm{Nand}_Y, \pi]$ are equivalent, whence the lemma. ∎

We will proceed by proving assertion (a) of Theorem 2.

LEMMA 2. *Let Y be a graph, $\pi = (i_1, \ldots, i_n)$, $\pi^* = (i_n, \ldots, i_1) \in S_n$, and*

$$\mathfrak{P}_Y = \{(\xi_j) \in \mathbb{F}_2^n \mid \forall\, j \in \mathbb{N}_n\colon\ \xi_j = 1 \ \Rightarrow\ \forall\, i \in S_1(j)\colon\ \xi_i = 0\}.$$

Then we have

$$\mathfrak{P}_Y = \mathrm{Per}[\mathrm{Nor}_Y, \pi] = [\mathrm{Nor}, \pi](\mathbb{F}_2^n).$$

Proof. First we observe that $\mathrm{Per}[\mathrm{Nor}_Y, \pi] \subset [\mathrm{Nor}, \pi](\mathbb{F}_2^n) \subset \mathfrak{P}_Y$ and it remains to show $\mathfrak{P}_Y \subset \mathrm{Per}[\mathrm{Nor}_Y, \pi]$. To prove this, we first note that $[\mathrm{Nor}_Y, \pi]' = \mathrm{res}_{\mathfrak{P}_Y}[\mathrm{Nor}_Y, \pi]\colon\ \mathfrak{P}_Y \longrightarrow \mathfrak{P}_Y$ is a well-defined mapping. We will show that $[\mathrm{Nor}_Y, \pi]'$ is invertible with inverse $[\mathrm{Nor}_Y, \pi^*]' = \mathrm{res}_{\mathfrak{P}_Y}[\mathrm{Nor}_Y, \pi^*]$. To prove invertibility, it suffices, in view of

$$[\mathrm{Nor}_Y, \pi^*] \circ [\mathrm{Nor}_Y, \pi] = \prod_{j=1}^{n} \mathrm{Nor}_{i_{n+1-j},\, Y} \circ \prod_{j=1}^{n} \mathrm{Nor}_{i_j,\, Y}$$

$$[\mathrm{Nor}_Y, \pi] \circ [\mathrm{Nor}_Y, \pi^*] = \prod_{j=1}^{n} \mathrm{Nor}_{i_j,\, Y} \circ \prod_{j=1}^{n} \mathrm{Nor}_{i_{n+1-j},\, Y}$$

to show

$$\forall\, (\xi_j) \in \mathfrak{P}_Y, i \in \mathbb{N}, \qquad \mathrm{Nor}_{i,\,Y} \circ \mathrm{Nor}_{i,\,Y}((\xi_j)) = (\xi_j). \tag{4.1}$$

Case (a). $\mathrm{Nor}_{i,\,Y}((\xi_j)) = (\xi_1, \ldots, 1, \ldots, \xi_n)$. Then, by definition of $\mathrm{Nor}_{i,\,Y}$, all coordinates ξ_k, $k \in B_1(i)$, have the property $\xi_k = 0$ and, clearly,

$$\mathrm{Nor}_{i,\,Y} \circ \mathrm{Nor}_{i,\,Y}((\xi_j)) = \mathrm{Nor}_{i,\,Y}((\xi_1, \ldots, 1, \ldots, \xi_n)) = (\xi_j).$$

Case (b). $\mathrm{Nor}_{i,\,Y}((\xi_j)) = (\xi_1, \ldots, \xi_{i-1}, 0, \xi_{i+1}, \ldots, \xi_n)$. By definition of $\mathrm{Nor}_{i,\,Y}$, we have either $\xi_i = 1$ or there exists at least one i-neighbor, k, such that $\xi_k = 1$. We conclude from $(\xi_j) \in \mathfrak{P}_Y$ that, in case of $\xi_i = 1$, i is the unique vertex in $B_1(i)$ with this property. Therefore we derive

$$\begin{aligned} &\mathrm{Nor}_{i,\,Y}\big((\xi_1, \ldots, \xi_{i-1}, 0, \xi_{i+1}, \ldots, \xi_n)\big) \\ &\quad = \begin{cases} (\xi_1, \ldots, \xi_{i-1}, 1, \xi_{i+1}, \ldots, \xi_n) & \text{if } k = i \\ (\xi_1, \ldots, \xi_{i-1}, 0, \xi_{i+1}, \ldots, \xi_n) & \text{otherwise,} \end{cases} \end{aligned}$$

whence $\mathrm{Nor}_{i,\,Y} \circ \mathrm{Nor}_{i,\,Y}((\xi_j)) = (\xi_j)$ and (4.1) follows. We immediately obtain from (4.1) that $[\mathrm{Nor}_Y, \pi]' \circ [\mathrm{Nor}_Y, \pi^*]' = [\mathrm{Nor}_Y, \pi^*]' \circ [\mathrm{Nor}_Y, \pi]' = \mathrm{id}$ holds, whence $\mathfrak{P}_Y \subset \mathrm{Per}[\mathrm{Nor}_Y, \pi]$ and the proof of the lemma is complete. ∎

In view of $\mathrm{Per}[\mathfrak{F}_Y, \pi] = \{(\xi_j) \in \mathbb{F}_2^n \mid \forall j \in \mathbb{N}_n\colon\ \xi_j = 1 \ \Rightarrow\ \forall\, i \in S_1(j)\colon\ \xi_i = 0\}$ we immediately observe that the mapping

$$\iota\colon \mathrm{Per}[\mathfrak{F}_Y, \pi] \longrightarrow \mathfrak{I}(Y), \qquad (\xi_j) \mapsto \{j \mid \xi_j = 1\},$$

is a bijection and assertion (a) follows. Obviously, $\mathrm{Per}[\mathrm{Nor}_Y, \pi] = [\mathrm{Nor}, \pi](\mathbb{F}_2^n)$ implies that each $\mathbb{G}[\mathrm{Nor}, \pi]$-vertex is either contained in a cycle or has in-degree 0. To complete the proof of assertion (b) it remains to show that (0) has maximal $\mathbb{G}[\mathrm{Nor}, \pi]$ in-degree.

LEMMA 3. *For $x \neq 0$ let $M(x) = \{h \mid x_h = 1\}$ and for $S \subset M(x)$ let x^S be the n-tuple with $x_j^S = x_j$ for $j \notin S$ and $x_j^S = 0$ for $j \in S$. Then we have*

$$\text{(4.2)}\quad \forall\, x \in \mathbb{F}_2^n,\ S \subset M(x), \qquad |[\mathrm{Nor}_Y, \sigma]^{-1}(x)| \leq |[\mathrm{Nor}_Y, \sigma]^{-1}(x^S)|$$

and in particular $|[\mathrm{Nor}_Y, \sigma]^{-1}(x)| \leq |[\mathrm{Nor}_Y, \sigma]^{-1}(0)|$ *holds.*

Proof. Obviously, (4.2) holds for any x with the property $|[\mathrm{Nor}_Y, \sigma]^{-1}(x)| = 0$. Thus we can w.l.o.g. assume that $|[\mathrm{Nor}_Y, \sigma]^{-1}(\xi)| > 0$ holds. Let $(0) \neq (\xi_j) \in \mathbb{F}_2^n$ with $(\eta_k) \in [\mathrm{Nor}_Y, \sigma]^{-1}(\xi_j)$ and $\xi_i = 1$. Writing $j <_\sigma k$ iff $\sigma^{-1}(j) < \sigma^{-1}(k)$, we can w.l.o.g. assume that i is maximal w.r.t. $<_\sigma$. Let $S_1^{>_\sigma}(h) = \{j \in S_1(h) \mid j >_\sigma h\}$ and $S_1^{>_\sigma}(h, \xi) = \{j \in S_1^{>_\sigma}(h) \mid \xi_h = 1\}$. By definition of $\mathrm{Nor}_{i,Y}$, $\xi_i = 1$ implies, for $j \in S_1^{>_\sigma}(i)$, $\eta_j = 0$. We set $\mathfrak{O} = \mathfrak{O}(Y)_\sigma$ and consider the mapping

$$r_{\mathfrak{O}}^{\xi,i}\colon \mathbb{F}_2^n \to \mathbb{F}_2^n, \qquad r_{\mathfrak{O}}^{\xi,i}(\eta)_k = \begin{cases} 1 & \text{for } k = i \vee k \in S_1^{>_\sigma}(i) \setminus \left(\bigcup_h S_1^{>_\sigma}(h, \xi)\right) \\ \eta_k & \text{else.} \end{cases}$$

For (χ_k) given by $\chi_i = 0$ and $\chi_k = \xi_k$ otherwise, $r_{\mathfrak{O}}^{\xi,i}$ induces by restriction an injective mapping

$$\text{(4.3)}\qquad \mathrm{res}(r_{\mathfrak{O}}^{\xi,i})\colon [\mathrm{Nor}_Y, \sigma]^{-1}(\xi_k) \longrightarrow [\mathrm{Nor}_Y, \sigma]^{-1}(\chi_k),$$

since, for $k \in S_1^{\geq_\sigma}(i)$, $\eta_k = 0$ holds. The rest is obvious. In particular we have

$$|[\mathrm{Nor}_Y, \sigma]^{-1}(\xi_k)| \leq |[\mathrm{Nor}_Y, \sigma]^{-1}(\chi_k)|$$

and (4.2) follows by induction on $|\{\xi_g \mid \xi_g = 1\}|$ successively replacing the coordinates $\xi_i = 1$ by 0. Clearly, (4.2) implies $|[\mathrm{Nor}_Y, \sigma]^{-1}(x)| \leq |[\mathrm{Nor}_Y, \sigma]^{-1}(0)|$. ∎

Finally we prove assertion (c) of Theorem 2. For this purpose we introduce

$$\text{(4.4)}\qquad M(Y, \sigma) = \Big\{x \mid x \text{ has maximal } \mathbb{G}[\mathrm{Nor}_Y, \sigma] \text{ in degree} \wedge [\mathrm{Nor}_Y, \sigma]^{-1}([\mathrm{Nor}_Y, \sigma](x)) = \{x\}\Big\}.$$

LEMMA 4. *Let* $[\mathrm{Nor}_Y, \sigma]$ *be a SDS and let* $M(Y, \sigma)$ *be given by* (4.4). *Then*

(i) *for any connected graph* Y, $(0) \in M(Y, \sigma)$ *holds*;

(ii) *for* $Y = \mathrm{Line}_n$ *or* $Y = \mathrm{Circ}_n$ *we have* $M(Y, \sigma) = \{(0)\}$;

(iii) *there exist graphs with the property* $|M(Y, \sigma)| > 1$.

Proof. *Ad* (i): Lemma 3 guarantees that (0) has maximal $\mathbb{G}[\mathrm{Nor}_Y, \sigma]$ in-degree for arbitrary $\sigma \in S_n$. Thus it suffices to prove $[\mathrm{Nor}_Y, \sigma]^{-1}([\mathrm{Nor}_Y, \sigma](0)) = \{(0)\}$. Suppose there exists some $\eta \neq 0$ such that $[\mathrm{Nor}_Y, \sigma](\eta) = [\mathrm{Nor}_Y, \sigma](0)$. Since $\eta \neq 0$ there exists some vertex i with $\eta_i = 1$ and hence $[\mathrm{Nor}_Y, \sigma](0)_i = 0$. By assumption we have, for any vertex k, $([\mathrm{Nor}_Y, \sigma^*] \circ [\mathrm{Nor}_Y, \sigma](0))_k = 0$, from which we can conclude that there exists a vertex $j \in S_1^{<_\sigma}(i)$ such that $[\mathrm{Nor}_Y, \sigma](0)_j = 1$. Now we have the following situation: there exists a vertex $j \in S_1^{<_\sigma}(i)$ with $[\mathrm{Nor}_Y, \sigma](\eta)_j = [\mathrm{Nor}_Y, \sigma](0)_j = 1$ and $\eta_i = 1$, which is impossible and thus $[\mathrm{Nor}_Y, \sigma]^{-1}([\mathrm{Nor}_Y, \sigma](0)) = \{(0)\}$ and (i) follows.

Suppose $(0) \neq x = (x_r) \in M$ and let i be a vertex such that $x_i \neq 0$. We can w.l.o.g. assume that the vertex i with $x_i = 1$ is minimal w.r.t. $<_\sigma$. To show assertions (ii) and (iii) we prove two claims:

Claim 1. For all $j \in S_1(i)$ we have $j <_\sigma i$.

We will prove the claim by contradiction. Suppose there exists some $j \in S_1(i)$ such that $j >_\sigma i$ holds and let $x^{(i)}$ be the n-tuple defined by $x_r^{(i)} = 0$ for $i \neq r$ and $x_i^{(i)} = 1$. Lemma 3 guarantees (a) $|[\mathrm{Nor}_Y, \sigma]^{-1}(x^{(i)})| = |[\mathrm{Nor}_Y, \sigma]^{-1}(0)|$ and (b) that the preimages of (0) correspond uniquely to preimages η' of $x^{(i)}$ having the property $\eta'_i = 1$ (see (4.3)). We now consider $\eta = (\eta_r)$ with $\eta_i = 0$ and $\eta_r = 1$, otherwise. Since there exists some $j >_\sigma i$ we have $[\mathrm{Nor}_Y, \sigma](\eta) = (0)$, with $\eta_i \neq 1$, contradicting Claim 1 in view of Lemma 3, since $|[\mathrm{Nor}_Y, \sigma]^{-1}(x^{(i)})| = |[\mathrm{Nor}_Y, \sigma]^{-1}(0)|$.

Since Y is connected there exists some j adjacent to i with $j <_\sigma i$.

Claim 2. $\exists k \in S_1(j);\ k <_\sigma j$.

Let us assume that, $\forall\, k \in S_1(j),\ j <_\sigma k$. Then we define $x' = (x'_r)$, where

$$x'_r = \begin{cases} 1 & r = j \\ x_r & r \neq j. \end{cases} \tag{4.5}$$

Clearly, we have $x \neq x'$ and since $x_i = 1$, $x_j = 0$ holds. By assumption $\forall\, k \in S_1(j)$ we have $j <_\sigma k$, from which we can conclude $[\mathrm{Nor}_Y, \sigma](x') = [\mathrm{Nor}_Y, \sigma](x)$, which is impossible, and Claim 2 follows.

Since i is minimal w.r.t. $<_\sigma$ with the property $x_i = 1$ we have $x_k = 0$ and there exists no $s <_\sigma k$ with the property $x_s = 1$.

Ad (ii): Let $(0) \neq x \in M$. For $Y = \mathrm{Line}_n$ or Circ_n we can conclude from $x_k = 0$ that, for any $\eta \in [\mathrm{Nor}_Y, \sigma]^{-1}(x)$, $\eta_j = 1$ holds. Again, $|[\mathrm{Nor}_Y, \sigma]^{-1}(x)| = |[\mathrm{Nor}_Y, \sigma]^{-1}(0)|$ implies that

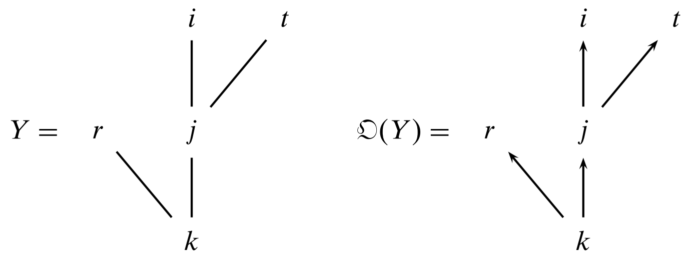
$$\mathrm{res}(r_{\mathfrak{O}})\colon [\mathrm{Nor}_Y, \sigma]^{-1}(x) \longrightarrow [\mathrm{Nor}_Y, \sigma]^{-1}(0) \tag{4.6}$$

is a bijection having the property $\mathrm{res}(r_{\mathfrak{O}})(\eta)_j = 0$. We now derive a contradiction by showing that there exists a preimage $\eta' = (\eta'_r)$ of (0) with the property $\eta'_j = 0$. For this purpose we define η' by

$$\eta'_r = \begin{cases} 0 & r = j \\ 1 & \text{otherwise.} \end{cases} \tag{4.7}$$

Clearly, we have $[\mathrm{Nor}_Y, \sigma](\eta') = (0)$, whence (ii).

Ad (iii): Let



We consider $x = (x_k, x_r, x_j, x_t, x_i)$, where $x_i = 1$, and $x_h = 0$, otherwise and $\sigma \in S_n$ such that $\mathfrak{O}(Y)_\sigma = \mathfrak{O}(Y)$. Then $[\mathrm{Nor}_Y, \sigma](x)_t = [\mathrm{Nor}_Y, \sigma](x)_k = 1$ and $[\mathrm{Nor}_Y, \sigma](x)_h = 0$, otherwise. For any $\eta \in [\mathrm{Nor}_Y, \sigma]^{-1}(x)$ we have $\eta_r = \eta_t = 1$, $\eta_i = 0$ and conclude that

$$\mathrm{res}_{\mathfrak{O}}\colon [\mathrm{Nor}_Y, \sigma]^{-1}(x) \to [\mathrm{Nor}_Y, \sigma]^{-1}(0), \quad \mathrm{res}_{\mathfrak{O}}(\eta_h) = \begin{cases} \eta_h & \text{for } h \neq i \\ 1 & h = i \end{cases}$$

is a bijection. Now let $\eta \in [\mathrm{Nor}_Y, \sigma]^{-1}([\mathrm{Nor}_Y, \sigma](x))$. Clearly we have $\eta_k = \eta_t = 0$ and, in view of $[\mathrm{Nor}_Y, \sigma](x)_k = 1$, $\eta_r = \eta_j = 0$. Finally, $[\mathrm{Nor}_Y, \sigma](x)_i = 0$ implies $\eta_i = 1$; i.e.,

$$[\mathrm{Nor}_Y, \sigma]^{-1}([\mathrm{Nor}_Y, \sigma](x)) = x,$$

proving (iii). ∎

It is clear that assertion (c) of Theorem 2 follows immediately from the above lemma since a digraph isomorphism preserves in-degrees.

Finally, to prove (d), let us assume that there exist $\lambda, \sigma, \pi \in S_n$ such that

$$[\mathrm{Nor}_{\lambda(Y)}, \lambda\sigma] = [\mathrm{Nor}_Y, \pi] \tag{4.8}$$

holds. Clearly, $\lambda \notin \mathrm{Aut}(Y)$ implies $Y \not\cong \lambda(Y)$ and there exists some Y-vertex i with the property $S_{1,\lambda(Y)}(i) \neq S_{1,Y}(i)$. Since $\mathrm{Aut}(Y)$ acts transitively, Y is regular and in particular we have $|S_{1,\lambda(Y)}(i)| = |S_{1,Y}(i)|$. Consequently, there exist vertices $k \in S_{1,Y}(i) \setminus S_{1,\lambda(Y)}(i)$ and $k' \in S_{1,\lambda(Y)}(i) \setminus S_{1,Y}(i)$.

Claim. We can w.l.o.g. assume that i is an $\mathfrak{O}(Y)_\pi$-origin.

By Proposition 2, (4.8) is equivalent to

$$\forall \gamma \in \mathrm{Aut(Y)}, \qquad [\mathrm{Nor}_{\gamma\lambda(Y)}, \gamma\lambda\sigma] = [\mathrm{Nor}_Y, \gamma\pi]. \tag{4.9}$$

Furthermore for any vertex i with the property $S_{1,\lambda(Y)}(i) \neq S_{1,Y}(i)$ we have

$$\gamma(S_{1,\lambda(Y)}(i)) = S_{1,\gamma\lambda(Y)}(\gamma(i)) \neq S_{1,Y}(\gamma(i)) = \gamma(S_{1,Y}(i))$$

and can therefore conclude

$$\forall\, i \in \mathrm{v}[Y], \quad S_{1,\lambda(Y)}(i) \neq S_{1,Y}(i) \quad \Longrightarrow \quad \forall\, \gamma \in \mathrm{Aut}(Y),\ \gamma(i),$$
$$S_{1,\gamma\lambda(Y)}(\gamma(i)) \neq S_{1,Y}(\gamma(i)).$$

To prove the lemma it then suffices to show $\gamma\lambda \in \mathrm{Aut}(Y)$. By assumption, $\mathrm{Aut}(Y)$ acts transitively and we can choose $\gamma \in \mathrm{Aut}(Y)$ such that $\gamma(i)$ is an $\mathfrak{O}(Y)_\pi$-origin, proving the claim.

For an index set M we set

$$(e_M)_j = \begin{cases} 1 & \text{if } j \in M \\ 0 & \text{otherwise.} \end{cases}$$

If i is an $\mathfrak{O}(\lambda(Y))_{\lambda\sigma}$-origin, we obtain the contradiction:

$$0 = ([\mathrm{Nor}_Y, \pi](e_k))_i \neq ([\mathrm{Nor}_{\lambda(Y)}, \lambda\sigma](e_k))_i = 1.$$

Thus we may assume that i is not an $\mathfrak{O}(\lambda(Y))_{\lambda\sigma}$-origin. We distinguish the two cases $\exists k' >_{\lambda\sigma} i$ and $\exists k' <_{\lambda\sigma} i$. In the first case we derive

$$1 = ([\mathrm{Nor}_Y, \pi](e_{k'}))_i \neq ([\mathrm{Nor}_{\lambda(Y)}, \lambda\sigma](e_{k'}))_i = 0,$$

which is impossible. For $k' <_{\lambda\sigma} i$ we consider the index set

$$M = \{h \mid h <_{\lambda\sigma} k' \ \wedge\ h \in S_{1,\lambda(Y)}(k') \setminus S_{1,Y}(i)\}.$$

Since i is an $\mathfrak{O}(Y)_\pi$-origin we have $([\mathrm{Nor}_Y, \pi](e_M))_i = 1$ and

$$\forall h \in S_{1,Y}(i), \qquad ([\mathrm{Nor}_Y, \pi](e_M))_h = 0 = ([\mathrm{Nor}_{\lambda(Y)}, \lambda\sigma](e_M))_h.$$

Therefore, $([\mathrm{Nor}_{\lambda(Y)}, \lambda\sigma](e_M))_{k'} = 1$ and since $k' \notin S_{1,Y}(i)$,

$$1 = ([\mathrm{Nor}_Y, \pi](e_M))_i \neq ([\mathrm{Nor}_{\lambda(Y)}, \lambda\sigma](e_M))_i = 0$$

holds. We finally prove $\mathfrak{O}(Y)_{\lambda\sigma} = \mathfrak{O}(Y)_\pi$. In view of (4.8) we have $[\mathrm{Nor}_{\lambda(Y)}, \lambda\sigma] = [\mathrm{Nor}_Y, \pi]$ and since $\lambda \in \mathrm{Aut}(Y)$ (2.5) guarantees

$$[\mathrm{Nor}_Y, \lambda\sigma] = [\mathrm{Nor}_Y, \pi]. \tag{4.10}$$

We immediately observe that h: $\mathrm{Acyc}(Y) \longrightarrow \{[\mathrm{Nor}_Y, \pi] \mid \pi \in S_n\}$, $\mathfrak{O}_\pi \mapsto [\mathrm{Nor}_Y, \pi]$ is bijective. Accordingly, (4.10) implies $\mathfrak{O}(Y)_{\lambda\sigma} = \mathfrak{O}(Y)_\pi$, whence (d) and the proof of Theorem 2 is complete.

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