Notes on Operator Algebras

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Chapter 1

Structure Theory I

1.1 Invertible Elements and Spectra

Definition. A C^* algebra is an involutive Banach algebra $\mathcal{A}$ with a norm satisfying the relations

$$\begin{aligned} \|AB\| &\leq \|A\| \, \|B\| \, , \\ \|A^*\| &= \|A\| \, , \\ \|A^*A\| &= \|A\|^2 \, . \end{aligned}$$

We denote the dual algebra by $\mathcal{A}'$. Then we can and often will use the weak topology for $\mathcal{A}$ and the weak-* topology for $\mathcal{A}'$. These properties are motivated by the study of bounded operators on Hilbert spaces. For example, to show that $\|A^*A\| = \|A\|^2$ holds in $\mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, we have the following calculation.

$$\begin{aligned} \|A\psi\|^2 &\leq \|\psi\|^2 \, \|A^*A\| \\ \implies \|A\|^2 &\leq \|A^*A\| \, . \end{aligned}$$

The first two properties are easy and they give

$$\begin{aligned} \|A^*A\| &\leq \|A^*\| \, \|A\| = \|A\|^2 \\ \implies \|A\|^2 &= \|A^*A\| \, . \end{aligned}$$

Remark. Some algebras come without a unit element. Such algebras are sometimes more easy to deal with if a unit element is appended in a formal way. A typical example of this is the appending of the Dirac delta function to a convolution algebra of smooth functions, there being no smooth representative which can play the role of the unit element in a convolution algebra.

Definition. Let $\mathcal{A}$ be a Banach algebra. Define the unitization of $\mathcal{A}$ to be

$$\mathcal{A}^+ = (\dashv, \lambda) \in \mathcal{A} \times \mathbb{C}$$

with

$$(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda\mu).$$

Definition. The resolvent of $A \in \mathcal{A}$, $\mathrm{Res}\,(A) \subseteq \mathbb{C}$, is the set $\{\lambda \in \mathbb{C} \,:\, (\lambda - A)^{-1} exists in \mathcal{A}\}$.

Definition. The spectrum of $A \in \mathcal{A}$, $\mathrm{Spec}\,(A)$, is the complement of $\mathrm{Res}\,(A)$.

1.1. Theorem (Spectral Radius).

1. $\mathrm{Res}\,(A)$ *is open.*
2. $\mathrm{Spec}\,(A)$ *is compact.*
3. $\sup_{\lambda \in \mathrm{Spec}(A)} |\lambda| = \lim_{n\to\infty} \|A^n\|^{1/n} = \inf_n \|A^n\|^{1/n}$.

Proof. The first two follow easily from the definitions. To prove the third, write $\alpha_n = \|A^n\|$, then $\alpha_{n+l} \leq \alpha_n \alpha_l$. Let $m \in \mathbb{Z}$, $m > 0$. We can write $n = p(n)m + q(n)$, where $p(n)$ and $q(n) < m$ are unique integers. Then

$$\alpha_n^{1/n} \leq \alpha_{p(n)m}^{1/n} \alpha_{q(n)}^{1/n} \leq \alpha_m^{p(n)/n} \alpha_{q(n)}^{1/n}.$$

Now

$$\limsup_{n\to\infty} \alpha_n^{1/n} \leq \lim_{n\to\infty} \alpha_m^{p(n)/n} \left(\sup\{\alpha_0, \alpha_1, \ldots, \alpha_m\}\right)^{1/n} \quad \leq \lim_{n\to\infty} \alpha_m^{p(n)/n} \leq \alpha_m^{1/m}, \text{ for any } m.$$

Therefore

$$\limsup_{n\to\infty} \alpha_n^{1/n} \leq \inf_n \alpha_n^{1/n} \leq \lim \inf_{n\to\infty} \alpha_n^{1/n},$$

so that the limit of the theorem exists and equals $\inf_n \|A^n\|^{1/n}$. To show that $\sup_{\lambda \in \mathrm{Spec}(A)} |\lambda| = \lim_n \|A^n\|^{1/n}$, let ζ be such that $|\zeta| > \lim_n \|A^n\|^{1/n}$, then $|\zeta| > \inf_n \|A^n\|^{1/n}$ so that the series $\sum (A/\zeta)^n$ converges in norm to $(1 - A/\zeta)^{-1}$ so $\zeta \notin \mathrm{Spec}\,(A)$. Furthermore, suppose that for all r in the interval $\left(\sup_{\lambda \in \mathrm{Spec}(A)} |\lambda|, \sup_n \|A^n\|^{1/n}\right)$, $(1 - A/r)^{-1}$ existed. Then $(1 - A/r)^{-1}$ would be analytic for $r > \sup_{\lambda \in \mathrm{Spec}(A)} |\lambda|$ and $(1 - A/r)^{-1} = \sum_n (A/r)^n$. But $\|A^n/r^n\|^{1/n} > \frac{\|A^n\|^{1/n}}{\inf_n \|A^n\|^{1/n}} \geq 1$. $\Rightarrow\Leftarrow$. $\square$

1.2. Theorem (Holomorphic Symbolic Calculus). *Let $\mathcal{A}$ be a C^* algebra, and $A \in \mathcal{A}$. Let f be holomorphic on $\mathcal{O} \supset$ $\mathrm{Spec}\,(A)$. Then we can define $f(A) \in \mathcal{A}$ such that*

1. $f \mapsto f(A)$ *is a homomorphism of the algebra of holomorphic functions on $\mathcal{O}$ to $\mathcal{A}$.*
2. *For the function $f(\lambda) \equiv \lambda$, $f(A) = A$.*
3. *For $f(\lambda) \equiv (\lambda - \lambda_0)^{-1}$, $f(A) = (A - \lambda)^{-1}$.*
4. $\mathrm{Spec}\,(f(A)) = f(\mathrm{Spec}\,(A))$. *[spectral mapping theorem]*

Proof. First we prove that $\frac{1}{2\pi i} \int_\Gamma (\alpha - \lambda)^n (\lambda - A)^{-1} d\lambda = (\alpha - A)^n$, where Γ is a contour surrounding $\mathrm{Spec}\,(A)$. Let y_n equal this integral expression. Then it is easy to show that $(\alpha - A) y_n = y_{n+1}$. Thus the formula will follow from the case $n = 0$.

By the Cauchy theorem for vector-valued integrals we can replace the contour Γ by a circle of radius $r > \|A\|$. Then integrate $(\lambda - A)^{-1} = \sum \lambda^{-n-1} A^n$ term by term. This proves the result for $n = 0$.

Now we can assert the statements of the theorem for rational functions f since the integrands will then always be of the given form by factorization.

Finally, the approximation of holomorphic functions by rational functions converges uniformly on compact sets. Therefore we can define $f(A)$ for holomorphic f by interchanging the limit and the integral. □

Remark. In order to state a more refined version of the symbolic calculus, valid for arbitrary continuous functions on $\mathrm{Spec}\,(A)$, we need to delve into the theory of commutative C^* algebras.

Definition. A subspace $\mathcal{I}$ of a commutative algebra $\mathcal{A}$ is called an ideal if, for any $A \in \mathcal{A}$, $AB \in \mathcal{I}$ whenever $B \in \mathcal{I}$.

Definition. A complex homomorphism, ϕ, of a Banach algebra is a linear functional with the property $\phi(AB) = \phi(A)\phi(B)$. It is also called a character of the algebra.

1.3. Theorem (Gleason, Kahane, Zelazko). *If ϕ is a linear functional on a Banach algebra such that $\phi(1) = 1$, and $\phi(A) \neq 0$ for any invertible A, then ϕ is a complex homomorphism.*

Proof. See [Rud91]. □

1.4. Theorem (Gelfand-Mazur). *Let $\mathcal{A}$ be a Banach algebra in which every nonzero element is invertible. Then $\mathcal{A}$ is isometrically isomorphic to $\mathbb{C}$.*

Proof. Let $A \in \mathcal{A}$ and $\lambda_1 \neq \lambda_2$; then at least one of $\lambda_1 - A$, $\lambda_2 - A$ is invertible by hypothesis. $\mathrm{Spec}\,(A)$ is nonempty by a standard result (a spectrum is never empty), so it follows that for each such A there is a unique $\lambda(A) \in \mathbb{C}$ in $\mathrm{Spec}\,(A)$. The mapping $A \mapsto \lambda(A)$ is an isomorphism since $A = \lambda(A) \cdot 1$; it is obviously an isometry. □

1.5. Theorem. *Let $\mathcal{A}$ be a commutative Banach algebra, and let Δ be the set of all complex homomorphisms of $\mathcal{A}$. Then*

1. *Every maximal ideal of $\mathcal{A}$ is the kernel of some $h \in \Delta$.*
2. *If $h \in \Delta$, $\ker\,(h)$ is a maximal ideal of $\mathcal{A}$.*
3. *$A \in \mathcal{A}$ invertible if and only if $h(A) \neq 0$ for all $h \in \Delta$.*
4. *$A \in \mathcal{A}$ invertible if and only if A lies in no proper ideal of $\mathcal{A}$.*
5. *$\lambda \in \mathrm{Spec}\,(A)$ if and only if $h(A) = \lambda$ for some $h \in \Delta$.*

Proof.

1. Let M be a maximal ideal of $\mathcal{A}$. Since the set of all invertible elements is open, maximal ideals are closed; so M is closed. Therefore $\mathcal{A}/M$ is a Banach algebra. Choose $x \in \mathcal{A}$, $x \notin M$, and set $J = \{ax + y \,:\, a \in \mathcal{A}, y \in M\}$. Then J is an ideal, and $x \in J$ so J is larger than M, so $J = \mathcal{A}$. Therefore, for some $A \in \mathcal{A}$ and $y \in M$ we have $Ax + y = 1$. Applying the quotient map $\pi : \mathcal{A} \longrightarrow \mathcal{A}/M$ we see that $\pi(A)\pi(x) = \pi(1)$, thus every nonzero element of $\mathcal{A}/M$ is invertible. So by the Gelfand-Mazur theorem $\mathcal{A}/M \cong \mathbb{C}$, $j : \mathcal{A}/M \longrightarrow \mathbb{C}$. Let $h = j \circ \pi$, then $h \in \Delta$ and $h(M) = 0$.

2. If $h \in \Delta$ then $h^{-1}(0)$ is an ideal in $\mathcal{A}$ with codimension 1. Therefore it is maximal.

3. If A is invertible in $\mathcal{A}$ and $h \in \Delta$, then $h(A)h(A^{-1}) = h(1) = 1$ and so $h(A) \neq 0$. If A is not invertible then $\{aA : a \in \mathcal{A}\} \cap \{1\} = \varnothing$, so $\{aA : a \in \mathcal{A}\}$ is a proper ideal which lies in a maximal ideal and so it is annihilated by some $h \in \Delta$ by the first result.

4. No invertible element lies in any proper ideal. The converse is proved in the previous item.

5. Apply the third item to $\lambda - A$ instead of A.

□

Remark. As an application of the above we have the following result on Fourier series.

1.6. Theorem (Wiener Lemma). *Suppose* $f : \mathbb{R}^n \longrightarrow \mathbb{C}$,

$$f(x) = \sum_{m \in \mathbb{Z}^n} a_m \exp im \cdot x, \quad \sum |a_m| < \infty,$$

If $f(x)$ *is never zero then*

$$1/f(x) = \sum_{m \in \mathbb{Z}^n} b_m \exp im \cdot x, \quad \sum |b_m| < \infty.$$

Proof. Let $\mathcal{A}$ be the commutative Banach algebra of functions of the form $\sum a_m \exp im \cdot x$ with the norm $\|f\| = \sum |a_m|$. For each $x \in \mathbb{R}^n$, the evaluation map $f \mapsto f(x)$ is a complex homomorphism. By assumption no evaluation gives zero. So if we can prove that all complex homomorphisms of $\mathcal{A}$ are evaluations for some $x \in \mathbb{R}^n$, then the third part of the structure theorem above will assert the existence of $1/f$ in $\mathcal{A}$.

Let h be any complex homomorphism of $\mathcal{A}$. Write $g_r(x) = \exp ix_r$, $r = 1, \ldots, n$; x_r is the r-th coordinate of $x \in \mathbb{R}^n$. Then $g_r \in \mathcal{A}$, $1/g_r \in \mathcal{A}$, and $\|g_r\| = \|1/g_r\| = 1$.

It is easy to see that if $\|A\| < 1$ then $|\phi(A)| < 1$ for any complex homomorphism ϕ, since for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ we know $(1 - A/\lambda)^{-1}$ exists and so $\phi(1 - A/\lambda) \neq 0$ and so $\phi(A) \neq \lambda$. So we see that $|h(g_r)| \leq 1$ and $|h(1/g_r)| = |1/h(g_r)| \leq 1$. Therefore $h(g_r) = \exp iy_r$ for some $y_r \in \mathbb{R}$, $r = 1, \ldots, n$.

Let P be any trigonometric polynomial. Then $h(P) = P(y_1, \ldots, y_n)$. But h is continuous and the trigonometric polynomials are dense in $\mathcal{A}$, so $h(f) = f(y)$ for all $f \in \mathcal{A}$ and so h is evaluation at y. □

1.2 Gelfand Transform

Definition. Given $A \in \mathcal{A}$ we can define a function $\widehat{A} : \Delta \longrightarrow \mathbb{C}$ by

$$\widehat{A}(h) = h(A).$$

$\widehat{A}$ is called the Gelfand transform of A. It is also sometimes called the spectrum of $\mathcal{A}$, though we will never use this terminology.

Definition. The Gelfand topology on Δ is the weakest topology such that $\widehat{A}$ is continuous for every $A \in \mathcal{A}$.

Definition. The radical of $\mathcal{A}$, $\mathrm{rad}\,(\mathcal{A})$, is the intersection of all the maximal ideals of $\mathcal{A}$.

1.7. Theorem. *Let Δ be equipped with the Gelfand topology. Then*

1. *Δ is a compact Hausdorff space.*
2. *The Gelfand transform is a homomorphism of $\mathcal{A}$ onto a subalgebra of the continuous functions on Δ, and the kernel of this homomorphism is $\mathrm{rad}\,(A)$. Thus the Gelfand transform is an isomorphism if and only if $\mathrm{rad}\,(A) = \{0\}$.*
3. *For all $A \in \mathcal{A}$, $\mathrm{Ran}(\widehat{A}) = \mathrm{Spec}\,(A)$, and $\left\|\widehat{A}\right\|_\infty = \sup_{\lambda \in \mathrm{Spec}(A)} |\lambda| \leq \|A\|$. Furthermore, $A \in \mathrm{rad}\,(A)$ if and only if $\sup_{\lambda \in \mathrm{Spec}(A)} |\lambda| = 0$.*

Proof. For a complete proof see [Rud91]. The second and third items follow from the structure theorem above, together with some computation. The first item follows from the Banach-Alaoglu theorem and a proof of the closure of Δ. The Gelfand topology is the restriction of the weak-* topology to Δ. □

Definition. The set $\widehat{\mathcal{A}} \subseteq \Delta$ is called the spectrum of $\mathcal{A}$. To avoid technical complications, the spectrum of $\mathcal{A}$ is actually defined as not to contain the zero homomorphism.

Remark. Here is an example that shows how the Gelfand transform is a generalization of the Fourier transform, in the L^1 context. Let $\mathcal{A} = L^1(\mathbb{R}^n)\,dx$, with unit attached. So members of $\mathcal{A}$ are $f + \alpha\delta$, where δ is the Dirac measure. Of course, the multiplication is convolution.

Let h be a complex homomorphism, $h \in \Delta$; then h is one of the following forms,

$$h_t(f + \alpha\delta) = \widehat{f}(t) + \alpha$$

or

$$h_\infty(f + \alpha\delta) = \alpha$$

We prove this as follows. If $h(f) = 0$ for all $f \in \mathcal{A}$ then $h = h_\infty$. Assume $h(f) \neq 0$ for some $f \in \mathcal{A}$. Then $h(f) = \int f\beta$ for some $\beta \in L^\infty(\mathbb{R}^n)\,dx$. Since $h(f \star g) = h(f)h(g)$, we can show that β coincides almost everywhere with a continuous function b which satisfies $b(x + y) = b(x)b(y)$. But every bounded solution of this functional equation is of the form $b(x) = \exp(-ixt)$. Thus $h(f) = \widehat{f}(t)$ and h is of the form h_t.

So $\Delta = \mathbb{R}^n \cup \{\infty\}$, say with the topology of the one-point compactification. Since $\widehat{f}(t) \to 0$ as $|t| \to \infty$, $\widehat{A} \subset C(\Delta)$. $\widehat{A}$ separates points on Δ so the weak topology induced on Δ by $\widehat{A}$ is as we have chosen.

1.8. Theorem (Gelfand-Naimark). *Let $\mathcal{A}$ be a commutative C^* algebra, and equip Δ with the Gelfand topology as usual. Then the Gelfand transform is an isometric *-isomorphism of $\mathcal{A}$ onto the algebra of continuous $\mathbb{C}$-valued functions on Δ, $C(\Delta)$.*

Proof. In order to show that the involution is preserved we need only show that for $A = A^*$, $h(A) \in \mathbb{R}$. So let $A = A^*$ and write $h(A) = \alpha + i\beta$. Calculate

$$\begin{gathered} h(A+it) = \alpha + i(\beta + t) \\ \alpha^2 + (\beta+t)^2 = |h(A+it)|^2 \le \|A+it\|^2 = \|(A+it)(A-it)\| \le \|A\|^2 + t^2 \\ \alpha^2 + \beta^2 + 2\beta t \le \|A\|^2 \quad \forall t \in \mathbb{R} \\ \Longrightarrow \beta = 0. \end{gathered}$$

By definition the elements of $\widehat{\mathcal{A}} \subseteq C(\Delta)$ separate points of Δ. Also, $\widehat{1} = 1$, so $1 \in \widehat{A}$. Therefore $\widehat{A}$ is dense in $C(\Delta)$ by the Stone-Weierstrass theorem.

Now we show that $\widehat{\cdot}$ is an isometry. Let $x \in \mathcal{A}$, $y = xx^*$. So $y = y^*$ and $\|y^2\| = \|y\|^2$ and $\|y^m\| = \|y\|^m$. Therefore by the spectral radius formula $\|\widehat{y}\|_\infty = \|y\|$. Since $y = xx^*$, $\widehat{y} = |\widehat{x}|^2$, and so $|\widehat{x}|^2 = \|\widehat{y}\|_\infty = \|y\| = \|xx^*\| = \|x\|^2$, proving the isometry. From this, $\widehat{A}$ is closed in $C(\Delta)$ and so $\widehat{A} = C(\Delta)$. □

1.9. Theorem (Inverse Gelfand-Naimark). *Let $\mathcal{A}$ be a commutative C^* algebra. Let $x \in \mathcal{A}$ be such that the polynomials in x and x^* are dense in $\mathcal{A}$. Then we can define an isometric isomorphism $\Psi : C(\mathrm{Spec}\,(x)) \longrightarrow \mathcal{A}$ by*

$$\widehat{(\Psi f)} = f \circ \widehat{x},$$

and we have

$$\Psi f^* = (\Psi f)^*.$$

Moreover if $f(\lambda) = \lambda$, then $\Psi f = x$.

Proof. Let Δ be equipped with the Gelfand topology. Then $\widehat{x}$ is a continuous function on Δ with $\mathrm{Ran}(\widehat{x}) = \mathrm{Spec}\,(x)$. Suppose we have h_1 and h_2 from Δ such that $h_1(x) = h_2(x)$. Then also $h_1(x^*) = h_2(x^*)$. By continuity, $h_1(y) = h_2(y)$ for all y in the algebra generated by polynomials in x and x^*, i.e. $\mathcal{A}$. Therefore $h_1 = h_2$. Therefore $\widehat{x}$ is one-to-one. Since Δ is compact, $\widehat{x}$ is a homeomorphism $\Delta \to \mathrm{Spec}\,(x)$. Therefore $f \mapsto f \circ \widehat{x}$ is an isometric isomorphism of $C(\mathrm{Spec}\,(x))$ onto $C(\Delta)$. By the Gelfand-Naimark theorem, $f \circ \widehat{x}$ is thus the Gelfand transform of a unique element in $\mathcal{A}$ which we denote Ψf, and $\|\Psi f\| = \|f\|_\infty$. If $f(\lambda) = \lambda$, then $f \circ \widehat{x} = \widehat{x}$ and $\Psi f = x$. □

Remark. This last theorem provides a continuous symbolic calculus for operators as long as they generate a commutative C^* algebra. So, for example, if x is a normal operator then we apply the above theorem to the algebra generated by x and x^*, and we get the continuous functional calculus for normal operators.

1.3 Local Algebras, Idempotents and Projections

Definition. A local Banach algebra is a dense subalgebra $\mathcal{A}$ of a Banach algebra $\overline{\mathcal{A}}$ where $\mathcal{A}$ is closed under holomorphic symbolic calculus in $\overline{\mathcal{A}}$.

Remark. Note that we need the explicit reference to the completion $\overline{\mathcal{A}}$, in order to define $f(a)$ for $a \in \mathcal{A}$ because f is required to be holomorphic on the spectrum of the element a, and the spectrum depends on the whole algebra. However, if $\mathcal{A}$ has a unit, then direct reference to $\overline{\mathcal{A}}$ is not necessary, as shown by the following.

1.10. Theorem. *Let $\mathcal{A}$ be a local Banach algebra with unit. Let $z \in \mathcal{A}$ be invertible in $\overline{\mathcal{A}}$. Then z is invertible in $\mathcal{A}$. Therefore the spectrum of any element is the same in $\mathcal{A}$ or $\overline{\mathcal{A}}$.*

Proof. Let z be invertible in $\overline{\mathcal{A}}$. So the domain of holomorphy of $f(\lambda) = \lambda^{-1}$ is contained in the $\overline{\mathcal{A}}$-spectrum of z, by definition. By definition, $\mathcal{A}$ is closed under action of f in its domain of holomorphy. $\square$

Definition. Let $\mathcal{A}$ be a local Banach algebra. An idempotent in $\mathcal{A}$ is an element x with $x^2 = x$. If idempotents x, y satisfy $xy = yx = 0$, they are said to be orthogonal, written $x \perp y$. If idempotents x, y satisfy $xy = yx = x$, we write $x \leq y$.

Definition. Let $\mathcal{A}$ be a local C^* algebra. Then two idempotents x, y are said to be orthogonal if, in addition to the above, we have $x^* y = y x^* = 0$.

Definition. Let $\mathcal{A}$ be a local C^* algebra. An idempotent is called a projection if it is self-adjoint.

1.11. Theorem. *The idempotents of $\mathcal{A}$ are dense in the idempotents of $\overline{\mathcal{A}}$.*

Proof. Let x be an idempotent of $\overline{\mathcal{A}}$, and let $\epsilon > 0$. Choose y in a neighbourhood of x such that $\|y - y^2\| = \|x - x^2 + \delta - 2x\delta - \delta^2\| < \epsilon$. So the spectrum of y is contained completely in an ϵ neighbourhood of 0 and 1. Construct the required idempotent by holomorphic calculus. **This is not so clear to me...**. $\square$

Chapter 2

von Neumann Algebras

2.1 Commutant and Bicommutant

Remark. Now we will introduce von Neumann algebras. These are defined in a concrete sense, explicitly as subalgebras of $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. Recall the zoo of topologies on $\mathcal{B}(\mathcal{H})$.

1. The norm topology or uniform topology.
2. The strong topology is the locally convex topology associated to the family of seminorms $\chi \to \|\chi v\|$, $v \in \mathcal{H}$.
3. The weak topology is the locally convex topology associated with the family of seminorms $\chi \to |\,(v, \chi w)\,|$, $v, w \in \mathcal{H}$.

Definition. A von Neumann algebra is a strongly closed C^*-subalgebra of $\mathcal{B}(\mathcal{H})$.

Definition. The commutant of $M \in \mathcal{B}(\mathcal{H})$ is the set $M^c = \{x \in \mathcal{B}(\mathcal{H}) \,:\, xy = yx \,\forall y \in M\}$. Clearly M^c is a weakly closed subalgebra.

2.1. Theorem (von Neumann). *Let $\mathcal{M}$ be a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, containing the identity. Then T.F.A.E.*

1. *$\mathcal{M} = \mathcal{M}^{cc}$.*
2. *$\mathcal{M}$ is weakly closed.*
3. *$\mathcal{M}$ is strongly closed.*

Proof. The second clearly follows from the first. To show that the second and third are equivalent, note the fact that each strongly closed convex set in $\mathcal{B}(\mathcal{H})$ is weakly closed.

To show that the last implies the first, let y be a fixed element of $\mathcal{M}^{cc}$. Let p be the projection onto the closed subspace of $p\mathcal{H} = \overline{\{x\xi \,:\, x \in \mathcal{M}\}}$ for some fixed $\xi \in \mathcal{H}$. Clearly $py = yp$ so $y\xi \in p\mathcal{H}$. Therefore there exists $x \in \mathcal{M}$ such that $\|(y - x)\xi\| < \epsilon$, for each $\epsilon > 0$.

Take $\xi_1, \xi_2, \ldots, \xi_n \in \mathcal{H}$ and put $\xi = \xi_1 \oplus \cdots \oplus \xi_n \in \mathcal{H} \oplus \cdots \oplus \mathcal{H}$. Now $y \oplus \cdots \oplus y \in (\mathcal{M} \oplus \cdots \oplus \mathcal{M})^{cc}$. Apply the construction above to find an x such that $\|(y \oplus \cdots \oplus y - x \oplus \cdots \oplus x)\xi\| < \epsilon$. Then

$$\sum_{k=1}^{n} \|(y-x)\xi_k\|^2 < \epsilon^2.$$

Therefore y is approximated by x in the strong topology, and so by hypothesis $y \in \mathcal{M}$. Therefore $\mathcal{M}^{cc} \subseteq \mathcal{M}$. The opposite inclusion is obvious. □

2.2. Theorem (Kaplansky Density). *Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ with strong closure $\mathcal{M}$. Then the unit ball $\mathcal{A}^1$ of $\mathcal{A}$ is strongly dense in the unit ball $\mathcal{M}^1$ of $\mathcal{M}$. If $1 \in \mathcal{A}$, then the unitary group of $\mathcal{A}$ is strongly dense in the unitary group of $\mathcal{M}$.*

Proof. See [Kap51]. □

2.2 Factors

Definition. Let $\mathcal{M}$ be a von Neumann algebra. Then the center of $\mathcal{M}$ is $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}^c$.

Definition. If $\mathcal{Z}(\mathcal{M}) = \{\alpha 1 \,:\, \alpha \in \mathbb{C}\}$, then $\mathcal{M}$ is called a factor.

Remark. A factor is a kind of algebraic counterpart of an irreducible representation. The factors play an important role in the classification of von Neumann algebras.

2.3 The Trace

Definition. Let $A \in \mathcal{B}(\mathcal{H})$, $A \geq 0$. The trace of A is

$$\mathrm{Tr}\,(A) = \sum_i (v_i, Av_i) \in [0, \infty].$$

If $\mathrm{Tr}\,(A) < \infty$ then A is called trace class.

Definition. Consider the family of seminorms $\|\cdot\|_i : A \mapsto |\mathrm{Tr}\,(AB_i)\,|$, for $\{B_i\}$ the set of trace class operators. The topology associated to this family is called the σ-weak topology or the ultra-weak topology.

Remark. Choosing a basis we see that every functional $A \mapsto \mathrm{Tr}\,(AB)$ can be written as $A \mapsto \sum (v_i, Aw_i)$, so the ultra-weak topology is stronger than the weak topology. However, these two topologies coincide on the unit ball of $\mathcal{B}(\mathcal{H})$.

Definition. A bounded functional ϕ on a von Neumann algebra $\mathcal{M}$ is called normal if for each bounded monotone increasing net $\{A_i\}$ in $\mathcal{M}_{sa}$ with limit A, the net $\{\phi(A_i)\}$ converges to $\phi(A)$. The set of normal functionals on a von Neumann algebra $\mathcal{M}$ is denoted by $\mathcal{M}_*$. $\mathcal{M}_*$ is a Banach space and $\mathcal{M}_*{}' = \mathcal{M}$. Thus $\mathcal{M}_*$ is called the pre-dual of $\mathcal{M}$.

2.3. Theorem. *Let ϕ be a bounded functional on a von Neumann algebra $\mathcal{M}$. Then T.F.A.E.*

1. *ϕ is normal.*

2. *ϕ is weakly continuous on the unit ball in $\mathcal{M}$.*

3. *ϕ is ultra-weakly continuous.*

4. *There is a trace class operator A such that $\phi(B) = \mathrm{Tr}\,(AB)$ for all $B \in \mathcal{M}$.*

Proof. See [Ped79]. □

2.4. Theorem. *Let ϕ be a positive normal functional on a von Neumann algebra $\mathcal{M}$. Then there exists a positive trace class element $A \in \mathcal{B}(\mathcal{H})$, such that $\phi(B) = \mathrm{Tr}\,(AB)$ for all $B \in \mathcal{M}$. Furthermore, $\|\phi\| = \mathrm{Tr}\,(A)$.*

Proof. See [Ped79]. □

2.4 Hyperfinite Algebras

Definition. A von Neumann algebra $\mathcal{M}$ is called hyperfinite if there exists an increasing sequence of finite-dimensional subalgebras whose union is weakly-dense in $\mathcal{M}$.

Chapter 3

States and Representations

3.1 GNS Construction

Definition. A linear functional on a C^* algebra $\mathcal{A}$ satisfying $\phi(A^*A) \geq 0$ for all $A \in \mathcal{A}$ is called a positive functional.

3.1. Theorem. *The following properties hold for positive functionals ϕ.*

1. *If A is self-adjoint then $\phi(A) \in \mathbb{R}$.*
2. $|\phi(A^*B)|^2 \leq \phi(A^*A)\phi(B^*B)$.
3. *ϕ is continuous w.r.t. the norm topology.*
4. $\phi(B^*AB) \leq \|A\| \, \phi(B^*B)$.

Proof. These are elementary properties following from the definition. $\square$

Definition. The state space for $\mathcal{A}$ is defined by

$$\mathcal{A}'_{1+} = \{\phi \in A' \, : \, \phi \geq 0, \|\phi\| = 1\} \, .$$

3.2. Theorem (Banach-Alaoglu). *The unit ball in $\mathcal{A}'$, $\mathcal{A}'_1$, is compact in the weak-* topology. And thus so is $\mathcal{A}'_{1+}$.*

Remark. $\mathcal{A}'_{1+}$ is obviously convex. Since it is compact, by the Krein-Milman theorem it is equal to the convex hull of its extreme points.

Definition. The extreme points of $\mathcal{A}'_{1+}$ are called pure states.

Definition. A representation of $\mathcal{A}$ on $\mathcal{B}(\mathcal{H})$ is a C^* algebra homomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$.

Definition. π is called irreducible if the only closed invariant subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$.

Definition. A vector $\Phi \in \mathcal{H}$ is cyclic for π if the set $\{\pi(A)\Phi \, : \, A \in \mathcal{H}\}$ is a total subset of $\mathcal{H}$.

3.3. Theorem. *T.F.A.E.*

1. *$\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is irreducible.*
2. *Every operator that commutes with all of $\pi(\mathcal{A})$ is a multiple of 1. [Schur]*
3. *Every $\Phi \in \mathcal{H}$ is cyclic for π.*

Definition. π_1 and π_2 are called unitarily equivalent if there exists an isomorphism $V : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ such that $\pi_2(A) = V\pi_1(A)V^{-1}$ for all $A \in \mathcal{A}$.

3.4. Theorem. *Let π_1 and π_2 be irreducible and not unitarily equivalent. Then for every bounded $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$, we have*

$$T\pi_1(A) = \pi_2(A)T \quad \forall A \in \mathcal{A} \quad \Longleftrightarrow \quad T = 0.$$

Proof. Consider T^*T and use the above result. □

Definition. π is faithful if one of the following equivalent statements holds.

1. $\pi(A) = \pi(B) \Longrightarrow A = B$.
2. $\pi(A) = 0 \Longrightarrow A = 0$.
3. $A \mapsto \pi(A)$ is an isomorphism.

3.5. Theorem (Gelfand-Naimark-Segal). *Let ϕ be a positive functional on $\mathcal{A}$. Then there exists a (cyclic) representation π_ϕ on a Hilbert space, with a cyclic vector Φ_ϕ such that*

$$\phi(A) = (\Phi_\phi, \pi_\phi(A)\Phi_\phi) \quad \forall A \in \mathcal{A}.$$

Furthermore, π_ϕ is unique up to unitary equivalence.

Proof. Let $\mathcal{F}_\phi = \{A \in \mathcal{A} : \phi(A^*A) = 0\}$. It is easy to show that $\mathcal{F}_\phi$ is a left ideal. Let $\xi_\phi(\mathcal{A}) = \mathcal{A}/\mathcal{F}_\phi$. Note that $\omega(A, B) = \phi(A^*B)$ defines a positive scalar product on $\xi_\phi(\mathcal{A})$. We write $\xi_\phi(A)$ for the projection of A onto $\mathcal{A}/\mathcal{F}_\phi$. By completing $\xi_\phi(\mathcal{A})$ we get a Hilbert space.

Define the representation π_ϕ by $\pi_\phi(A)\xi_\phi(B) = \xi_\phi(AB)$. It follows from $(\xi_\phi(B), \pi_\phi(A)\xi_\phi(B)) = \phi(B^*AB)$ that $\pi_\phi(A)$ is bounded; $\|\pi_\phi(A)\| \leq \|A\|$. Thus by continuity it can be extended to all of $\mathcal{H}$. Set $\Phi_\phi = \xi_\phi(1)$.

All that remains is the proof of uniqueness. Let π be another representation of $\mathcal{A}$ on $\mathcal{H}$ with cyclic vector Φ and such that $\phi(A) = (\Phi, \pi(A)\Phi)$ for all $A \in \mathcal{A}$. The sets $\pi(\mathcal{A})\Phi$ and $\pi_\phi(\mathcal{A})\Phi_\phi$ are each everywhere dense subspaces of $\mathcal{H}$. Thus we can define V by $V\pi_\phi(A)\Phi_\phi = \pi(A)\Phi$ for all $A \in \mathcal{A}$, and V extends by continuity to an isomorphism $\mathcal{H} \to \mathcal{H}$. Now we can use this to prove $\pi_\phi(A) = V\pi(A)V^{-1}$ for all $A \in \mathcal{A}$. □

3.6. Theorem (Gelfand-Naimark). *For all $A \in \mathcal{A}$, $\|A\| = \sup_\pi \|\pi(A)\|$.*

Definition. Let π_ϕ be the GNS representation for the state ϕ. The folium of the representation π_ϕ is the set of all states of the form

$$\phi_\rho(A) = \mathrm{Tr}\left(\rho\pi_\phi(A)\right), \quad A \in \mathcal{A},$$

for $\rho \in \mathcal{B}(\mathcal{H}_\phi)$ trace class and positive.

3.7. Theorem. *The folium of a faithful representation of a C^* algebra $\mathcal{A}$ is weakly dense in $\mathcal{A}'_{1+}$.*

Proof. See Ref. [Fel60]. □

Definition. Let γ be an automorphism of the C^* algebra $\mathcal{A}$. We say that the positive linear functional ϕ on $\mathcal{A}$ is invariant with respect to γ if $\phi(\gamma(A)) = \phi(A)$ for all $A \in \mathcal{A}$.

3.8. Theorem (Unitary Representations). *Let ϕ be as above. Let $(\pi_\phi, \mathcal{H}_\phi, \Phi_\phi)$ be given by the GNS construction. Then there exists a unitary operator U such that*

$$\pi_\phi(\gamma(A)) = U\pi_\phi(A)U^{-1} \quad \forall A \in \mathcal{A}.$$

Proof. Define U by using

$$U\pi_\phi(A)\Phi_\phi = \pi_\phi(\gamma(A))\Phi_\phi.$$

□

Remark. Note that if we demand $U\Phi_\phi = \Phi_\phi$ then U is determined uniquely.

Definition. A subspace of the state space, $F \subseteq \mathcal{A}'_{1+}$, is said to be separating for $\mathcal{A}$ if

$$\mathcal{A} \ni A \text{ positive and } \phi(A) = 0 \quad \forall \phi \in F \Longrightarrow A = 0.$$

3.9. Theorem. *Let $\mathcal{A}$ be separable. Then the state $\sum 2^{-n}\phi_n$ is separating for any dense sequence $\{\phi_n\} \subseteq \mathcal{A}'_{1+}$.*

Proof. $\mathcal{A}$ is separable, therefore the unit ball of $\mathcal{A}'$ is second countable, since it is weak-*-metrizable and compact. Therefore $\mathcal{A}'_{1+}$ is second countable, and the result follows. □

Definition. It is customary to say that ϕ is faithful if it is separating. This is sensible by the result of the next theorem. For each $F \subseteq \mathcal{A}'_{1+}$ we form $\mathcal{H}_F = \oplus_{\phi\in F}\mathcal{H}_\phi$ and $\pi_F = \oplus_{\phi\in F}\pi_\phi$.

3.10. Theorem. *Let $F \subseteq \mathcal{A}'_{1+}$ be a separating family of states for $\mathcal{A}$. Then π_F is a faithful representation of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$.*

Proof. Let A be positive and $A \in \ker(\pi_F)$. Then $\phi(A) = (\pi_\phi(A)\Phi_\phi, \Phi_\phi) = 0$ for all $\phi \in F$. Therefore $A = 0$ and $\ker(\pi_\phi) = \{0\}$. □

3.2 Basic Structure of Representations

Definition. The universal Hilbert space for a C^* algebra $\mathcal{A}$ and the universal representation for $\mathcal{A}$ are defined to be

$$\mathcal{H}_{\mathcal{A}'_{1+}}, \quad \pi_{\mathcal{A}'_{1+}}.$$

Definition. The enveloping von Neumann algebra for $\mathcal{A}$ is the strong closure of $\pi_{\mathcal{A}'_{1+}}(A)$ in $\mathcal{B}(\mathcal{H}_{\mathcal{A}'_{1+}})$. The enveloping von Neumann algebra is conveniently denoted by $\mathcal{A}^{cc}$. By the following we can also denote it as $\mathcal{A}''$.

3.11. Theorem. *Let $\mathcal{A}$ be a C^* algebra. Then the enveloping von Neumann algebra of $\mathcal{A}$ is isomorphic as a Banach space to the second dual of $\mathcal{A}$.*

Proof. Each state of $\mathcal{A}$ is a vector state in $\mathcal{H}_{\mathcal{A}'_{1+}}$, and therefore a normal state on $\mathcal{A}^{cc}$. Obviously each element of A' is a linear combination of elements of $\mathcal{A}'_{1+}$. Therefore we can define a map from $\mathcal{A}'$ into the pre-dual of $\mathcal{A}^{cc}$.

Now $\mathcal{A}$ is ultra-weakly dense in $\mathcal{A}^{cc}$, so this map is a linear isometry and each ϕ in the pre-dual of $\mathcal{A}^{cc}$ will be the image of $\phi|_{\mathcal{A}}$ in $\mathcal{A}'$. Thus $\mathcal{A}'$ is the pre-dual of $\mathcal{A}^{cc}$. Therefore $\mathcal{A}'' = \mathcal{A}^{cc}$. □

Definition. Given a (non-degenerate) representation $(\pi, \mathcal{H})$ of a C^* algebra $\mathcal{A}$, we can find a projection in the enveloping von Neumann algebra, $\mathcal{A}^{cc}$, which takes us down to the image of π. In other words, this is the projection onto the block $(\pi, \mathcal{H})$ inside the enveloping von Neumann algebra, which contains all representation elements. This projection is called the central cover of the representation $(\pi, \mathcal{H})$. Denote this projection by $c(\pi)$.

3.12. Theorem (Central Projections). *Let $(\pi_1, \mathcal{H}_1)$, $(\pi_2, \mathcal{H}_2)$ be two representations of a C^* algebra $\mathcal{A}$. These representations are equivalent if and only if $c(\pi_1) = c(\pi_2)$. The map $(\pi, \mathcal{H}) \mapsto c(\pi)$ is a bijection between nonzero central projections in $\mathcal{A}^{cc}$ and equivalence classes of representations of $\mathcal{A}$.*

Proof. For each central projection $p \neq 0$ in $\mathcal{A}^{cc}$, we can form a representation for $\mathcal{A}$ with the map $A \mapsto Ap$, $A \in \mathcal{A}$. The Hilbert space for the representation is $p\mathcal{H}_{\mathcal{A}'_{1+}}$, and its central cover is p. Thus we associate a representation with each central projection. Now if $(\pi, \mathcal{H})$ is a representation then clearly it is equivalent to the representation $\tilde{\pi} : A \mapsto Ac(\pi)$ on $c(\pi)\mathcal{H}_{\mathcal{A}'_{1+}}$. □

Remark. It is important to know when a separable C^* algebra has a representation on a separable Hilbert space. In particular, the enveloping von Neumann algebra acts on a generically non-separable space, and we would like to know how this interacts with representations.

Definition. A von Neumann algebra $\mathcal{M}$ is called σ-finite or countably decomposable if each set of pairwise orthogonal non-zero projections in $\mathcal{M}$ is countable. A projection p on $\mathcal{M}$ is called σ-finite if $p\mathcal{M}p$ is σ-finite. If $\mathcal{M}$ acts on a separable Hilbert space then it is σ-finite. A partial converse of this is true.

3.13. Theorem. *A von Neumann algebra $\mathcal{M}$ has a faithful normal representation on a separable Hilbert space if and only if $\mathcal{M}$ is σ-finite and contains a strongly dense sequence (is countably generated).*

Proof. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ separable, then $\mathcal{M}$ is σ-finite. Since the unit ball in $\mathcal{B}(\mathcal{H})$ is second countable for the strong topology, the unit ball in $\mathcal{M}$ is second countable and so $\mathcal{M}$ is separable in the strong topology.

Conversely, for each $v \in \mathcal{H}$ define $[\mathcal{M}^c v] \in \mathcal{M}$ to be the projection onto the closure of the subspace $\mathcal{M}^c v$. Let $\{[\mathcal{M}^c v]\}$ be a maximal family of these projections, so $\sum [\mathcal{M}^c v] = 1$. If $\mathcal{M}$ is σ-finite then $\{v_n\}$ is countable. Let $\phi(A) = \sum 2^{-n} (Av_n, v_n)$, then ϕ is a normal state on $\mathcal{M}$. It is also clear that ϕ is faithful. $\mathcal{M}$ is countably generated therefore there exists a C^* algebra $\mathcal{A}$ which is separable and strongly dense in $\mathcal{M}$. $\mathcal{H}_\phi$ contains a dense separable subspace, so it is separable. □

3.14. Corollary. *A representation $(\pi, \mathcal{H})$ of a separable C^* algebra $\mathcal{A}$ is equivalent to a separable representation if and only if $c(\pi)$ is σ-finite in $\mathcal{A}^{cc}$.*

3.15. Theorem. *Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be non-degenerate representations of a C^* algebra $\mathcal{A}$. Then T.F.A.E.*

1. $c(\pi_1) \perp c(\pi_2)$.
2. $(\pi_1 \oplus \pi_2)\mathcal{A}^{cc} = \pi_1^{cc}(\mathcal{A}) \oplus \pi_2^{cc}(\mathcal{A})$
3. $(\pi_1 \oplus \pi_2)\mathcal{A}^{c} = \pi_1^{c}(\mathcal{A}) \oplus \pi_2^{c}(\mathcal{A})$
4. *There are no equivalent subrepresentations of $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$.*

Proof.

1⟹2

$$\begin{aligned} &\ker{(\pi_1 \oplus \pi_2)''} = \mathcal{A}''(1 - c(\pi_1) - c(\pi_2)) \\ \Longrightarrow\ &((\pi_1 \oplus \pi_2)''\mathcal{A})^{cc} \cong \mathcal{A}^{cc}(c(\pi_1) + c(\pi_2)) \\ \Longrightarrow\ &((\pi_1 \oplus \pi_2)\mathcal{A})^{cc} \cong \pi_1(\mathcal{A})^{cc} \oplus \pi_2(\mathcal{A})^{cc}. \end{aligned}$$

2⟹3 Follows from von Neumann's bicommutant theorem.

3⟹4 Assume there exists an isometry $u : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$. By definition of equivalence $u^*u \in \pi_1(\mathcal{A})^c$, $uu^* \in \pi_2(\mathcal{A})^c$, and $u^*(\pi_2(A)uu^*)u = \pi_1(A)u^*u$ for all $A \in \mathcal{A}$. Now

$$\begin{aligned} (\pi_1 \oplus \pi_2)(A)u &= (\pi_1(A) + \pi_2(A))u \\ &= u\pi_1(A) = \pi_2(A)u \end{aligned}$$

regarding u as an element of $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$

$$\begin{aligned} &= u(\pi_1(A) + \pi_2(A)) \\ &= u(\pi_1 \oplus \pi_2)(A) \\ \Longrightarrow &u \in ((\pi_1 \oplus \pi_2)(\mathcal{A}))^c \end{aligned}$$

By assumption $((\pi_1 \oplus \pi_2)(\mathcal{A}))^c \subset \mathcal{B}(\mathcal{H}_1) \oplus \mathcal{B}(\mathcal{H}_2)$. So $u = 0$.

4⟹1 If 1 does not hold then consider subrepresentations with central cover $c(\pi_1)c(\pi_2)$.

□

Definition. Representations satisfying the properties of the previous theorem are called disjoint representations.

Definition. A non-degenerate representation $(\pi, \mathcal{H})$ of a C^* algebra $\mathcal{A}$ is called a factor representation when $\pi(\mathcal{A})^{cc}$ is a factor. Note that $(\pi, \mathcal{H})$ is a factor representation if and only if $c(\pi)$ is a minimal projection in the center of $\mathcal{A}''$. Two factor representations are either equivalent or disjoint.

Definition. Let $(\pi, \mathcal{H})$ be a representation of a C^* algebra $\mathcal{A}$. If $K \subset \mathcal{H}$ is a linear subspace with $\pi(\mathcal{A})K \subset K$, then K is called reducing for π. Representations satisfying the conditions of the following theorem are called irreducible.

3.16. Theorem (Irreducible Representations). *Let $(\pi, \mathcal{H})$ be a nonzero representation of a C^* algebra $\mathcal{A}$. Then T.F.A.E.*

1. *There are no non-trivial reducing subspaces for π.*
2. $\pi(\mathcal{A})^c = \{\alpha 1\}$.
3. $\pi(\mathcal{A})$ *is strongly dense in $\mathcal{B}(\mathcal{H})$.*
4. *Each non-zero $v \in \mathcal{H}$ is cyclic for $\pi(\mathcal{A})$.*
5. $(\pi, \mathcal{H})$ *is equivalent to a cyclic representation associated with a pure state.*

Proof. The proof is straightforward computation. See [Ped79]. □

3.17. Corollary. *Two irreducible representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of $\mathcal{A}$ are either disjoint or equivalent.*

Proof. If they are not disjoint then they have equivalent subrepresentations by a previous result. But irreducible representations have only trivial subrepresentations by the above. □

3.18. Theorem (Repelling Representations). *Let ϕ and ψ be pure states of a C^* algebra $\mathcal{A}$. If $\|\phi - \psi\| < 2$ then $(\pi_\phi, \mathcal{H}_\phi)$ and $(\pi_\psi, \mathcal{H}_\psi)$ are equivalent. If they are equivalent, then $\psi = \phi(u^* \cdot u)$ for some unitary $u \in \mathcal{A}$.*

Proof. Assume $(\pi_\phi, \mathcal{H}_\phi)$ and $(\pi_\psi, \mathcal{H}_\psi)$ are not equivalent. Then they are disjoint, $c(\pi_\phi) \perp c(\pi_\psi)$. Now $\phi(c(\pi_\phi)) = 1$ and $\psi(c(\pi_\psi)) = 1$, so $\phi(c(\pi_\psi)) = 0$ and $\psi(c(\pi_\phi)) = 0$, so $\|\phi - \psi\| \geq (\phi - \psi)(c(\pi_\phi) - c(\pi_\psi)) = 2$.

To prove the second part, assume the representations are equivalent. Then for every $A \in \mathcal{A}$ we have

$$\psi(A) = (\pi_\phi(A)\xi, \xi) \text{ for some unitary } \xi \in \mathcal{H}_\phi.$$

Let u be the unitary which takes ξ to Φ_ϕ, $\pi_\phi(u)\Phi_\phi = \xi$. Then

$$\begin{aligned}\psi(A) &= (\pi_\phi(A)\pi_\phi(u)\Phi_\phi, \pi_\phi(u)\Phi_\phi) \\ &= (\pi_\phi(u^*Au)\Phi_\phi, \Phi_\phi) \\ &= \phi(u^*AU).\end{aligned}$$

□

Remark. Previously we introduced the idea of the Gelfand transform of a commutative Banach algebra. This was a map from algebra elements to functions h, $h : \Delta \longrightarrow \mathbb{C}$. The generalization of this to the non-commutative case is connected to representation theory.

Definition. Let $\mathrm{Irr}\,(\mathcal{A})$ be the set of irreducible representations of $\mathcal{A}$. Define the spectrum of $\mathcal{A}$, $\widehat{\mathcal{A}}$, to be the set of equivalence classes of irreducible representations of $\mathcal{A}$.

Remark. When $\mathcal{A}$ is commutative, all the irreducible representations are one-dimensional. Then $\widehat{\mathcal{A}}$ is nothing but the set of non-zero complex homomorphisms of $\mathcal{A}$, which is the Gelfand transform of $\mathcal{A}$. One approach to the non-commutative case is through the so-called decomposition theory. The basic object in decomposition theory is the atomic representation.

Definition. Let $\mathcal{A}$ be a C^* algebra. For each $t \in \widehat{\mathcal{A}}$ choose pure state ϕ_t with representation $(\pi_t, \mathcal{H}_t)$. Define the atomic representation to be $(\pi_a, \mathcal{H}_t)$ with

$$\pi_a = \oplus_{t \in \widehat{\mathcal{A}}} \pi_t \qquad \mathcal{H}_a = \oplus_{t \in \widehat{\mathcal{A}}} \mathcal{H}_t.$$

Remark. The above definition involves a choice, but the equivalence for different choices is easy to show, so the atomic representation is essentially unique.

3.19. Theorem.

$$\pi_a(\mathcal{A})^{cc} = \bigoplus_{t \in \widehat{\mathcal{A}}} \mathcal{B}(\mathcal{H}_t).$$

Proof. By a previous theorem $\pi_t(\mathcal{A})$ is strongly dense in $\mathcal{B}(\mathcal{H}_t)$ for each t. Therefore $\pi_t(\mathcal{A})^{cc} = \mathcal{B}(\mathcal{H}_t)$. The π_t's are mutually disjoint so the result follows. □

Remark. The decomposition theory proceeds by making $\widehat{\mathcal{A}}$ into a measure space, beginning with the so-called D-Borel structure. The first major result is a classification of equivalent representations according to the measures which are associated to them via their states. See [Kad57]. The converse construction, building representations up from their measures leads to the theory of the direct integral.

Chapter 4

Structure Theory II

4.1 Weights and Traces

Definition. Let $\mathcal{A}$ be a C^* algebra. A weight on $\mathcal{A}$ is a function $\phi : \mathcal{A}_+ \longrightarrow [0, \infty]$ such that

1. $\phi(\alpha A) = \alpha\phi(A) \quad \forall A \in \mathcal{A}_+, \alpha \in \mathbb{R}_+$.
2. $\phi(A + B) = \phi(A) + \phi(B) \quad \forall A, B \in \mathcal{A}_+$,

where $\mathcal{A}_+$ is the set of positive elements of $\mathcal{A}$.

Definition. A weight is said to be densely defined if the set $\mathcal{A}_+^\phi = \{A \in \mathcal{A}_+ \,:\, \phi(A) < \infty\}$ is dense in $\mathcal{A}_+$.

Definition. Let $\mathcal{M}$ be a von Neumann algebra. We say that ϕ is semi-finite if $\mathcal{M}_+^\phi$ is weakly dense in $\mathcal{M}$. For von Neumann algebras this coincides with the notion of σ-finite.

Definition. A weight ϕ on a von Neumann algebra $\mathcal{M}$ is called σ-normal if there exists a sequence $\{\phi_n\}$ of sequentially normal positive functionals on $\mathcal{M}$ such that $\phi(x) = \sum \phi_n(x)$ for all $x \in \mathcal{M}_+$.

Definition. ϕ is called lower semi-continuous if for each $\alpha \in \mathbb{R}_+$ the set $\{A \in \mathcal{A}_+ \,:\, \phi(A) \leq \alpha\}$ is closed.

Definition. A trace on a C^* algebra $\mathcal{A}$ is a weight ϕ such that $\phi(u^*Au) = \phi(A)$ for all $A \in \mathcal{M}_+$ and u unitary.

4.1. Theorem (Radon-Nikodym). *Let ϕ and ψ be normal functionals on a von Neumann algebra $\mathcal{M}$ such that $0 \leq \psi \leq \phi$. Then for each $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 1/2$ there is an element $h \in \mathcal{M}_+^1$ such that*

$$\psi = \lambda\phi(h\cdot) + \lambda^*\phi(\cdot h).$$

If ϕ is faithful then h is unique.

Proof. Let $N = \left\{\lambda\phi(h\cdot) + \lambda^*\phi(\cdot h) \,:\, h \in \mathcal{M}_+^1\right\}$. N is compact and convex since $\mathcal{M}_+^1$ is convex and ultra-weakly compact. N is a subset of the pre-dual of $\mathcal{M}$. If $\psi \notin N$, then there is an element in the self-adjoint part of $\mathcal{M}$, $\mathcal{M}_{sa}$, say $a \in \mathcal{M}_{sa}$, and a $t \in \mathbb{R}$ such that $\psi(a) > t$, $N(a) \leq t$. Let $a = a_+ - a_-$ and take $h = [a_+]$. Then

$$\psi(a_+) \geq \psi(a_+ - a_-) > t \geq 2\mathrm{Re}\,\lambda\phi(a_+) \geq \phi(a_+) \quad \Rightarrow\Leftarrow .$$

If ϕ is faithful and if $\psi = \lambda\phi(k\cdot) + \lambda^*\phi(\cdot k)$ for some $k \in \mathcal{M}_{sa}$, then since

$$(\lambda + \lambda^*)(h-k)^2 = \lambda h(h-k) + \lambda^*(h-k)h - \lambda k(h-k) - \lambda^*(h-k)k,$$

we have

$$2\mathrm{Re}\,\lambda\phi((h-k)^2) = \psi(h-k) - \psi(h-k) = 0,$$
$$\Longrightarrow h = k.$$

□

Definition. Let $\mathcal{M}$ be a von Neumann algebra on a separable Hilbert space. We have the following nomenclature.

- $\mathcal{M}$ is called finite if it admits a faithful, normal, finite trace.
- $\mathcal{M}$ is called semi-finite if it admits a faithful, normal, semi-finite trace.
- $\mathcal{M}$ is called properly infinite if it does not admit a non-zero, normal, finite trace.
- $\mathcal{M}$ is called purely infinite if it does not admit a non-zero, normal, semi-finite trace.

4.2. Theorem (First Decomposition). *Let $\mathcal{M}$ be a von Neumann algebra. Then $\mathcal{M}$ has a unique decomposition*

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3,$$

where

- *$\mathcal{M}_1$ is finite,*
- *$\mathcal{M}_2$ is semi-finite but not properly infinite,*
- *$\mathcal{M}_3$ is purely infinite.*

Proof. Let ϕ be a normal trace on $\mathcal{M}$, so ϕ is weakly lower semi-continuous. Therefore $\mathcal{N}_\phi = \{x \in \mathcal{M} : \phi(x^*x) = 0\}$ is a weakly closed ideal of $\mathcal{M}$. Therefore $\mathcal{N}_\phi = (1-p)\mathcal{M}$ for some central projection $p \in \mathcal{M}$, and ϕ is faithful on $p\mathcal{M}$.

Also, the weak closure of $\mathcal{M}^\phi = \{x \in \mathcal{M} : \phi(x) < \infty\}$ is an ideal of $\mathcal{M}$, so there is a central projection q such that ϕ is semi-finite on $q\mathcal{M}$ and purely infinite on $(1-q)\mathcal{M}$. Therefore ϕ is faithful and semi-finite on $pq\mathcal{M}$.

Let $\{\phi_n, p_n\}$ be a maximal family of normal finite traces ϕ_n and pairwise orthogonal projections p_n such that ϕ_n is faithful on $p_n\mathcal{M}$. $\mathcal{M}$ is σ-finite [Ped79] so $\{\phi_n, p_n\}$ is countable. Define

$$\phi(x) = \sum 2^{-n}\phi_n(1)^{-1}\phi_n(p_n x).$$

If $p = \sum p_n$ then ϕ is faithful, normal, and finite on $p\mathcal{M}$, and by maximality of $\{\phi_n, p_n\}$, $(1-p)\mathcal{M}$ is properly infinite.

Let $\{\psi_n, q_n\}$ be a maximal family of normal, semi-finite traces ψ_n and pairwise orthogonal projections $q_n \leq 1-p$ such that ψ_n is faithful on $q_n\mathcal{M}$. Let $\psi(x) = \sum \psi_n(q_n x)$, $q = \sum q_n$. Then $q \perp p$, ψ is faithful, normal, and semi-finite on $q\mathcal{M}$. $(1-q-p)\mathcal{M}$ is purely infinite by maximality of $\{\psi_n, q_n\}$. □

4.3. Theorem. *If $\mathcal{M}$ is a semi-finite von Neumann algebra on a separable Hilbert space, then $\mathcal{M}^c$ is semi-finite.*

Proof. The proof requires a somewhat technical result. See [Ped79]. □

4.2 Types

Definition. Let p and q be projections in a C^* algebra $\mathcal{A}$. If there exists a partial isometry $v \in \mathcal{A}$ such that $v^*v = p$ and $vv^* = q$, we say that p is equivalent to q, writing $p \sim q$. Recall that u is a partial isometry if u^*u (and thus uu^*) is a projection. This coincides with the previous notion of projection equivalence.

Remark. As an example, suppose $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then two projections are equivalent if and only if $p\mathcal{H}$ and $q\mathcal{H}$ have the same dimension. Thus the equivalence classes of projections on a von Neumann algebra are a sort of "generalized dimension" set.

Definition. Let x be in $\mathcal{M}_{sa}$. The central cover of x, $c(x)$, is the infimum of all $z \in \mathcal{Z}_{sa}$ with $z \geq x$. It exists because $\mathcal{Z}_{sa}$ is a complete lattice.

Definition. A projection p is called abelian if $p\mathcal{A}p$ is a commutative algebra.

Definition. A von Neumann algebra $\mathcal{A}$ is called type I if there is an abelian projection $p \in \mathcal{M}$ with $c(p) = 1$.

4.4. Theorem. *Let $\mathcal{M}$ be a von Neumann algebra of type I, on a separable Hilbert space, and let p be an abelian projection with $c(p) = 1$. Then there is a faithful, normal, semi-finite trace ϕ on $\mathcal{M}$ with $\phi(p) = 1$.*

Proof. $p\mathcal{M}p$ is a commutative von Neumann algebra on a separable Hilbert space, therefore $p\mathcal{M}p \cong L^\infty(T)\,\mu$ for some locally compact, Hausdorff, second countable measure space T with measure μ. Take any finite measure on T equivalent to μ as ϕ on $p\mathcal{M}p$. Normalize to $\phi(p) = 1$. ϕ extends to a normal semi-finite trace on $\mathcal{M}$, and since $c(p) = 1$, ϕ is faithful on $\mathcal{M}$. □

Definition. A von Neumann algebra $\mathcal{M}$ is said to be homogeneous of degree n if $1 = \sum_{i=1}^n p_i$, for some $\{p_i\}$ a set of n orthogonal, equivalent, abelian projections.

4.5. Theorem. *Let $\mathcal{M}$ be a von Neumann algebra of type I on a separable Hilbert space. Then $\mathcal{M}$ has a unique decomposition*

$$\mathcal{M} = \bigoplus \mathcal{M}_n, \quad 1 \leq n \leq \infty,$$

with $\mathcal{M}_n$ homogeneous of degree n.

Proof. First we show that every type I $\mathcal{M}$ contains a nonzero, homogeneous, central summand. Let $\{q_i\}$ be a maximal family of orthogonal abelian projections in $\mathcal{M}$ with $c(q_i) = 1$. The family is non-empty by definition of type I. Let $z = 1 - c(1 - \sum q_i)$. If $z = 0$, then there would be an abelian projection $q \leq 1 - \sum q_i$ with $c(q) = c(1 - \sum q_i) = 1$, contradicting the maximality of $\{q_i\}$. Therefore $c(z - \sum zq_i) = zc(1 - \sum q_i) = 0$, but $z \neq 0$. Therefore $z = \sum zq_i$, and so $\mathcal{M}z$ is homogeneous.

Let $\{z_j^{(n)}\}$ be the maximal family of orthogonal central projections each of which is a sum of n orthogonal, equivalent, abelian projections $\{p_{ji}\}$. $c(p_{ji}) = z_j$ and $p_i = \sum_j p_{ji}$ is abelian for each $1 \leq i \leq n$. Let $e_n = \sum_j z_j$. Then $c(p_i) = e_n$, so $\{p_i\}$ is a family of orthogonal, equivalent, abelian projections since equivalence of projections follows from equality of their central covers.

Now $\sum_i p_i = \sum_{ij} p_{ji} = \sum_j z_j = e_n$, so $\mathcal{M}e_n$ is homogeneous of degree n. For $n \neq m$, $e_n e_m = 0$ by the well-defined-ness of the degree of homogeneity.

Since $\{z_j\}$ is maximal, $\mathcal{M}(1 - \sum e_n)$ contains no homogeneous central summand. But $\mathcal{M}(1 - \sum e_n)$ is clearly of type I, which is a contradiction. □

4.6. Corollary. *Let $\mathcal{M}$ be a factor of type I. Then $\mathcal{M}$ is isomorphic to $\mathcal{B}(\mathcal{H})$ where $\dim \mathcal{H}$ is the degree of homogeneity of $\mathcal{M}$.*

Proof. Let p be a non-zero abelian projection in $\mathcal{M}$. Since $\mathcal{M}$ is a factor, p is minimal and $c(p) = 1$. Let ϕ be a normal state with $\phi(p) = 1$. Then ϕ is pure and so $(\pi_\phi, \mathcal{H}_\phi)$ is irreducible. Thus $\pi_\phi(\mathcal{M}) = \mathcal{B}(\mathcal{H}_\phi)$. The degree of homogeneity of $\pi_\phi(\mathcal{M})$ is obviously equal to that for $\mathcal{M}$. □

4.7. Lemma. *Let $\mathcal{M}$ be a von Neumann algebra of type I on a Hilbert space $\mathcal{H}$. Then $\mathcal{M}^c$ is isomorphic to a von Neumann algebra with abelian commutant.*

Proof. Let p be an abelian projection with $c(p) = 1$. Then $\mathcal{M}^c \cong \mathcal{M}^c p$, and $(\mathcal{M}^c p)^c$ on $p\mathcal{H}$ is $p\mathcal{M}p$. □

4.8. Lemma. *Let $\mathcal{M}$ be a commutative von Neumann algebra on a Hilbert space $\mathcal{H}$. Then $\mathcal{M}^c$ is of type I.*

Proof. Let q be a non-zero projection in $\mathcal{M}^c$ and choose a unit vector $v \in q\mathcal{H}$. Let p be the cyclic projection on the closed subspace $[\mathcal{M}v]$. Then $p \in \mathcal{M}^c$ and $p \leq q$.

Now $\mathcal{M}p$ is commutative and has a cyclic vector, so it is maximal commutative on $p\mathcal{H}$. Therefore $\mathcal{M}p = (\mathcal{M}p)^c = p\mathcal{M}^c p$. Therefore p is an abelian projection and $\mathcal{M}^c$ is type I. □

4.9. Theorem. *Let $\mathcal{M}$ be a von Neumann algebra. Then T.F.A.E.*

1. *$\mathcal{M}$ is of type I.*
2. *$\mathcal{M}^c$ is of type I.*
3. *$\mathcal{M}$ is isomorphic to a von Neumann algebra with abelian commutant.*
4. *$\mathcal{M}^c$ is isomorphic to a von Neumann algebra with abelian commutant.*

Proof. 1 $\Longrightarrow$4, 2$\Longrightarrow$3 follow from the first lemma.
4 $\Longrightarrow$2, 3$\Longrightarrow$1 follow from the second lemma. □

Definition. $\mathcal{M}$ is said to be type II if it is semi-finite, but contains no non-zero abelian projections.

Definition. $\mathcal{M}$ is said to be type III if it is purely infinite.

Remark. Notice that $\mathcal{M}$ is finite if it is homogeneous of degree n with $n < \infty$. It is properly infinite if $n = \infty$. We thus subdivide type I into type I_n, $1 \leq n \leq \infty$. If $\mathcal{M}$ is type II, then we say it is type II_1 if it is finite and type II_∞ if it is properly infinite. The following theorem is a restatement of this classification.

4.10. Theorem (Second Decomposition). *Let $\mathcal{M}$ be a von Neumann algebra on a separable Hilbert space. Then $\mathcal{M}$ has a unique decomposition into central summands of each type,*

$$\mathcal{M} = \mathcal{M}_I \bigoplus \cdots \bigoplus \mathcal{M}_{I_\infty} \bigoplus \mathcal{M}_{II_1} \bigoplus \mathcal{M}_{II_\infty} \bigoplus \mathcal{M}_{III}.$$

Remark. We would like to extend the notion of type to C^* algebras which are not necessarily von Neumann algebras. This is what we do in the following.

Definition. Let $\mathcal{A}$ be a C^* algebra, and let $A \in \mathcal{A}$. The hereditary algebra generated by A is the norm closure of $A\mathcal{A}A$.

Definition. A positive element $A \in \mathcal{A}$ is called abelian if the hereditary algebra generated by A is a commutative algebra.

Definition. $\mathcal{A}$ is a type I C^* algebra if each non-zero quotient of $\mathcal{A}$ contains a non-zero abelian element. If $\mathcal{A}$ is actually generated by its abelian elements, then we say $\mathcal{A}$ is of type I_0.

Definition. If $\mathcal{A}$ contains no non-zero abelian elements, then we say it is antiliminary.

Remark. A von Neumann algebra of type I is not in general a C^* algebra of type I. As an example, let $\mathcal{M} = \mathcal{B}(\mathcal{H})$ for an infinite-dimensional $\mathcal{H}$. Let $\mathcal{K}(\mathcal{H})$ denote the compact operators. Then $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ contains no abelian elements.

4.11. Lemma. *Let A be a positive element of a C^* algebra $\mathcal{A}$. Then A is abelian if and only if $\dim \pi(A) \leq 1$ for every irreducible representation $(\pi, \mathcal{H})$ of $\mathcal{A}$.*

Proof. Suppose A is abelian and $(\pi, \mathcal{H})$ is an irreducible representation. Then $\pi(A)$ is abelian in $\pi(\mathcal{A})$, so $\pi(A)\mathcal{B}(\mathcal{H})\pi(A)$ is commutative and so $\dim \pi(A) \leq 1$.

Conversely, let A be positive in $\mathcal{A}$ and suppose $\dim \pi(A) \leq 1$ for each irreducible representation π. Then $A\mathcal{A}A$ is commutative in the atomic representation, which is faithful. So A is abelian. □

4.12. Lemma. *Let $\mathcal{A}$ be a C^* algebra acting irreducibly on a Hilbert space $\mathcal{H}$ such that $\mathcal{A} \cap \mathcal{K}(\mathcal{H}) \neq 0$. Then $\mathcal{K}(\mathcal{H}) \subset \mathcal{A}$ and each faithful irreducible representation of $\mathcal{A}$ is unitarily equivalent to the identity map.*

Proof. $\mathcal{A} \cap \mathcal{K}(\mathcal{H}) \neq 0$, therefore there is a finite-dimensional projection in $\mathcal{A} \cap \mathcal{K}(\mathcal{H})$ and a one-dimensional projection $p \in \mathcal{A}$. If ξ is a unit vector in $p\mathcal{H}$ then for any $\eta \in \mathcal{H}$ there is an $A \in \mathcal{A}$ such that $A\xi = \eta$; $\mathcal{A}$ acts irreducibly on $\mathcal{H}$. Therefore A^*pA is the projection on $\mathbb{C}\eta$. Thus $\mathcal{A}$ contains all the one-dimensional projections, so $\mathcal{K}(\mathcal{H}) \subset \mathcal{A}$.

Furthermore, let ϕ be a pure state on $\mathcal{A}$. Then $(\pi_\phi, \mathcal{H}_\phi)$ is faithful. $\phi|_{\mathcal{K}(\mathcal{H})}$ is non-zero since $(\pi_\phi, \mathcal{H}_\phi)$ is faithful, so it is a state for $\mathcal{K}(\mathcal{H})$. Since the dual of $\mathcal{K}(\mathcal{H})$ is the set of trace class operators, $\phi(x) = (x\xi, \xi)$, some $\xi \in \mathcal{H}$, for all $x \in \mathcal{K}(\mathcal{H})$.

But the extension of a state from an ideal to the whole algebra is unique, so ϕ on $\mathcal{A}$ is a vector state $\phi(A) = (A\xi, \xi)$. Thus any such ϕ determines a cyclic representation with a cyclic vector ξ_ϕ and any such ϕ is equal to the identity representation on cyclic vectors, and thus unitarily equivalent to the identity representation. □

4.13. Theorem. *Let $\mathcal{A}$ be a C^* algebra of type I. Then $\mathcal{K}(\mathcal{H}) \subset \pi(\mathcal{A})$ for each irreducible representation $(\pi, \mathcal{H})$ of $\mathcal{A}$.*

Proof. $(\pi, \mathcal{H})$ is irreducible. By the first lemma, for A abelian in $\mathcal{A}$ with norm 1, there is a one-dimensional projection on $\mathcal{H}$. By the second lemma $\mathcal{K}(\mathcal{H}) \subset \pi(\mathcal{A})$. □

4.14. Corollary. *Let $\mathcal{A}$ be a type-I_0 C^* algebra. Then for every irreducible representation $(\pi, \mathcal{H})$ of $\mathcal{A}$, $\pi(\mathcal{A}) = \mathcal{K}(\mathcal{H})$.*

Proof. $\pi(A)$ is generated by its abelian elements, $\pi(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{H})$. But $\mathcal{K}(\mathcal{H}) \subset \pi(\mathcal{A})$. □

Definition. A C^* algebra $\mathcal{A}$ is called liminary if $\pi(\mathcal{A}) = \mathcal{K}(\mathcal{H})$ for each irreducible representation $(\pi, \mathcal{H})$ of $\mathcal{A}$. Thus each type-I_0 C^* algebra is liminary, but the converse is false.

Remark. The following are useful properties which we state without proof.

4.15. Theorem. *A liminary C^* algebra is of type I.*

Proof. See [Ped79]. □

Definition. If $\mathcal{A}$ is a C^* algebra, we define a composition series to be a strictly increasing family of closed ideals $\{\mathcal{I}_\alpha\}$ indexed by $\alpha \in [0, \beta]$, a segment of the ordinals, with $\mathcal{I}_0 = 0$, $\mathcal{I}_\beta = \mathcal{A}$, and such that for each limit ordinal γ we have

$$\mathcal{I}_\gamma = \text{norm closure}\left(\bigcup_{\alpha<\gamma} \mathcal{I}_\alpha\right).$$

4.16. Theorem. *Let $\mathcal{A}$ be a C^* algebra. Then T.F.A.E.*

1. *$\mathcal{A}$ is of type I.*
2. *$\mathcal{A}$ has a composition series $\{\mathcal{I}_\alpha\}$, $\alpha \in [0, \beta]$, such that $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is type I_0 for each $\alpha < \beta$.*
3. *$\mathcal{A}$ has a composition series $\{\mathcal{I}_\alpha\}$, $\alpha \in [0, \beta]$, such that $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is liminary for each $\alpha < \beta$.*
4. *$\mathcal{A}$ has a composition series $\{\mathcal{I}_\alpha\}$, $\alpha \in [0, \beta]$, such that $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is type I for each $\alpha < \beta$.*

Proof. See [Ped79]. □

Chapter 5

Matrices

5.1 Inductive Limits

Definition. Define a directed set I to be a partially ordered set such that if $\alpha, \beta \in I$ then there exists $\gamma \in I$ with $\alpha < \gamma$, $\beta < \gamma$.

Definition. Let I be a directed set. Let X^α be locally convex spaces with α varying in I. Let $X = \bigcup_{\alpha \in I} X^\alpha$; X is a locally convex space. Suppose that $\alpha < \beta$ if and only if $X^\alpha \subset X^\beta$ and that the inclusion is continuous. Suppose also that for any convex $V \subset X$, V is a nbhd. of $U \subset X$ if and only if $\forall \alpha \in I$ $V \cap X^\alpha$ is a nbhd. of $U \in X^\alpha$. When all the above conditions hold we say that X is the inductive limit of X^α.

Definition. When the X^α are Banach spaces, in particular Banach algebras, the inclusions in the definition of inductive limit are bounded linear maps. When these inclusions also satisfy $\limsup_\beta \|\phi_{alpha\beta}\| < \infty$ for all α, the system is called a normed inductive system, and the limit X is called a normed inductive limit. This extra uniformity condition implies that $\|x\| = \limsup_\beta \left\|\phi_{\alpha\beta(x)}\right\|$, for $x \in X^\alpha \subset X$, is a seminorm on X. Quotienting by elements of zero seminorm and completing gives a Banach space, which will also be called the inductive limit of X^α, and again we will write $X = \lim_{\rightarrow} X^\alpha$.

5.2 Glimm Algebras

Remark. Now we construct some antiliminary algebras which are interesting both as examples of non-intuitive algebras and as physical fermion algebras. These are the Glimm algebras. We need the notion of inductive limits for locally convex spaces.

Definition. Let M_m be the C^* algebra of $m \times m$ matrices, identified with $\mathcal{B}(\mathcal{H}_m)$. Suppose $i : M_m \longrightarrow M_n$ is a morphism of M_m into M_n with $i(1) = 1$. Let $d = \operatorname{Tr}\left(i(v_{11}^{(m)})\right)$, where $v_{11}^{(m)}$ is the matrix with 1 in the $(1,1)$ place and zeroes elsewhere. We have $md = n$.
Let $\{s(n) : n \in \mathbb{N}\}$ be a sequence of natural numbers, greater than one. Let $s(n)! = \prod_{k=1}^n s(k)$. Then consider the inductive system

$$M_{s(1)} \to M_{s(2)!} \to \cdots \to M_{s(n)!} \to \cdots ,$$

with the inclusion map i. The inductive limit $M_\infty = \cup M_{s(n)!}$ is not necessarily norm complete, but its completion, $\mathcal{A}_\infty$, is a C^* algebra which is called the Glimm algebra of rank $\{s(n)\}$.

Definition. The fermion algebra is the Glimm algebra for which $s(n) = 2\,\forall n \in \mathbb{N}$.

5.1. Theorem. *Every Glimm algebra is a separable, simple (contains no non-zero closed ideals) C^* algebra and has a unique tracial state.*

Proof. Each $M_{s(n)!}$ is separable and M_∞ is dense in $\mathcal{A}_\infty$, so $\mathcal{A}_\infty$ is separable. If π is a non-zero morphism of $\mathcal{A}_\infty$, then $\pi|_{M_{s(n)!}}$ is an isometry for each n. Therefore π_{M_∞} is an isometry and π is an isometry. Thus $\mathcal{A}_\infty$ is simple (every morphism is an isometry).

Let τ_n be the normalized trace on $M_{s(n)!}$. Then $\tau_{n+1} \circ i = \tau_n$, so there is a unique tracial state on M_∞, and so it is tracial on $\mathcal{A}_\infty$. Conversely if ϕ is tracial on $\mathcal{A}_\infty$ then from the uniqueness of the trace on $\mathcal{B}(\mathcal{H}_{s(n)!})$ $\phi = \tau$. $\square$

5.2. Theorem. *There exists a factor of type II_1.*

Proof. $\mathcal{M} = \pi_\tau(\mathcal{A}_\infty)^{cc}$ has a non-zero finite normal trace, the extension of τ. $\ker(\tau)$ is a central projection, so we can assume that τ is faithful on $z\mathcal{M}$ for some central projection z; τ is faithful on $\mathcal{M}_\infty$ so $z \neq 0$.

Since τ is the unique tracial state on $\mathcal{A}_\infty$, the center of $z\mathcal{M}$ is trivial, so $z\mathcal{M}$ is a factor. This factor is finite but not finite-dimensional, so it is of type II_1. $\square$

Remark. Let $\mathcal{F}$ denote the fermion algebra. For each $\lambda \in [0, 1/2]$ we can construct a state on $\mathcal{F}$ as follows.

Let $\{\Lambda^n\}$ be a sequence of convex combinations each of length 2, i.e. $\Lambda^1 = 2$, $\Lambda_1^2 + \Lambda_2^2 = 2$, etc. Note that $M_{s(n)!} = M_{s(1)} \otimes \cdots \otimes M_{s(n)}$, so each element of $M_{s(n)!}$ can be written

$$x = x^{(1)} \otimes \cdots \otimes x^{(n)}, \quad x^{(k)} \in M_{s(k)}.$$

Let

$$\phi_\Lambda(x) = \prod_{k=1}^{n} \left(\sum_{i=1}^{s(k)} \Lambda_i^k x_{ii}^{(k)} \right).$$

This extends to a unique state on $\mathcal{F}$. It is called the product state on $\mathcal{F}$. Note that the tracial state on $\mathcal{F}$ is the product state with $\Lambda_i^n = 1/2$ for all $i \leq 2, n \in \mathbb{N}$. note that if $x \in M_{s(n)!}$ and $y \in {M_{s(n)!}}^c$, then $\phi_\Lambda(xy) = \phi_\Lambda(x)\phi_\Lambda(y)$.

Remark. For each $\lambda \in [0, 1/2]$ we choose $\Lambda = \{\Lambda^n\}$ to be the sequence of convex combinations $\Lambda_1^n = \lambda$, $\Lambda_2^n = 1 - \lambda$, for all $n \in \mathbb{N}$.

Let ϕ_λ be the product state associated with Λ and let $(\pi_\lambda, \mathcal{H}_\lambda, \xi_\lambda)$ be the cyclic representation of $\mathcal{F}$ associated with ϕ_λ. We already know that

$$\pi_0(\mathcal{F})^{cc} = \mathcal{B}(\mathcal{H}_0),$$
$$\pi_{1/2}(\mathcal{F})^{cc} = \text{factor of type } II_1.$$

5.3. Theorem. *Each product state of a Glimm algebra is factorial, i.e. gives rise to a factor representation.*

Proof. Let ϕ_Λ be a product state on $\mathcal{A}_\infty$ of rank $\{s(n)\}$. Let $z \in \mathcal{Z}\left(\pi_\Lambda(\mathcal{A}_\infty)^{cc}\right)$. By Kaplansky's density theorem there exists a sequence $\{y_k\}$ in $\mathcal{A}_\infty$ with $\|y_k\| \leq \|z\|$ and $\pi_\Lambda(y_k) \to z$ weakly. Let $U_{s(n)!}$ denote the unitary group of $M_{s(n)!}$. This is a compact group with Haar measure du. Let

$$z_k = \int_{U_{s(n)!}} du\, u y_k u^*, \quad \text{for fixed } n.$$

z_k commutes with $M_{s(n)!}$, and for each $u \in U_{s(n)!}$ we have $\pi_\Lambda(u y_k u^*) \to \pi_\Lambda(u) z \pi_\Lambda(u^*) = z$ weakly. Therefore $\pi(z_k) \to z$ weakly by the Lebesgue dominated convergence theorem.

So for every $x, y \in M_{s(n)!}$ we have

$$\begin{aligned}
(z\xi_x, \xi_y) &= \lim\left(\pi(z_k)\xi_x, \xi_y\right) \\
&= \lim \phi_\Lambda(y^* z_k x) \\
&= \lim \phi_\Lambda(z_k)\phi_\Lambda(y^* x) \\
&= \left(z\xi_\Lambda, \xi_\Lambda\right)\left(\xi_x, \xi_y\right).
\end{aligned}$$

This holds for any n so $z = \left(z\xi_\Lambda, \xi_\Lambda\right)1$, so that z is a multiple of 1. Therefore $\pi_\Lambda(\mathcal{A}_\infty)^{cc}$ is a factor. ☐

Remark. Let Π be the group of permutations of $\mathbb{N}$ which leave all but a finite number of elements fixed. For a $t \in \Pi$ we define a unitary operator by

$$\begin{aligned}
&u_t : \mathcal{H}_{2^n} \longrightarrow \mathcal{H}_{2^n} \\
&u_t(v_1 \otimes \cdots \otimes v_n) = v_{t(1)} \otimes \cdots \otimes v_{t(n)}.
\end{aligned}$$

Fix a sequence $\{u_n\} \subset \{u_t : t \in \Pi\}$ such that the permutation t_n corresponding to u_n satisfies $t_n(i) > n \quad \forall i \leq n$. It is clear that $\phi_\lambda(u_t x u_t^*) = \phi_\lambda \quad \forall x \in \mathcal{F}, t \in \Pi$.

5.4. Lemma. *Let $x \in \mathcal{F}$, then $\pi_\lambda(u_n x u_n^*) \to \phi_\lambda(x)1$ weakly.*

Proof.

$$\begin{aligned}
\phi_\lambda(z^* u_n x u_n^* y) &= \phi_\lambda(u_n x u_n^* z^* y) \\
&= \phi_\lambda(u_n x u_n^*)\phi_\lambda(z^* y)
\end{aligned}$$

for all $x, y, z \in M_{2^k}$ and $n \geq k$. By continuity $\phi_\lambda(z^* u_n x u_n^* y) \to \phi_\lambda(x)\phi_\lambda(z^* y)$ for all $x, y, z \in \mathcal{F}$. Therefore $(\pi_\lambda(u_n x u_n^*)\xi_y, \xi_z) \to \phi_\lambda(x)\left(\xi_y, \xi_z\right)$. Now $\{\pi_\lambda(u_n x u_n^*)\}$ is bounded and weakly convergent on a dense set of vectors. Therefore $\pi_\lambda(u_n x u^*) \to \phi_\lambda(x)1$ weakly. ☐

5.5. Lemma. *Let ψ be a positive functional on $\mathcal{F}$ with a normal extension to $\pi_\lambda(\mathcal{F})^{cc}$ for some $\lambda \in [0, 1/2]$. Suppose also that $\psi(u_t x u_t^*) = \psi(x)\ \forall x \in \mathcal{F}, t \in \Pi$. Then ψ is a scalar multiple of ϕ_λ.*

Proof. ψ is weakly continuous on bounded sets in $\pi_\lambda(\mathcal{F})$ so by the lemma $\psi(x) = \psi(u_n x u_n^*) \to \phi_\lambda(x)\psi(1)$, therefore $\psi = \psi(1)\phi_\lambda$. ☐

5.6. Theorem. *The von Neumann algebras $\mathcal{M}_\lambda = \pi_\lambda(\mathcal{F})^{cc}$, for $0 < \lambda < 1/2$, are factors of type III.*

Proof. Each $\mathcal{M}_\lambda$ is a factor by a previous result. Let τ be a normal, faithful, semi-finite trace on $\mathcal{M}_\lambda$. Thus there exists a unique positive operator h on $\mathcal{H}_\lambda$ such that $\phi_\lambda(x) = \tau(hx)$. Now h is unique and

$$\begin{aligned}\tau(\pi_\lambda(u_t^*)h\pi_\lambda(u_t)x) &= \tau(h\pi_\lambda(u_t)x\pi_\lambda(u_t^*)) \\ &= \phi_\lambda(\pi_\lambda(u_t)x\pi_\lambda(u_t^*)) \\ &= \phi_\lambda(x),\end{aligned}$$

so $\pi_\lambda(u_t^*)h\pi_\lambda(u_t) = h$ for all $t \in \Pi$.

Pick $\epsilon > 0$ and put $\psi_\epsilon(\cdot) = \tau(h(\epsilon + h)^{-1}\cdot)$. $h(\epsilon+h)^{-1}$ commutes with all $\pi_\lambda(u_t)$ so we have $\psi_\epsilon(\pi_\lambda(u_t x u_t^*)) = \psi_\epsilon(\pi_\lambda(x))$. Therefore $\psi_\epsilon = \psi_\epsilon(1)\phi_\lambda$.

Choose $x \in (\mathcal{M}_\lambda)_+$ such that $\tau(x) < \infty$ and $\phi_\lambda(x) = \infty$. This is possible since τ is semi-finite. Then

$$\psi_\epsilon(1)\phi_\lambda(x) = \psi_\epsilon(x) = \tau(h(\epsilon + h)^{-1}x) \to \tau(x).$$

Therefore $\psi_\epsilon(1) \to \alpha < \infty$ as $\epsilon \to 0$, and for any $x \in (\mathcal{M}_\lambda)_+$ we have

$$\begin{aligned}\alpha\phi_\lambda(x) &= \lim \psi_\epsilon(1)\phi_\lambda(x) \\ &= \lim \tau(h(\epsilon+h)^{-1}x) \\ &= \tau(x).\end{aligned}$$

Therefore $\tau(x) = \alpha\phi_\lambda$. But we know that ϕ_λ is not a trace when $\lambda \neq 1/2$, so there is a contradiction. Therefore $\mathcal{M}_\lambda$ is of type III. ☐

5.3 $\mathrm{Mat}_n(\mathcal{A})$

Definition. Let $\mathcal{A}$ be a Banach algebra. Let $\mathrm{Mat}_n(\mathcal{A})$ denote the $n \times n$ matrix algebra over $\mathcal{A}$. Then $\mathrm{Mat}_n(\mathcal{A})$ can be made into a Banach algebra in a number of equivalent ways.

Definition. Define $GL_n(\mathcal{A})$ to be the group of invertible elements in $\mathrm{Mat}_n(\mathcal{A})$ which are congruent to 1_n mod $\mathrm{Mat}_n(\mathcal{A})$.

Remark. If $\mathcal{A}$ has a unit, then $GL_n(\mathcal{A})$ is isomorphic to the group of invertible elements of $\mathrm{Mat}_n(\mathcal{A})$.

Definition. Let $\mathcal{A}$ be a C^* algebra. Define $U_n(\mathcal{A})$ to be the group of unitary elements in $\mathrm{Mat}_n(\mathcal{A}^+)$ which are congruent to 1_n mod $\mathrm{Mat}_n(\mathcal{A})$.

Definition. $\mathrm{Mat}_\infty(\mathcal{A})$ is the inductive limit $\mathrm{Mat}_\infty(\mathcal{A}) = \lim_\to \mathrm{Mat}_n(\mathcal{A})$ with the obvious choice of isometric inclusions.

Definition. $GL_\infty(\mathcal{A}) = \lim_\to GL_n(\mathcal{A})$.

Chapter 6

Automorphism Groups

6.1 Automorphisms and Invariant States

Remark. It is necessary to fix some notation and basic ideas from the theory of locally compact groups. Let $M(G)$ denote the Banach space of bounded complex Radon measures on a locally compact group G, identified with $C_0(G)'$. $M(G)$ possesses convolution and involution but is not in general a C^* algebra;

$$\int f(s)d(\mu \times \nu)(s) = \iint f(ts)d\nu(s)d\mu(t), \quad f \in C_0(G)$$
$$\int f(s)d\mu^*(s) = \int \left[f^*(s^{-1})d\mu(s)\right]^*, \quad f \in C_0(G).$$

Definition. A unitary representation $(u, \mathcal{H})$ of G is a homomorphism $t \mapsto u_t$ of G into the unitary group of $\mathcal{B}(\mathcal{H})$, which is continuous in the weak topology on $\mathcal{B}(\mathcal{H})$. Note that the weak, ultra-weak, and strong topologies coincide on the unitary group of $\mathcal{B}(\mathcal{H})$. The representation is called uniformly continuous if it is continuous in the norm topology for $\mathcal{B}(\mathcal{H})$.

Definition. The universal representation $(\pi_u, \mathcal{H}_u)$ of $L^1\,(G)$dg is the direct sum of all non-degenerate representations of $L^1\,(G)$dg. The group C^* algebra of G, $C^*(G)$, is the norm closure of $\pi_u(L^1\,(G)\,dg)$ in $\mathcal{B}(\mathcal{H}_u)$.

Remark. By a representation of a Banach algebra, here we mean an involution-preserving homomorphism into $\mathcal{B}(\mathcal{H})$, for a Hilbert space $\mathcal{H}$.

Definition. For each $\mu \in M(G)$ and $f \in L^2\,(G)\,dg$ the convolution $\mu \times f$ is in $L^2\,(G)$dg. Define the map

$$\lambda : M(G) \longrightarrow \mathcal{B}(L^2\,(G)\,dg)$$
$$\lambda(\mu)f = \mu \times f.$$

It is easy to check that it is a representation of $M(G)$. We call it the regular representation.

Remark. We can identify the points of G with the point measures δ_s, $s \in G$. The restriction of λ to the point measures is thus identified with the unitary representation $s \to \lambda_s$ of G on $L^2\,(G)$dg given by

$$(\lambda_s f)(t) = f(s^{-1}t), \quad f \in L^2\,(G)\,dg$$

Definition. The group von Neumann algebra for G, $\mathcal{M}(G)$, is the weak closure of $\lambda(L^1(G)\,dg)$ in $\mathcal{B}(L^2(G)\text{dg})$.

Definition. Suppose $\mathcal{A}$ is a C^* algebra with $\mathcal{A} \subset L^\infty(G)\,dg$ and that $\mathcal{A}$ is invariant under left translation. We say that a state m on $\mathcal{A}$ is a left-invariant mean if $m(\lambda_s f) = m(f)$ for all $f \in \mathcal{A}$.

Definition. If there exists a left-invariant mean on $L^\infty(G)$dg then we say that G is amenable.

Definition. Let $UC^\flat(G)$ denote the algebra of bounded uniformly continuous functions on G. Let $C^\flat(G)$ denote the algebra of bounded continuous functions on G.

6.1. Theorem. *Let G be a locally compact group. Then T.F.A.E.*

1. *G is amenable.*
2. *There exists a left-invariant mean on $C^\flat(G)$.*
3. *There exists a left-invariant mean on $UC^\flat(G)$.*
4. *There exists a state on $L^\infty(G)$dg, m, such that $m(\mu \times f) = \mu(G)m(f)$ for each $\mu \in M(G)$ and $f \in L^\infty(G)\,dg$.*

Proof. See [Ped79]. □

6.2. Theorem. *G is amenable if and only if the regular representation is faithful on $C^*(G)$.*

Proof. See [Ped79]. □

Remark. When we speak of the Haar measure on G, we mean, for example, the left Haar measure so that $d(ts) = ds$. There is also a right Haar measure, and it is connected to the left Haar measure by the modular function $\Delta : G \longrightarrow \mathbb{R}_+$, $d(st) = \Delta(t)ds$, $d(s^{-1}) = \Delta(s)^{-1}ds$.

Definition. A C^*-dynamical system is a triple $(\mathcal{A}, G, \alpha)$ with a C^* algebra $\mathcal{A}$, a locally compact group G, and a continuous homomorphism $\alpha : G \longrightarrow \mathrm{Aut}\,(\mathcal{A})$. $\mathrm{Aut}\,(\mathcal{A})$ is equipped with the topology of pointwise convergence, so for each $A \in \mathcal{A}$ $\alpha(A) : G \longrightarrow \mathrm{Aut}\,(\mathcal{A})$ given by $t \mapsto \alpha_t(A)$ is continuous. When G and $\mathcal{A}$ are separable we call this a separable dynamical system.

Remark. When $\mathcal{M}$ is a von Neumann algebra we consider the topology of pointwise weak convergence on $\mathrm{Aut}\,(\mathcal{M})$. This is equivalent to pointwise ultra-weak convergence and pointwise strong convergence since these coincide on the unitary group of $\mathcal{M}$ and the unitary group is stable under $\mathrm{Aut}\,(\mathcal{A})$ and generates $\mathcal{M}$ linearly.

Definition. A W^*-dynamical system is a triple $(\mathcal{M}, G, \alpha)$ with $\alpha : G \longrightarrow \mathrm{Aut}\,(\mathcal{M})$ continuous in the topology of pointwise weak convergence.

Definition. A covariant representation of a C^*-dynamical system $(\mathcal{A}, G, \alpha)$ is a triple $(\pi, u, \mathcal{H})$ where $(\pi, \mathcal{H})$ is a representation of $\mathcal{A}$, and $(u, \mathcal{H})$ is a unitary representation of G, and we have $\pi(\alpha_t(A)) = u_t\pi(A)u_t^*$.

Remark. Let $(\mathcal{A}, G, \alpha)$ be a C^*-dynamical system. Let $K(G, \mathcal{A})$ be the space of continuous functions of compact support from G to $\mathcal{A}$. Define involution and convolution on $K(G, \mathcal{A})$ by

$$y^*(t) = \Delta(t)^{-1}\alpha_t(y(t^{-1})^*),$$
$$(y \times z)(t) = \int y(s)\alpha_s(z(s^{-1}t))ds.$$

Define $\|y\|_1 = \int \|y\| \, dt$. Then $K(G, \mathcal{A})$ is a normed algebra with isometric involution. Denote its completion by $L^1\,(G)\mathcal{A}$.

Let $A \in \mathcal{A}$ and $f \in L^1\,(G)\,dg$. We can define $A \otimes f \in L^1\,(G)\,\mathcal{A}$ such that $(A \otimes f)(t) = Af(t)$. The span of such elements is dense in $L^1\,(G)\mathcal{A}$.

6.3. Theorem. *If $(\pi, u, \mathcal{H})$ is a covariant representation of $(\mathcal{A}, G, \alpha)$, then there exists a non-degenerate representation $(\pi \times u, \mathcal{H})$ of $L^1\,(G)\mathcal{A}$ such that*

$$(\pi \times u)(y) = \int \pi(y(t))u_t dt \quad \textit{for all } y \in K(G, \mathcal{A}).$$

Moreover, the correspondence $(\pi, u, \mathcal{H}) \to (\pi \times u, \mathcal{H})$ is a bijection onto the set of non-degenerate representations of $L^1\,(G)\mathcal{A}$.

Proof. See [Ped79]. □

Definition. The universal representation $(\pi_u, \mathcal{H}_u)$ of $L^1\,(G)\mathcal{A}$ is the direct sum of all non-degenerate representations of $L^1\,(G)\mathcal{A}$.

Definition. The crossed product of $(\mathcal{A}, G, \alpha)$ is the norm closure of $\pi_u(L^1\,(G)\,\mathcal{A})$ in $\mathcal{B}(\mathcal{H}_u)$. It is denote by $G \times_\alpha \mathcal{A}$.

Remark. Now we will introduce some ideas due to Störmer which have direct physical relevance. See Refs. [Sto69, Sto67, DKS69].

Definition. Let $\mathrm{Conv}\,(W)$ denote the smallest convex subset of the vector space $V \supset W$ containing W. We say that G is represented as a large group of automorphisms of $\mathcal{A}$ if, for each G-invariant state ϕ, we have

$$\text{weak closure}\,\{\pi_\phi(\mathrm{Conv}\,(\alpha_G(\mathcal{A})))\} \bigcap \pi_\phi(\mathcal{A})^c \neq \varnothing, \quad \forall A \in \mathcal{A}.$$

Definition. We say that the C^*-dynamical system $(\mathcal{A}, G, \alpha)$ is asymptotically abelian if there is a net $\Lambda \subset G$ such that

$$\|A\alpha_t(B) - \alpha_t(B)A\| \to 0 \quad \text{as } t \to \infty \text{ in } \Lambda.$$

We say that it is weakly asymptotically abelian if we have

$$\phi(A\alpha_t(B) - \alpha_t(B)A) \to 0 \text{ as } t \to \infty \text{ in } \Lambda, \text{ for any } \phi \in \mathcal{A}'.$$

6.4. Lemma. *Let $(\mathcal{A}, G, \alpha)$ be a weakly asymptotically abelian C^*-dynamical system. Then G is a large group of automorphisms.*

Proof. Let ϕ be an invariant state and $A \in \mathcal{A}$. Let z denote any weak limit of the bounded net $\{\pi_\phi(\alpha_t(A)) \, : \, t \in \Lambda\}$. Then clearly z is in the weak closure of $\{\pi_\phi(\mathrm{Conv}\,(\alpha_G(\mathcal{A}))$. Moreover, for any $B \in \mathcal{A}$ and ψ in the pre-dual of $\pi_\phi(\mathcal{A})^{cc}$, we have

$$\begin{aligned} \psi(z\pi_\phi(B) - \pi_\phi(B)z) &= \lim \psi(\pi_\phi(\alpha_t(A)B - B\alpha_t(A))) \\ &= 0 \end{aligned}$$

Therefore $z \in \pi_\phi(\mathcal{A})^c$. So G is large. □

Definition. If ϕ is a G-invariant state of $\mathcal{A}$, then we say that ϕ is asymptotically multiplicative with respect to the net Λ if

$$\phi(\alpha_t(A)B) \to \phi(A)\phi(B) \text{ as } t \to \infty \text{ in } \Lambda.$$

Such states are also called strongly clustering or strongly mixing.

6.5. Theorem. *Let $(\mathcal{A}, G, \alpha)$ be a weakly asymptotically abelian C^*-dynamical system and consider a G-invariant state ϕ on $\mathcal{A}$, with covariant cyclic representation $(\pi_\phi, u^\phi, \mathcal{H}_\phi, \xi_\phi)$. Then T.F.A.E.*

1. *ϕ is asymptotically multiplicative.*
2. *ϕ is an extreme point of the set of G-invariant states on $\mathcal{A}$, and for each $A \in \mathcal{A}$ the net $\{\pi_\phi(\alpha_t(A)) \, : \, t \in \Lambda\}$ is weakly convergent to $\phi(A) \cdot 1$ in $\mathcal{B}(\mathcal{H}_\phi)$.*
3. *The net $\left\{u_t^\phi \, : \, t \in \Lambda\right\}$ is weakly convergent in $\mathcal{B}(\mathcal{H}_\phi)$ to the one-dimensional projection on $\mathbb{C}\xi_\phi$.*

Proof. See [Sto69, Sto67, DKS69]. □

6.6. Corollary. *Let $(\mathcal{A}, G, \alpha)$ and ϕ be as above. If ϕ is a factor state, then it is asymptotically multiplicative.*

6.7. Theorem. *Let $(\mathcal{A}, G, \alpha)$ be a C^*-dynamical system with G a large group of automorphisms. Let ϕ be a G-invariant factor state with cyclic covariant representation $(\pi_\phi, u^\phi, \mathcal{H}_\phi, \xi_\phi)$. Let $\mathcal{M} = \pi_\phi(\mathcal{A})^{cc}$, and let $\tilde{\phi}$ be the vector state on $\mathcal{B}(\mathcal{H}_\phi)$ determined by ξ_ϕ. Then*

1. *$\mathcal{M}$ is finite $\Longleftrightarrow \tilde{\phi}$ is a trace on $\mathcal{M}$.*
2. *$\mathcal{M}$ is semi-finite but infinite $\Longleftrightarrow \tilde{\phi}$ is a trace on $\mathcal{M}^c$, but not on $\mathcal{M}$.*
3. *$\mathcal{M}$ is type $III \Longleftrightarrow \tilde{\phi}$ is not a trace on $\mathcal{M}^c$.*

6.8. Theorem. *Let $(\mathcal{A}, G, \alpha)$ be a weakly asymptotically abelian C^*-dynamical system with G abelian, and let ϕ be a G-invariant factor state of $\mathcal{A}$ with cyclic covariant representation $(\pi_\phi, u^\phi, \mathcal{H}_\phi, \xi_\phi)$. Let $\mathcal{M} = \pi_\phi(\mathcal{A})^{cc}$, and let $\tilde{\phi}$ be the vector state on $\mathcal{B}(\mathcal{H}_\phi)$ determined by ξ_ϕ. Then*

1. *$\mathcal{M} = \mathbb{C} \cdot 1 \Longleftrightarrow \phi$ is multiplicative.*
2. *$\mathcal{M} = \mathcal{B}(\mathcal{H}_\phi)$, $\dim \mathcal{H}_\phi = \infty \Longleftrightarrow \phi$ is a pure state but not multiplicative.*

3. *$\mathcal{M}$ is type II_1 $\Longleftrightarrow \phi$ is a trace but not multiplicative.*

4. *$\mathcal{M}$ is type II_∞ $\Longleftrightarrow \phi$ is a trace on $\mathcal{M}^c$ but ϕ is neither pure nor a trace.*

5. *$\mathcal{M}$ is type III $\Longleftrightarrow \tilde{\phi}$ is not a trace on $\mathcal{M}^c$.*

Remark. In the case that G is abelian we have a zoo of results. These results can be thought of as harmonic analysis on operator algebras. They will lead us to the Tomita-Takesaki theory, the introduction of complex function theoretic techniques for one-parameter groups (KMS states, etc.), and a classification of factors of type III.

Remark. First we want to associate subsets of the dual group of G to subspaces of a Banach space X, when G acts as isometries on X. When X is a C^* algebra and G acts as automorphisms we will be able to construct a spectral measure μ on the dual group with $\mu(\Omega)$ corresponding to the support projection of the subspace associated to Ω. When this happens we will be able to construct a unitary representation of G which, under certain conditions, is covariant for the automorphism representation of G.

Definition. Let X and X_* be two Banach spaces in duality via a bilinear form $\langle \cdot, \cdot \rangle$. This means

- If $x \in X$ then $\langle x, \cdot \rangle \in {X_*}'$.
- If $\xi \in X_*$ then $\langle \cdot, \xi \rangle \in X'$.
- The maps $x \mapsto \langle x, \cdot \rangle$ and $\xi \mapsto \langle \cdot, \xi \rangle$ are isometries of X and X_* onto weak-* dense subspaces of ${X_*}'$ and X' respectively.

Let $B_\sigma(X)$ and $B_\sigma(X_*)$ denote the bounded linear operators which are continuous in the $\sigma(X, X_*)$ and $\sigma(X_*, X)$ topologies. Note that if $U \in B(X)$ then $U \in B_\sigma(X)$ if and only if $U^T \in B(X_*)$.

A representation of a locally compact G on X is a $\sigma(X, X_*)$ continuous homomorphism $t \mapsto \alpha_t$ of G onto the group of invertible elements in $B_\sigma(X)$. We say that α is an integrable representation if for each $\mu \in M(G)$ there is a (necessarily unique) $\alpha_\mu \in B\sigma(X)$ such that

$$\langle \alpha_\mu(x), \xi \rangle = \int \langle \alpha_t(x), \xi \rangle \, d\mu(t), \quad \forall x \in X, \xi \in X_*.$$

Note that α^T is integrable whenever α is integrable.

6.9. Lemma. *Let X be a Banach space and $X_* = X'$. Let $h : t \mapsto \alpha_t$ be a homomorphism of a locally compact group G into the group of invertible isometries on X such that $t \mapsto \alpha_t(x)$, $x \in X$, is norm continuous. Then h is an integrable representation of G on X.*

Proof. See the appendix of [Ped79]. □

Definition. Let G be a locally compact abelian group and let Γ denote its dual group. Denote the unit in Γ by θ. For $t \in G$ and $\tau \in \Gamma$ let (t, τ) denote the value of τ at t and write $\widehat{\mu}(t) = \int (t, \tau) d\mu(t)$ for each $\mu \in M(G)$, i.e. the inverse Fourier transform.

Let $K^1(G)$ be the dense ideal of $L^1(G)$dg consisting of functions such that $\widehat{f}$ has compact support in Γ. Let X and X_* be as introduced previously and let α be an integrable representation of G on X. For each open $\Omega \subseteq \Gamma$ we define the spectral R-subspace

$$R^\alpha(\Omega) = \sigma(X, X_*) - \text{closure in } X \text{ of the linear subspace } \left\{ \alpha_f(x) \, : \, x \in X, f \in K^1(G), \operatorname{supp} \widehat{f} \subseteq \Omega \right\}.$$

For each closed $\Lambda \subseteq \Gamma$ the spectral M-subspace is

$$M^{\alpha}(\Lambda) = \text{ annihilator of } R^{\alpha'}(\Gamma|\Lambda).$$

In other words,

$$x \in M^{\alpha}(\Lambda) \Longleftrightarrow \left\langle x, \alpha_f^T(\xi)\right\rangle = 0 \quad \forall \xi \in X_*, f \in K^1(G) \text{ with } \operatorname{supp}\widehat{f} \subseteq \Gamma|\Lambda.$$

6.10. Theorem. *Let R^α and M^α be as above. Then*

1. *If $\Omega_1 \subset \Omega_2$ then $R^\alpha(\Omega_1) \subset R^\alpha(\Omega_2)$.*
2. *If $\Lambda_1 \subset \Lambda_2$ then $M^\alpha(\Omega_1) \subset M^\alpha(\Omega_2)$.*
3. *The σ-closure of $\sum_i R^\alpha(\Omega_i)$ is equal to $R^\alpha(\cup_i \Omega_i)$.*
4. *$\cap_i M^\alpha(\Lambda_i) = M^\alpha(\cap_i \Lambda_i)$.*
5. *If $\Omega \subset \Lambda$ then $R^\alpha(\Omega) \subset M^\alpha(\Lambda)$.*
6. *If $\Lambda \subset \Omega$ then $M^\alpha(\Lambda) \subset R^\alpha(\Omega)$.*
7. *If $\Lambda = \cap_i \Omega_i = \cap_i \bar{\Omega}_i$ then $M^\alpha(\Lambda) = \cap_i R^\alpha(\Omega_i)$.*
8. *If $\Omega = \cup_i \Lambda_i = \cup_i \operatorname{int}\Lambda_i$ then $R^\alpha(\Omega) = \sigma$-closure $\sum_i M^\alpha(\Lambda_i)$.*
9. *$R^\alpha(\varnothing) = M^\alpha(\varnothing) = \{0\}$; $R^\alpha(\Gamma) = M^\alpha(\Gamma) = X$.*

Proof. See [Ped79]. □

Definition. From the fourth point above, there exists a smallest closed set $\Lambda \subset \Gamma$ such that $M^\alpha(\Lambda) = X$. We call Λ the Arveson spectrum of α and denote it $\operatorname{Spec}(\alpha)$.

6.11. Theorem. *Let α be an integrable representation of G on X. For each $\sigma \in \Gamma$, T.F.A.E.*

1. *$\sigma \in \operatorname{Spec}(\alpha)$.*
2. *$R^\alpha(\Omega) \neq 0$ for every nbhd. Ω of σ.*
3. *There exists a net $\{x_i\}$ in the unit sphere of X such that $\|\alpha_t(x_i) - (t,\tau)x_i\| \to 0$ uniformly on compact subsets of G.*
4. *For every $\mu \in M(G)$ we have $|\widehat{\mu}| \leq \|\alpha_\mu\|$.*
5. *For every $f \in L^1(G)\,dg$ we have $|\widehat{f}(\sigma)| \leq \|\alpha_f\|$.*
6. *If $f \in L^1(G)\,dg$ and $\alpha_f = 0$ then $\widehat{f}(\sigma) = 0$.*

Proof.

1⟹2 If $R^\alpha(\Omega) = 0$, then $\operatorname{Spec}(\alpha) \subset \Gamma\backslash\Omega$. Conversely if $\sigma \notin \operatorname{Spec}(\alpha)$ then there exists an open nbhd. Ω of σ with $\operatorname{Spec}(\alpha) \cap \Omega = \varnothing$. Therefore $R^\alpha(\Omega) = 0$.

2⟹3 This follows from the following technical lemma which we do not prove: For $\sigma \in \Gamma$, $\epsilon > 0$, and K a compact subset of G, there exists a compact nbhd. Λ of σ such that $\|\alpha_t(x) - (t,\sigma)x\| < \epsilon\,\|x\|\ \forall t \in K, x \in M^\alpha(\Lambda)$.

3⟹4 Given μ and $\epsilon > 0$ there is a compact $K \subset G$ such that $|\mu(G\backslash K)| < \epsilon$. Assume $x_i \in X$ and $\|\alpha_t(x_i) - (t,\sigma)x_i\| < \epsilon$ for all $t \in K$; then

$$\begin{aligned}
|\widehat{\mu}(\sigma)| &= \|\widehat{\mu}(\sigma)x_i\| \\
&= \left\| \int (t,\sigma)x_i d\mu(t) \right\| \\
&\le \left\| \int (\alpha_t(x_i) - (t,\sigma)x_i) d\mu(t) \right\| + \left\| \int \alpha_t(x_i) d\mu(t) \right\| \\
&\le \epsilon|\mu(K)| + 2|\mu(G\backslash K) + \|\alpha_\mu(x_i)\| \\
&\le \epsilon\,\|\mu\| + 2\epsilon + \|\alpha_\mu\|\,.
\end{aligned}$$

Therefore $\widehat{\mu}(\sigma) \le \|\alpha_\mu\|$.

4⟹5 Obvious.

5⟹6 Obvious.

6⟹2 Let Ω be a nbhd. of σ. There exists $f \in K^1(G)$ with $\operatorname{supp}(\widehat{f}) \subset \Omega, \widehat{f}(\sigma) = 1$. Then by assumption $\alpha_f(x) \neq 0$ for some $x \in X$, and so $R^\alpha(\Omega) \neq 0$.

□

6.12. Theorem. *Let α be an integrable representation of G on X. If $\mathcal{A}$ is the commutative Banach algebra in $B(X)$ generated by $\alpha_f, f \in L^1(G)\,dg$, then the Arveson spectrum of α is homeomorphic to the Gelfand spectrum of $\mathcal{A}$.*

Proof. The dual of the homomorphism $\alpha : L^1(G)\,dg \longrightarrow \mathcal{A}$ defines a continuous injection $\alpha_* : \widehat{\mathcal{A}} \longrightarrow \Gamma$ since Γ is the spectrum of $L^1(G)$dg. $\widehat{\mathcal{A}}$ is locally compact so α_* is a homeomorphism onto its image. From the previous proposition then $\sigma \in \alpha_*(\widehat{\mathcal{A}})$ if and only if $\sigma \in \operatorname{Spec}(\alpha)$. □

6.13. Theorem (Compact Arveson Spectrum). *Let α be an integrable representation of G on X. Then T.F.A.E.*

1. $\operatorname{Spec}(\alpha)$ *is compact.*
2. α *is uniformly continuous, i.e.* $\|1 - \alpha_t\| \to 0$ *as* $t \to 0$.

Proof.

1⟹2 Let $f \in K^1(G)$ with $\widehat{f} = 1$ on an open set Ω containing $\operatorname{Spec}(\alpha)$. Then $\alpha_f(x) = x, \forall x \in R^\alpha(\Omega)$, and since $R^\alpha(\Omega) = X$, $\alpha_f = 1$. But then

$$\begin{aligned}
&\|1 - \alpha_t(x)\| \le \|f - \delta_t \times f\|_1\,\|x\|\,, \quad \forall x \\
&\Longrightarrow \|1 - \alpha_t\| \to 0 \text{ as } t \to 0.
\end{aligned}$$

2$\Longrightarrow$1 Let (f_λ) be an approximate identity for $L^1(G)$dg.

$$\begin{aligned}\|x-\alpha_t(x)\| &\leq \int \|\alpha_t(x)-x\| \, f_\lambda(t)dt \\ &\leq \int \|\alpha_t - 1\| \, f_\lambda(t)dt \, \|x\| \, .\end{aligned}$$

Therefore $\alpha_{f_\lambda} \to 1$ so the Banach algebra generated by $\alpha(L^1(G)\,dg)$ contains the identity, and so $\mathrm{Spec}\,(\alpha)$ is compact by the previous theorem.

$\square$

6.14. Theorem (Stone). *Let $t \mapsto u_t$ be a unitary representation of an abelian group G on a Hilbert space $\mathcal{H}$. There exists a unique spectral measure μ on the Borel sets of Γ, with values in $\mathcal{B}(\mathcal{H})$, such that*

$$u_t = \int (t,\tau) d\mu(\tau), \quad \forall t \in g.$$

Proof. Let $\pi(f) = \int u_t f(t)dt$ for any $f \in L^1(G)\,dg$. Then π is a *-representation of $L^1(G)$dg into $\mathcal{B}(\mathcal{H})$. Since each $\pi(f)$ is a normal operator $\|\pi(f)\| \leq \left\|\widehat{f}\right\|$. Therefore π extends by continuity to a representation of the C^* algebra $C_0(\Gamma)$. Restricting π to the projections in the Borel functions on Γ we obtain a spectral measure μ on Γ satisfying the required relation. $\square$

Definition. Let $\mathcal{I}$ be an ideal in the C^* algebra $\mathcal{A}$. We say that $\mathcal{I}$ is essential in $\mathcal{A}$ if each non-zero closed ideal of $\mathcal{A}$ has a non-zero intersection with $\mathcal{I}$.

Remark. Remember that the groups in this section are abelian. If $\mathcal{B}$ is a G-invariant C^* algebra of $\mathcal{A}$, then we can consider the dynamical system $(\mathcal{B}, G, \alpha|_{\mathcal{B}})$. Clearly $\mathrm{Spec}\,(\alpha|_{\mathcal{B}}) \subset \mathrm{Spec}\,(\alpha)$.

Definition. Let $H^\alpha(\mathcal{A})$ denote the set of G-invariant, hereditary, non-zero C^*-subalgebras of $\mathcal{A}$. Let $H^\alpha_B(\mathcal{A})$ denote the subset consisting of algebras $\mathcal{B}$ in $H^\alpha(\mathcal{A})$ such that the closed ideal of $\mathcal{A}$ generated by $\mathcal{B}$ is essential in $\mathcal{A}$.

The Connes spectrum of α is

$$\Gamma(\alpha) = \bigcap \mathrm{Spec}\,(\alpha|_{\mathcal{B}})\,, \quad \mathcal{B} \in H^\alpha(\mathcal{A}).$$

The Borchers spectrum of α is

$$\Gamma_B(\alpha) = \bigcap \mathrm{Spec}\,(\alpha|_{\mathcal{B}})\,, \quad \mathcal{B} \in H^\alpha_B(\mathcal{A}).$$

Obviously $\Gamma(\alpha) \subset \Gamma_B(\alpha)$.

Definition. If $(\mathcal{M}, G, \alpha)$ is a W^*-dynamical system then we define

$$\Gamma(\alpha) = \bigcap \mathrm{Spec}\,(\alpha|_{p\mathcal{M}p})\,, \quad p \in \{\text{non-zero } G\text{-invariant projections}\},$$

$$\Gamma_B(\alpha) = \bigcap \mathrm{Spec}\,(\alpha|_{p\mathcal{M}p})\,, \quad p \in \{\text{non-zero } G\text{-invariant projections with } c(p)=1\}.$$

6.15. Theorem (Connes Subgroup). *Let $(\mathcal{A}, G, \alpha)$ be an abelian C^*-dynamical system. If $\sigma_1 \in \Gamma(\alpha)$ and $\sigma_2 \in \mathrm{Spec}\,((\alpha)$, then $\sigma_1 + \sigma_2 \in \mathrm{Spec}\,(\alpha)$. Moreover, $\Gamma(\alpha)$ is a closed subgroup of Γ.*

Proof. Let Ω be a nbhd. of $\sigma_1 + \sigma_2$. Then there are nbhds. Ω_1 and Ω_2 of σ_1 and σ_2 such that $\Omega_1 + \Omega_2 \subset \Omega$. Now $R^\alpha(\Omega_2)$ is non-trivial by assumption; let $x_2 \neq 0$ be from $R^\alpha(\Omega_2)$. Let $\mathcal{B}$ denote the hereditary C^*-subalgebra of $\mathcal{A}$ generated by the orbit $\{\alpha_t(x_2^* x_2) : t \in G\}$. If $x \in \mathcal{B}, x \neq 0$, then $\alpha_t(x_2^* x_2) \neq 0$ for some $t \in G$. $\mathcal{B}$ is G-invariant so there is a non-zero element $x \in R^{\alpha|_\mathcal{B}}(\Omega_1)$. Thus $\alpha_t(x_2)x_1 \neq 0$ fro some $t \in G$.

$\alpha_t(x_2) \in R^\alpha(\Omega_2) \Longrightarrow \alpha_t(x_2)x_1 \in R^\alpha(\Omega_1 + \Omega_2) \Longrightarrow R^\alpha(\Omega) \neq 0$. This holds for every Ω a nbhd. of $\sigma_1 + \sigma_2$, so $\sigma_1 + \sigma_2 \in \mathrm{Spec}\,(\alpha)$.

Now, if $\sigma_1, \sigma_2 \in \Gamma(\alpha)$, by the above construction we know $\sigma_1 + \sigma_2 \in \mathrm{Spec}\,(\alpha|_\mathcal{B})$ for all $\mathcal{B} \in H^\alpha(\mathcal{A})$. Therefore $\sigma_1 + \sigma_2 \in \Gamma(\alpha)$. Since $\Gamma(\alpha)$ is the intersection of symmetric, closed sets, it is a closed subgroup of Γ. ☐

6.16. Theorem ($\mathbb{Z}$ subgroups of Γ_B). *Let $(\mathcal{A}, G, \alpha)$ be an abelian C^*-dynamical system. If $\sigma \in \Gamma_B(\alpha)$ then $n\sigma \in \Gamma_B(\alpha), \forall n \in \mathbb{Z}$.*

Proof. We will prove by induction that for any nbhd. Ω of σ, any $\mathcal{B} \in H^\alpha_\mathcal{B}(\mathcal{A})$, and any $n \in \mathbb{Z}$ there exist elements $x_1, \ldots, x_n$ in $R^\alpha(\Omega) \cap \mathcal{B}$ such that $x_1 x_2 \cdots x_n \neq 0$. This is true for $n = 1$ since $\sigma \in \Gamma_B(\alpha)$.

Assume the induction step for n. Let $\{\mathcal{C}_i\}$ be the maximal collection of algebras in $H^\alpha(\mathcal{B})$ such that the ideals generated by the $\mathcal{C}_i$ are mutually orthogonal and such that for each i there is an $x_i \in R^\alpha(\Omega)$ such that $\mathcal{C}_i$ is the hereditary C^*-algebra generated by the orbit $\{\alpha_t(x_i^* x_i) : t \in G\}$. Let $\mathcal{C} = \oplus \mathcal{C}_i$. Either $\mathcal{C}_i \in H^\alpha_B(\mathcal{B})$ or we can find (by maximality) a closed, G-invariant ideal $\mathcal{I} \in \mathcal{B}$, orthogonal to the ideal generated by $\mathcal{C}$ such that $\mathcal{C} + \mathcal{I} \in H^\alpha_\mathcal{B}(\mathcal{B})$. In either case, $\mathcal{I} = 0$ or $\mathcal{I} \neq 0$, we must have $R^\alpha(\Omega) \cap \mathcal{I} = 0$. Otherwise we contradict maximality of $\{\mathcal{C}_i\}$.

$\mathcal{C} + \mathcal{I} \in H^\alpha_B(\mathcal{B})$ and $\mathcal{B} \in H^\alpha_\mathcal{B}(\mathcal{B})$, so $\mathcal{C} + \mathcal{I} \in H^\alpha(\mathcal{B})$. By the induction hypothesis there exist $x_1, \ldots, x_n$ in $R^\alpha(\Omega) \cap \mathcal{C} + \mathcal{I}$ such that $y = x_1 x_2 \cdots x_n \neq 0$. Since $R^\alpha(\Omega) \cap \mathcal{I} = 0$, $x_k \in R^\alpha(\Omega) \cap \mathcal{C} \forall k$. Thus $y \in \mathcal{C}$. But then $\alpha_t(x_i)y \neq 0$ for some $t \in G$ and some i since $\mathcal{C} = \oplus \mathcal{C}_i$. Since $\alpha_t(x_i) \in R^\alpha(\Omega) \cap \mathcal{B}$ we have established the claim for $n + 1$, and thus for all $n \in \mathbb{N}$.

Now assume $n > 0$, since $\Gamma_B(\alpha)$ is a symmetric set. Let Ω_n be a nbhd. of $n\sigma$, and choose Ω a nbhd. of σ such that $\Omega + \cdots + \Omega \subset \Omega_n$. Given $\mathcal{B} \in H^\alpha_B(\mathcal{B})$ we obtain $x_1, \ldots, x_n$ in $R^\alpha(\Omega) \cap \mathcal{B}$ such that $y = x_1 x_2 \cdots x_n \neq 0$. Then $y \in R^\alpha(\Omega + \cdots + \Omega) \cap \mathcal{B} \subset R^\alpha(\Omega_n) \cap \mathcal{B}, \forall n$. Therefore $n\sigma \in \mathrm{Spec}\,(\alpha|\mathcal{B})$. But $\mathcal{B}$ was arbitrary. ☐

6.2 KMS States

Remark. Now we will further specialize to the case $G = \mathbb{R}$. States will be characterized by the behaviour of their correlation functions in the complex frequency plane. Roughly speaking, the growth at $\mathrm{Im}\omega > 0$ controls the growth for $t < 0$. This will introduce complex function techniques.

Definition. Let $(\mathcal{A}, G, \alpha)$ be a C^*-dynamical system. We say that $A \in \mathcal{A}$ is analytic for α if the function $t \mapsto \alpha_t(A)$ has an extension to an analytic function $\zeta \mapsto \alpha_\zeta(A)$, $\zeta \in \mathbb{C}$.

6.17. Lemma. *The set of analytic elements of a C^*-dynamical system for $\mathcal{A}$ forms a dense *-subalgebra of $\mathcal{A}$. The set of analytic elements of a W^*-dynamical system for $\mathcal{M}$ forms a σ-weakly dense *-subalgebra of $\mathcal{M}$.*

Proof. Density follows from the approximation, for any $A \in \mathcal{A}$,

$$A_n = \pi^{-1/2} n^{1/2} \int \alpha_t(A) e^{-nt^2} dt,$$

$A_N \to A$ as $n \to \infty$, and A_n is analytic. Similarly for the W^* case. $\square$

Definition. Given a C^*-dynamical system $(\mathcal{A}, \mathbb{R}, \alpha)$ we say that a state ϕ on $\mathcal{A}$ is a KMS state for β, $\beta \in (0, \infty)$, if for any $A \in \mathcal{A}_{analytic}$, $B \in \mathcal{A}$,

$$\phi(B\alpha_{\zeta+i\beta}(A)) = \phi(\alpha_\zeta(A)B), \quad \zeta \in \mathbb{C}.$$

ϕ is called KMS for $\beta = 0$ if it is an α-invariant trace. (chaotic state)
ϕ is called KMS for $\beta = \infty$ if $|\phi(B\alpha_\zeta(A))| \leq \|A\| \, \|B\|$ for $\operatorname{Im}\zeta \geq 0$. (ground state)

6.18. Theorem. *Let $(\mathcal{A}, \mathbb{R}, \alpha)$ be a C^*-dynamical system. Fix $\beta \in (0, \infty]$. Then ϕ is a β-KMS state if and only if for every $A, B \in \mathcal{A}$ there exists a bounded continuous function $f : \Omega_\beta \longrightarrow \mathbb{C}$, $\Omega_\beta = \{\zeta \in \mathbb{C} : 0 \leq \operatorname{Im}\zeta \leq \beta\}$, such that f is holomorphic on $\operatorname{int}(\Omega_\beta)$ and one of the following is true.*

- *If $\beta < \infty$ $f(t) = \phi(B\alpha_t(A))$, $f(t + i\beta) = \phi(\alpha_t(A)B)$.*
- *If $\beta = \infty$ $f(t) = \phi(B\alpha_t(A)), t \in \mathbb{R}$, $\|f\| \leq \|x\| \, \|y\|$.*

Proof. Obviously the second part implies that ϕ is β-KMS. Assume ϕ is β-KMS. Let $\{A_n\}$ be the sequence of analytic elements converging to $A \in \mathcal{A}$ and let $B \in \mathcal{A}$. Define $f_n(\zeta) = \phi(B\alpha_\zeta(A))$ for $\beta < \infty$. Then the $\{f_n\}$ are analytic and $f_n(\zeta + i\beta) = \phi(\alpha_\zeta(A)B)$.

Now each f_n is bounded on Ω_β, $|f_n(\zeta)| \leq \|B\| \, \|\alpha_{it}(A_n)\|$. By the Phragmen-Lindelöf theorem we have

$$\begin{aligned} |f_n(\zeta) - f_m(\zeta)| &\leq \sup_{z \in \partial\Omega_\beta} |f_n(z) - f_m(z)| \\ &\leq \sup_t |\phi(B\alpha_t(A_n - A_m))| \vee |\phi(\alpha_t(A_n - A_m)B)| \\ &\leq \|B\| \, \|A_n - A_m\| \, . \end{aligned}$$

Therefore the $\{f_n\}$ are uniformly convergent to a function bounded and continuous on Ω_β and holomorphic on $\operatorname{int}(\Omega_B)$. On the boundary $f(t) = \phi(B\alpha_t(A)), f(t + i\beta) = \phi(\alpha_t(A)B), t \in \mathbb{R}$.

If $\beta = \infty$ define $f_n(\zeta) = \phi(B\alpha_t(A))$ and the KMS condition at ∞ gives $|f_n(\zeta) - f_m(\zeta)| \leq \|A_n - A_m\| \, \|B\|$ for $\operatorname{Im}\zeta \geq 0$ and again the f_n converge to an f with the required properties. $\square$

6.19. Theorem. *Let $(\mathcal{A}, \mathbb{R}, \alpha)$ be a C^*-dynamical system and let ϕ be a β-KMS state on $\mathcal{A}$. Then ϕ is α-invariant.*

Proof. We have $\phi(\alpha_{\zeta+i\beta}(A)) = \phi(\alpha_\zeta(A))$ for $\beta > 0$. Thus $f : \zeta \mapsto \phi(\alpha_\zeta(A))$ is bounded on Ω_β (previous result) and periodic with period $i\beta$. Therefore f is a constant. Since the analytic elements are dense in $\mathcal{A}$, ϕ is α-invariant by continuity.

If $\beta = 0$ the α-invariance is by definition.

If $\beta = \infty$ we have $f : \zeta \mapsto \phi(\alpha_\zeta(A))$ is such that $|f(\zeta)| \leq \|A\|$ when $\operatorname{Im}\zeta \geq 0$. Now $\phi = \phi^*$, so $\phi(\alpha_\zeta(A))^* = \phi(\alpha_{\zeta^*}(A^*))$, so for $\operatorname{Im}\zeta < 0$ we have $|f(\zeta)| \leq \phi(\alpha_{\zeta^*}(A^*))^* \leq \|A\|$. Therefore f is bounded, therefore f is a constant and ϕ is α-invariant. $\square$

6.20. Theorem (Ground States and Hamiltonians). *Let $(\mathcal{A}, \mathbb{R}, \alpha)$ be a C^*-dynamical system and let ϕ be a state on $\mathcal{A}$. Then T.F.A.E.*

1. *ϕ is a KMS state with $\beta = \infty$ (ground state).*
2. *There exists a positive operator h on $\mathcal{H}_\phi$, not necessarily bounded, with $h\xi_\phi = 0$, and $\exp(ith)\pi_\phi(A)\exp($ $\pi_\phi(\alpha_t(A)), \forall t \in \mathbb{R}, A \in \mathcal{A}$.*
3. *ϕ is α-invariant, and if $(\pi_\phi, u^\phi, \mathcal{H}_\phi, \xi_\phi)$ is the cyclic covariant representation associated with ϕ then $\operatorname{Spec}\left(u^\phi\right) \subset \mathbb{R}_+$.*

Proof.

2⟹1 Since $h \geq 0$, for any $A, B \in \mathcal{A}$ we can define a function f on the upper half plane, Ω_∞, holomorphic and continuous on the boundary,

$$f(\zeta) = (\exp(i\zeta h)\xi_A, \xi_B)\,.$$

Clearly $|f| \leq \|A\|\,\|B\|$. Also, $f(t) = \phi(B^*\alpha_t(A))$ for all $t \in \mathbb{R}$, so ϕ satisfies the KMS condition with $\beta = \infty$.

1⟹3 We know that we can write $u_t = \exp(ith)$ for some self-adjoint h. If A is analytic then ξ_A is analytic for $\exp(ith)$ so $f : \zeta \mapsto (\exp(i\zeta h)\xi_A, \xi_A)$ is analytic; $f(\zeta) = \phi(A^*\alpha_\zeta(A))$. By assumption $f(\zeta) \leq \|A\|^2$ if $\operatorname{Im}\zeta \geq 0$ then $((\exp(-h))^s\xi_A, \xi_A) \leq \|A\|^2$ for any $s \geq 0$. Therefore $\exp(-h) \leq 1$, and so $h \geq 0$ and $\operatorname{Spec}\left(u^\phi\right) \subset \mathbb{R}_+$.

3⟹2 A computation shows that with $u_t^\phi = \exp^{ith}$, ξ_ϕ is in the domain of h and $h\xi_\phi = 0$.

□

Remark. The above theorem says something which can be readily accepted by anyone familiar with renormalization, but only after some realignment of religious ideas. It says that the Hamiltonian generating the time evolution depends on the state chosen, and that it does not really exist independently. As an example, consider a spin system. In the ordered phase the Hamiltonian contains an interaction with an external field. However, in the disordered phase this interaction is irrelevant (in the technical sense), and the construction corresponding to the above theorem would show this. The "renormalization physics" is in some way already contained inside the algebraic approach.

Definition. Let $(\mathcal{A}, \mathbb{R}, \alpha)$ be a C^*-dynamical system. We say that α is approximately inner if there is a net $\{h_\lambda\} \subseteq \mathcal{A}$, h_λ self-adjoint, such that

$$\lim_\lambda \|\alpha_\zeta(A) - \exp(i\zeta h_\lambda)A\exp(-i\zeta h_\lambda)\| = 0$$

uniformly on compact subsets of $\mathbb{C}$.

6.21. Theorem. *Let $(\mathcal{A}, \mathbb{R}, \alpha)$ be a C^*-dynamical system and assume α is approximately inner and $1 \in \mathcal{A}$. Then $\mathcal{A}$ has a ground state (KMS state with $\beta = \infty$).*

Proof. Let $\{h_\lambda\}$ be the net in the definition of approximately inner. Let $\alpha_t^\lambda = \mathrm{Ad}\,(\exp(ith_\lambda))$. Without loss, by adding a multiple of 1 if necessary, $h_\lambda \geq 0$ and $0 \in \mathrm{Spec}\,(h_\lambda)$ for all λ.

Now there is a net of states $\{\phi_\lambda\}$ such that $\phi_\lambda(h_\lambda) = 0$ for each λ. Since the state space is compact we can assume $\{\phi_\lambda\}$ is weak-* convergent to some ϕ. $|\phi_\lambda(B\alpha_\zeta^\lambda(A))| \leq \|A\|\,\|B\|$, if $\mathrm{Im}\,\zeta \geq 0$. Therefore ϕ_λ is a ground state for α^λ, applying theorem 6.18.

Moreover,

$$\begin{aligned} |\phi(B\alpha_\zeta(A))| &\leq |(\phi - \phi_\lambda)(B\alpha_\zeta(A))| + \|B\|\,\left\|\alpha_\zeta(A) - \alpha_\zeta^\lambda(A)\right\| + \|A\|\,\|B\| \\ &\leq \|A\|\,\|B\| \quad \text{in the limit of } \lambda. \end{aligned}$$

Therefore ϕ is a ground state for α. □

Remark. KMS states are physically interesting because the KMS condition can be substituted for the Gibbs ansatz, and it makes sense immediately in infinite volume, without requiring a limiting process. To see the equivalence for finite systems, let $\mathcal{A}$ be a finite-dimensional matrix algebra with a canonical trace $\mathrm{Tr}\,(\cdot)$. Consider the state

$$\phi(A) = \mathrm{Tr}\,(\rho A)\,/\mathrm{Tr}\,(\rho)\,.$$

The automorphism group is $\alpha_t(A) = e^{ith}Ae^{-ith}$. By elementary calculation, ϕ satisfies the KMS condition for some β, if and only if $\rho = \exp(-\beta h)$.

6.3 Modular Group

Remark. It is a remarkable fact that von Neumann algebras carry hidden within themselves a kind of "dynamical" information, in the form of an $\mathbb{R}$-action. How this arises is the subject of the following. This will lead to the classification of type III factors.

Definition. Let $\mathcal{M}$ be a von Neumann algebra on a separable Hilbert space $\mathcal{H}$. Let T be a closed operator on $\mathcal{H}$. T is said to be affiliated to $\mathcal{M}$ if

$$A\,\mathrm{Dom}(T) \subseteq \mathrm{Dom}(T), \quad TA \supseteq AT, \quad \forall\, A \in \mathcal{M}^c.$$

6.22. Lemma. *Let $T = U|T|$ be the polar decomposition of T. Then T.F.A.E.*

1. *U and the spectral projections $E_{|T|}(\cdot)$ belong to $\mathcal{M}$.*
2. *T is affiliated to $\mathcal{M}$.*

Proof. □

Definition. Let $\mathcal{M}$ be a von Neumann algebra on a separable Hilbert space $\mathcal{H}$. Let $\Omega \in \mathcal{H}$ be cyclic and separating for $\mathcal{M}$. Define two anti-linear operators S_0, F_0 by

$$\begin{aligned} S_0 A\Omega &\equiv A^*\Omega, \quad A \in \mathcal{M}, \\ F_0 B\Omega &\equiv B^*\Omega, \quad B \in \mathcal{M}^c. \end{aligned}$$

S_0 and F_0 are closable. Denote their closures by S and F respectively. S is called the Tomita operator for $(\mathcal{M}, \Omega)$. Furthermore we have

$$\begin{aligned} S_0^* &= F \qquad & F_0^* &= S, \\ S^{-1} &= S, \qquad & F^{-1} &= F, \end{aligned}$$

See Ref. [BW92, p. 32].

Definition. Let $S = J\Delta^{1/2}$ be the polar decomposition of the Tomita operator. The anti-unitary operator J is called the modular conjugation and the non-negative operator Δ is called the modular operator.

6.23. Lemma.

1. $\Delta = FS$
2. $\Delta^{-1} = SF$
3. $F = J\Delta^{-1/2}$
4. $J = J^*$
5. $J^2 = 1$
6. $\Delta^{-1/2} = J\Delta^{1/2}J$

Proof.

- $\Delta = S^*S = FS$. $(FS)^{-1} = S^{-1}F^{-1} = SF = \Delta^{-1}$.
- $S = S^{-1} = \Delta^{-1/2}J^* = J^*J\Delta^{-1/2}J^*$. Therefore, by uniqueness of the polar decomposition, $J = J^*$ and $J\Delta^{-1/2}J^* = J\Delta^{-1/2}J = \Delta^{1/2}$.
- $J^* = J \Longrightarrow J^2 = 1$.

□

Definition. The strongly continuous unitary group defined by

$$\Delta^{it} = \exp(it \ln \Delta).$$

is called the modular group.

Example. Let $\mathcal{H} = L^2\left([0,1]\right) dx$ and let $\mathcal{M}$ be the algebra of functions bounded a.e. on $[0,1]$ with pointwise multiplication. $\mathcal{M}$ acts on $\mathcal{H}$ as a commutative algebra of multiplication operators. $\Omega(x) = 1$ is a cyclic and separating vector for $\mathcal{M}$. Then the Tomita operator is complex conjugation, $SA\Omega = A^*\Omega$, and $\Delta = 1$.

Example. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces of dimension n. Let $\{f_1, \ldots, f_n\}$ and $\{g_1, \ldots, g_n\}$ be orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$ respectively. Let $\mathcal{M} = \mathcal{B}(\mathcal{H}) \otimes \mathbb{C}1_{\mathcal{K}}$, acting on the total space $\mathcal{H} \otimes \mathcal{K}$. Define the unit vector $\Omega \in \mathcal{H} \otimes \mathcal{K}$ by

$$\Omega = \sum_{j=1}^{n} a_j f_j \otimes g_j, \quad a_j > 0, \quad \sum_j |a_j|^2 = 1.$$

Note that Ω is not an arbitrary unit vector in $\mathcal{H} \otimes \mathcal{K}$, but is diagonal in the obvious basis. Ω is cyclic and separating for $\mathcal{M}$. The action of $\mathcal{M}$ on $\mathcal{H} \otimes \mathcal{K}$ is generated by the following operators which shuffle basis elements in the first factor,

$$A_{j,s} : f_p \otimes g_l \mapsto \delta_{j,p} f_s \otimes g_l.$$

The Tomita operator is given by

$$SA_{j,s}\Omega = A^*_{j,s}\Omega = A_{s,j}\Omega = a_s f_j \otimes g_s = S(a_j f_s \otimes g_j),$$

and so

$$S(f_s \otimes g_j) = \frac{a_s}{a_j}(f_j \otimes g_s).$$

From this we have

$$\Delta(f_s \otimes g_j) = \left(\frac{a_s}{a_j}\right)^2 (f_s \otimes g_j),$$
$$J(f_s \otimes g_j) = (f_j \otimes g_s).$$

Then the spectrum of Δ is

$$\mathrm{Spec}\,(\Delta) = \mathrm{Spec}\,\left(\Delta^{-1}\right) = \bigcup_{s,j=1}^{n} \left(\frac{a_s}{a_j}\right)^2.$$

Example. Let $\mathcal{M}(G)$ be the group von Neumann algebra for a locally compact group G. It is a result that there is a σ-normal and σ-finite weight ϕ_e on $\mathcal{M}(G)$ such that $\phi_e(x^*x) < \infty$ if and only if there is a left bounded element $f \in L^2\,(G)\,dg$ with $\lambda(f) = x$, and in this case $\phi_e(x^*x) = \|f\|_2^2$. Furthermore the representation associated to ϕ_e is spatially equivalent to the regular representation. See Ref. [Ped79, p. 236]. Now the unitary group associated to ϕ_e is given by

$$(u_t\xi)(s) = \Delta^{it}\xi(s), \quad \xi \in L^2\,(G)\,dg, \quad t \in \mathbb{R},$$

where Δ is the modular function of the group G, which links left and right Haar measures.

Remark. The following is the fundamental result of Tomita-Takesaki theory. The most self-contained proof is probably in [BR87, p. 94], which is what we follow. One lemma is required. A slightly different formalism is used in [Ped79, p. 377]. Ref. [BW92, p. 387] gives a proof in the approximately finite dimensional (AF) case, and seems to follow [BR87] in exposition.

6.24. Lemma. *Let $\lambda \in \mathbb{C}$, $-\lambda \notin \mathbb{R}_+$. Let $B \in \mathcal{M}^c$. Then there exists an element $A_\lambda \in \mathcal{M}$ such that*

$$A_\lambda^* \Omega = (\Delta + \lambda 1)^{-1} B \Omega.$$

Furthermore we have

$$JBJ = \Delta^{-1/2} A_\lambda \Delta^{1/2} + \overline{\lambda} \Delta^{1/2} A_\lambda \Delta^{-1/2},$$

as a relation between bilinear forms on $\mathrm{Dom}(\Delta^{1/2}) \cap \mathrm{Dom}(\Delta^{-1/2})$.

Proof. See Ref. [BR87, p. 91-94]. □

6.25. Theorem (Tomita-Takesaki). *Let $\mathcal{M}$ be a von Neumann algebra with a cyclic and separating vector Ω. Let Δ,J be the associated modular operator and modular conjugation. Then*

$$J\mathcal{M}J = \mathcal{M}^c,$$
$$\Delta^{it} \mathcal{M} \Delta^{-it} = \mathcal{M}, \quad \forall\, t \in \mathbb{R}.$$

Proof. Given $\lambda > 0$ and $B \in \mathcal{B}(\mathcal{H})$, define a quadratic form

$$I_\lambda(B) = \lambda^{-1/2} \int_{-\infty}^{\infty} dt\, \frac{\lambda^{it}}{e^{\pi t} + e^{-\pi t}} \Delta^{it} B \Delta^{-it}.$$

If $\phi, \psi \in \mathrm{Dom}(\Delta^{1/2}) \cap \mathrm{Dom}(\Delta^{-1/2})$, define the function

$$\begin{aligned} f(\lambda) &= \left(\Delta^{-1/2}\psi, I_\lambda(B)\Delta^{1/2}\phi\right) + \lambda \left(\Delta^{1/2}\psi, I_\lambda(B)\Delta^{-1/2}\phi\right) \\ &= \int_{-\infty}^{\infty} dt\, \frac{\lambda^{it}}{e^{\pi t} + e^{-\pi t}} \left[\lambda^{-1/2} \left(\Delta^{-1/2-it}\psi, B\Delta^{1/2-it}\phi\right) + \lambda^{1/2} \left(\Delta^{1/2-it}\psi, B\Delta^{-1/2-it}\phi\right)\right]. \end{aligned}$$

Let $\Delta = \int dE_\Delta(\mu)\mu$ be a spectral decomposition for Δ. Then we have

$$\begin{aligned} f(\lambda) &= \int_{-\infty}^{\infty} dt\, \frac{\lambda^{it}}{e^{\pi t} + e^{-\pi t}} \int d^2(E_\Delta(\mu)\psi, BE_\Delta(\rho)\phi) \left\{ \left(\frac{\rho}{\mu\lambda}\right)^{1/2} + \left(\frac{\mu\lambda}{\rho}\right)^{1/2} \right\} \int_{-\infty}^{\infty} \frac{dt}{e^{\pi t} + e^{-\pi t}} \left(\frac{\mu\lambda}{\rho}\right)^{it} \\ &= \int d^2(E_\Delta(\mu)\psi, BE_\Delta(\rho)\phi) \\ &= (\psi, B\phi)\,. \end{aligned}$$

Therefore, as equality of bilinear forms on $\mathrm{Dom}(\Delta^{1/2}) \cap \mathrm{Dom}(\Delta^{-1/2})$ we have

$$B = \Delta^{-1/2} I_\lambda(B) \Delta^{1/2} + \lambda \Delta^{1/2} I_\lambda(B) \Delta^{-1/2}.$$

From the lemma we have the existence of $A_1 \in \mathcal{M}^c$ with

$$JA_1J = \Delta^{-1/2} A_\lambda \Delta^{1/2} + \overline{\lambda} \Delta^{1/2} A_\lambda \Delta^{-1/2}.$$

Then the above expression gives an inverse relation

$$A_\lambda = I_\lambda(JA_1J).$$

If $B' \in \mathcal{M}^c$ then, since $A_\lambda \in \mathcal{M}$,

$$(\psi, [B', I_\lambda(JA_1J)]\,\phi) = 0,$$
$$\int_{-\infty}^{\infty} dt\, \frac{e^{ipt}}{e^{\pi t} + e^{-\pi t}} \left(\psi, \left[B', \Delta^{it} JA_1J\Delta^{-1/2}\right] \phi\right) = 0, \quad \forall p \in \mathbb{R}.$$

Therefore

$$\Delta^{it} JA_1J\Delta^{-it} \in \mathcal{M}^{cc} = \mathcal{M}.$$

Setting $t = 0$ gives $J\mathcal{M}^cJ \subseteq \mathcal{M}$. The symmetries of the conjugation then give $J\mathcal{M}J \subseteq \mathcal{M}^c$. Using $J^2 = 1$ gives the first result. Finally, since $J\mathcal{M}^cJ \subseteq \mathcal{M}$, any $A \in \mathcal{M}$ has the form $A = JA_1J$ for some $A_1 \in \mathcal{M}^c$. Since $\Delta^{it}JA_1J\Delta^{-it} \in \mathcal{M}$, we then have $\Delta^{it}A\Delta^{-it} \in \mathcal{M}$, which proves the second claim. ☐

Remark. The formal calculation here is that the Fourier transform provides an inverse for the map $A \mapsto JAJ \in \mathcal{M}^c$. The proof justifies this statement.

6.26. Theorem (Exterior Equivalence). *Let ϕ and ψ be faithful normal states on a von Neumann algebra $\mathcal{M}$. Let $\sigma^\phi_{\mathbb{R}}$ and $\sigma^\psi_{\mathbb{R}}$ be the associated modular groups. Then there exists a strongly continuous one-parameter family of unitary operators u_t in $\mathcal{M}$ such that*

1. $\sigma^\psi_t(x) = u_t\sigma^\phi_t(x)u_t^*$, $x \in \mathcal{M}$, $t \in \mathbb{R}$.
2. $u_{t+s} = u_t\sigma^\phi_t(u_s)$, $s, t \in \mathbb{R}$.

Proof. Consider the von Neumann algebra $\mathcal{M} \otimes \mathrm{Mat}_2(\mathbb{C})$. Define a faithful normal state ρ by

$$\rho\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) = \frac{1}{2}\left(\phi(x_{11}) + \psi(x_{22})\right).$$

Let $\sigma^\rho_{\mathbb{R}}$ denote the modular group We have $\sigma^\rho_t(x \otimes e_{11}) = \alpha_t(x) \otimes e_{11}$ and $\sigma^\rho_t(y \otimes e_{22}) = \beta_t(y) \otimes e_{22}$ for $x, y \in \mathcal{M}$, where α_t and β_t satisfy the KMS condition for the states ϕ and ψ. By the KMS uniqueness result then $\alpha_t = \sigma^\phi_t$ and $\beta_t = \sigma^\psi_t$. Define

$$W_t \equiv \sigma^\rho_t \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_t & c_t \\ u_t & b_t \end{pmatrix} \in \mathcal{M} \otimes \mathrm{Mat}_2(\mathbb{C}).$$

$\sigma^\phi_t(1) = \sigma^\psi_t(1) = 1$, so

$$W_t^*W_t = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad W_tW_t^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore $a_t = b_t = c_t = 0$ and $u_t \in \mathcal{M}$ is unitary. Furthermore

$$\begin{pmatrix} 0 & 0 \\ u_{s+t} & 0 \end{pmatrix} = \sigma^\rho_{t+s} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sigma^\rho_s \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_t & 0 \\ 0 & 0 \end{pmatrix}\right].$$

Therefore $u_{t+s} = u_s\sigma^\rho_s(u_t)$. Furthermore

$$\begin{pmatrix} 0 & 0 \\ 0 & \sigma^\psi_t(x) \end{pmatrix} = \sigma^\rho_t \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right],$$

so $\sigma^\psi_t(x) = u_t\sigma^\phi_t u_t^*$. Therefore u_t is the desired unitary family. ☐

Definition. A unitary family u_t satisfying the second condition of the above theorem is called a unitary cocycle. Two automorphism groups connected by a unitary cocycle as above are called exterior equivalent.

6.4 Type III Factors

Remark. Let $\mathcal{M}$ be a σ-finite von Neumann algebra. Then $\mathcal{M}$ admits a faithful normal state ϕ, with associated modular group $\sigma^\phi_\mathbb{R}$. By the exterior equivalence theorem, any other modular group associated with a faithful normal state on $\mathcal{M}$ will produce an exterior-equivalent dynamical system. Therefore the Connes spectrum $\Gamma(\sigma^\phi_\mathbb{R})$ is independent of ϕ. So we can denote it by $\Gamma(\mathcal{M})$. It is an algebraic invariant for $\mathcal{M}$.

Remark. The following theorem shows that the Connes spectrum is not sensitive to factors other than type III.

6.27. Theorem. *If $\mathcal{M}$ is semifinite then* $\Gamma(\mathcal{M}) = \{0\}$.

Remark. By a previous result $\Gamma(\mathcal{M})$ is a closed subgroup of $\mathbb{R}$. There are three cases.

1. $\Gamma(\mathcal{M}) = \{0\}$.
2. $\Gamma(\mathcal{M}) = \{n \log \lambda \,:\, n \in \mathbb{Z}\}$, with $\lambda \in (0, 1)$.
3. $\Gamma(\mathcal{M}) = \mathbb{R}$.

As a matter of notation, call the first case $\lambda = 0$ and the last case $\lambda = 1$, since the subgroups of the second case increase in size as $\lambda \to 1$. Therefore we have assigned a real number $\lambda \in [0, 1]$ to every factor of type III. We say that $\mathcal{M}$ is of type III_λ. This is the classification of (σ-finite) type III factors.

Definition. Suppose ϕ is a normal state of a von Neumann algebra $\mathcal{M}$. Since ϕ is normal there is a smallest projection $p \in \mathcal{M}$ such that $\phi(p) = 1$. Then ϕ is faithful on $p\mathcal{M}p$. Denote the modular operator associated to $\phi|_{p\mathcal{M}p}$ by Δ_ϕ and the modular group by $\sigma^\phi_\mathbb{R}$. Then define

$$S(\mathcal{M}) = \bigcap \mathrm{Spec}\,(\Delta_\phi)\,,$$

where the intersection is over all normal states ϕ. The following theorem shows that, like the Connes spectrum, $S(\cdot)$ is not sensitive to factors other than type III.

6.28. Theorem. *Let $\mathcal{M}$ be a von Neumann algebra. If $\mathcal{M}$ is type III then $0 \in S(\mathcal{M})$. Otherwise $S(\mathcal{M}) = \{1\}$.*

Proof. If $0 \notin S(\mathcal{M})$ then there is some normal state ϕ with associated projection p such that $0 \notin \mathrm{Spec}\,(\Delta_\phi)$. Since $\Delta^{-1} = F\Delta J$, we have $\mathrm{Spec}\,(\Delta_\phi) = \mathrm{Spec}\left(\Delta_\phi^{-1}\right)$. Since $0 \notin \mathrm{Spec}\,(\Delta_\phi)$, Δ_ϕ must be bounded. Then $\sigma^\phi_\mathbb{R}$ is clearly uniformly continuous on $p\mathcal{M}p$; therefore $\sigma^\phi_\mathbb{R}$ is inner. So $p\mathcal{M}p$ is semifinite. Since $p \neq 0$, $\mathcal{M}$ is not type III.

Conversely suppose $\mathcal{M}$ is not type III. Then it contains a nonzero finite projection p, and without loss p is the support of some normal tracial state ϕ. Then $\Delta_\phi = 1$, and so $S(\mathcal{M}) = \{1\}$. ☐

Remark. The following lemma and theorem show that there is an explicit relation between the Connes spectrum and $S(\cdot)$, again for the σ-finite case.

6.29. Lemma. *Let ξ_0 be a cyclic and separating vector for a von Neumann algebra $\mathcal{M}$. Let Δ be the modular operator and $\sigma_{\mathbb{R}}$ be the modular group associated with ξ_0. Then for any $s \in \mathbb{R}$ we have $s \in \operatorname{Spec}(\sigma)$ if and only if $e^s \in \operatorname{Spec}(\Delta)$.*

Proof. Let $f \in L^1(\mathbb{R})\, dx$ and $x \in \mathcal{M}$. Then

$$\begin{aligned} \widehat{f}(\log \Delta) x\xi_0 &= \int dt\, f(t) \exp(it \log(\Delta)) x\xi_0 \\ &= \int dt\, f(t) \Delta^{it} x\xi_0 \\ &= \int dt\, f(t) \sigma_t(x) \xi_0 \\ &= \sigma_f(x)\xi_0. \end{aligned}$$

Therefore $\sigma_f(x) = 0$ for all $x \in \mathcal{M}$ if and only if $\widehat{f}(\log \Delta) = 0$. But $s \in \operatorname{Spec}(\sigma)$ if and only if $\sigma_f \neq 0$ for all $f \in L^1(\mathbb{R})\, dx$ satisfying $\widehat{f}(\log(e^s)) = \widehat{f}(s) \neq 0$. ☐

6.30. Theorem. *Let $\mathcal{M}$ be a σ-finite von Neumann algebra and let $s \in \mathbb{R}$. Then $s \in \Gamma(\mathcal{M})$ if and only if $e^s \in S(\mathcal{M})$.*

Proof. Let ϕ be a normal state of $\mathcal{M}$ with support projection p, modular group $\sigma^\phi_{\mathbb{R}}$, and modular operator Δ_ϕ. Choose ψ a faithful normal state on $(1-p)\mathcal{M}(1-p)$ and consider the faithful state $\rho = \frac{1}{2}(\phi + \psi)$ on $\mathcal{M}$. From the uniqueness of the modular group on $p\mathcal{M}p$ we have

$$\sigma^\rho_t(x) = \sigma^\phi_t(x), \quad x \in p\mathcal{M}p.$$

p is fixed by $\sigma^\rho_{\mathbb{R}}$, so

$$\Gamma(\mathcal{M}) = \Gamma(\sigma^\rho) \subset \Gamma(\sigma^\rho|_{p\mathcal{M}p}) = \Gamma(\sigma^\phi).$$

Suppose $s \in \Gamma(\mathcal{M})$. Then by the lemma $e^s \in \operatorname{Spec}(\Delta_\phi)$. But ϕ was arbitrary so $e^s \in S(\mathcal{M})$.

Conversely, let ϕ be a faithful normal state on $\mathcal{M}$ with modular group $\sigma^\phi_{\mathbb{R}}$. For each nonzero projection p which is fixed by $\sigma^\phi_{\mathbb{R}}$ we have that $\sigma^\phi|_{p\mathcal{M}p}$ is the modular group associated with the faithful normal state $\phi(\cdot)/\phi(p)$ on $p\mathcal{M}p$. So if $s \in S(\mathcal{M})$ then it follows from the lemma that $s \in \operatorname{Spec}\left(\sigma^\phi|_{p\mathcal{M}p}\right)$. Since p was arbitrary, $s \in \Gamma(\sigma^\phi) = \Gamma(\mathcal{M})$. ☐

6.31. Theorem (Type III Glimm Algebras). *Let $\mathcal{F}$ be the fermion algebra. Let ϕ_λ be a permutation invariant product state on the fermion algebra, as in the theorem on the existence of type III Glimm algebras, $\lambda \in (0, 1/2)$. Then the factor $\mathcal{M}_\lambda$ arising from ϕ_λ is of type $III_{\lambda'}$, where $\lambda' = \lambda(1-\lambda)^{-1}$.*

Proof. See Ref. [Ped79, p. 392]. ☐

6.5 Hyperfiniteness Again

6.32. Theorem (Murray-von Neumann). *Up to algebraic isomorphism, there exists a unique hyperfinite II_1 factor.*

6.33. Theorem (Connes). *Let G be amenable. Then $\mathcal{M}(G)$ is hyperfinite.*

Chapter 7

Extensions

Definition. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{E}$ be C^* algebra s. Then a short exact sequence

$$0 \to \mathcal{B} \to \mathcal{E} \to \mathcal{A} \to 0$$

is called an extension of $\mathcal{A}$ by $\mathcal{B}$. As such, $\mathcal{B}$ is isomorphic to an ideal in $\mathcal{E}$, and $\mathcal{E}/\mathcal{B} \cong \mathcal{A}$.

Remark. The natural goal at this point is to introduce a notion of equivalence for extensions and then classify extensions of $\mathcal{A}$ by $\mathcal{B}$.

Example. Let $\mathcal{A} = \mathbb{C}$ and $\mathcal{B} = C_0(0,1)$. Then there are four extensions of $\mathcal{A}$ by $\mathcal{B}$;

$$\begin{aligned}
\mathcal{E}_1 &= \mathbb{C} \oplus C_0(0,1) \\
\mathcal{E}_2 &= C_0(0,1] \\
\mathcal{E}_3 &= C_0[0,1) \\
\mathcal{E}_4 &= C(S^1)
\end{aligned}$$

Definition. Define Busby invariant ...

Definition. Define $\mathrm{Ext}(\mathcal{A},\mathcal{B})$ ...

Chapter 8

K-Theory

8.1 Introduction

This chapter at best provides a few hints as to the nature of the subject. For treatments of the topics here, see Refs. [Bro96, Lod92, WO93, Bla86, Cun].

8.2 Commutative K-Theory

Remark. Recall that a commutative Banach algebra is always $C(X)$ for some compact Hausdorff space X, see theorem 1.8. This result for functions can be fruitfully generalized to include sections of vector bundles. Commutative K-theory is also called topological K-theory because of this connection. We will shortly see what topology it describes.

Definition. The Whitney sum of two topological vector bundles $p : E \longrightarrow X$, $q : F \longrightarrow X$ is the vector bundle

$$E \oplus F \cong \{(e, f) \in E \times F \,:\, p(e) = q(f)\}\,.$$

The set of isomorphism classes of complex vector bundles over a space X is a commutative semigroup with identity, with the operation given by Whitney sum. Denote this semigroup by $V_{\mathbb{C}}(X)$. Similarly define $V_{\mathbb{R}}(X)$.

8.1. Theorem (Swan). *Let E be a vector bundle over a compact Hausdorff space X. Then there is a bundle F over X such that $E \oplus F$ is trivial.*

Proof. See [Hus74]. □

Definition. A free module of rank n is a module which has a basis, and for which any two such basis sets have the same cardinality, n. When the module is actually a vector space over a division ring, the rank is usually called the dimension. See [Hun74].

Definition. A direct summand of a free module is called a projective module. Specifically, the module P is projective if and only if there exists a free module F and a module K such that $F \cong K \oplus P$. See [Hun74].

Remark. Every free module over a ring with identity is projective. See [Hun74].

Example. $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are both modules over $\mathbb{Z}_6$. However, neither of these is free, since neither posesses a nonempty basis. To see this explicitly, note that the equation $r + 1 = 0$ can be satisfied for nonzero $r \in \mathbb{Z}_6$, so that one cannot find any appropriate linearly independent sets with which to build a basis. However, both $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are projective modules over $\mathbb{Z}_6$ because $\mathbb{Z}_6$ is free as module over itself and we have the module isomorphism $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6$. From [Hun74].

Remark. Let E be a vector bundle over a compact space X. Let $\Gamma(E)$ be the set of sections of E. Then $\Gamma(E)$ is a module over the ring $C(X)$. If E is trivial of dimension n, then $\Gamma(E)$ is a free module of rank n. By Swan's theorem $\Gamma(E)$ is projective for any E. By compactness of X and finite-dimensionality of the fibers of E, $\Gamma(E)$ is finitely generated.

8.2. Theorem. *M is a finitely generated projective module over a commutative Banach algebra $\mathcal{A}$ with unit, if and only if $M = \Gamma(E)$ for some vector bundle E on a compact space X.*

Proof. One direction follows from the above easy remark. For the converse, assume M is as stated. Since M is finitely generated, we know there is a module W such that $M \oplus W = \mathcal{A}^n$, for some n. The projection onto M in $\mathcal{A}^n$ is an element of the matrix algebra $M_n(\mathcal{A})$ and it is an idempotent. In this way, each finitely-generated projective module is associated with an idempotent in $M_n(\mathcal{A})$, unique up to similarity. Since $\mathcal{A}$ is a commutative Banach algebra with unit, it is $C(X)$ for some compact Hausdorff space X. We identify $M_n(\mathcal{A})$ with $C(X, M_n)$. Let p be the idempotent associated to M above, so $p \in C(X, M_n)$. Define $E = \{(x, \nu) \in X \times \mathbb{C}^n : \nu \in \mathrm{Ran}(p_x)\}$. $\square$

Remark. Roughly speaking we can imagine a big "infinite-dimensional" bundle over X from which all other bundles are obtained by cutting out sub-bundles using a continuous projection valued function on X, which projects onto a finite-dimensional subspace at each fiber. Think of a matrix with a finite number of non-vanishing entries at each point of X.

Remark. The above characterization shows that $V_{\mathbb{C}}(X)$ or $V_{\mathbb{R}}(X)$ are equivalent to the isomorphism classes of finitely-generated projective modules over $C_{\mathbb{C}}(X)$ or $C_{\mathbb{R}}(X)$, or equivalently to the equivalence classes of idempotents in $M_\infty(C(X))$. Denote the latter by $V(C(X))$.

Definition. Let H be an abelian semigroup with identity. Define the Grothendieck group of H, $\mathrm{Groth}(H)$, to be the quotient of $H \times H$ by the equivalence relation $(x_1, y_1) \sim (x_2, y_2)$ iff there is a z with $x_1 + y_2 + z = x_2 + y_1 + z$.

Remark. As the prototype example, $\mathrm{Groth}(\mathbb{N}) = \mathbb{Z}$.

Definition. Let X be a compact Hausdorff space. Define $K^0_{\mathbb{C}}(X) = \mathrm{Groth}(V_{\mathbb{C}}(X))$ and $K^0_{\mathbb{R}}(X) = \mathrm{Groth}(V_{\mathbb{R}}(X))$. $K^0_{\mathbb{C}}$ and $K^0_{\mathbb{R}}$ are contravariant functors from compact Hausdorff spaces to abelian groups.

Example. Let $X = [0, 1]$ or $X = \{*\}$. Then $K^0_{\mathbb{R}}(X) \cong K^0_{\mathbb{C}}(X) \cong \mathbb{Z}$.

Example. $K^0_{\mathbb{R}}(S^1) \cong \mathbb{Z} \times \mathbb{Z}_2$.

Example. $K^0_{\mathbb{C}}(S^1) \cong \mathbb{Z} \times \mathbb{Z}$.

Remark. The definition of the K^0 groups for non-compact spaces is more involved. This is because the definition must behave well for relative spaces, which are required for the development of the long exact sequence.

Definition. Let X be a locally compact Hausdorff space. Let Y be a closed subspace of X. Let E and F be vector bundles over X which are isomorphic when restricted to Y, by an isomorphism α. Identify two such triples (E, F, α) and (E', F', α') if $E \cong E'$, $F \cong F'$, by isomorphisms which intertwine α and α' when restricted to Y. The set of such triples so identifiedforms an abelian semigroup, $V(X, Y)$. Define the relative K group $K^0(X, Y) = \mathrm{Groth}(V(X, Y))$.

Definition. Let X be a locally compact Hausdorff space. Define the K group

$$K^0(X) \equiv K^0(X^+, +),$$

where X^+ is the one-point compactification of X.

8.3. Theorem. *Let X be a locally compact Hausdorff space and Y be a closed subspace of X; let $U = X \backslash Y$. Let $q : X^+ \longrightarrow U^+$ be the identity on U and send $X^+ \backslash U$ to the point at infinity. Then $K^0(X, Y) \equiv K^0(U)$, and the sequence*

$$K^0(U) \to K^0(X) \to K^0(Y)$$

is exact in the middle, $\mathrm{Im}\,(q^*) = \ker\,(i^*)$.

Remark. It is not generically true that q^* is injective or that i^* is surjective.

Definition. $K^{-n}(X) \equiv K^0(X \times \mathbb{R}^n)$ for $n > 0$.

8.4. Theorem. *Let X, Y, U be as above. Then the following long sequence is exact.*

$$\cdots \to K^{-n}(U) \to K^{-n}(X) \to K^{-n}(Y) \to K^{-n+1}(U) \to \cdots \to K^0(Y).$$

Proof. □

8.5. Theorem (Chern Character). *Let X be a compact Hausdorff space. Let $\mathrm{H}^n\,(X; \mathbb{Q})$ be the n-th Alexander or Cech cohomology group of X. Then*

$$K^0_{\mathbb{C}} \otimes \mathbb{Q} \cong \oplus_{n \text{ even}} \mathrm{H}^n\,(X; \mathbb{Q})$$
$$K^{-1}_{\mathbb{C}} \otimes \mathbb{Q} \cong \oplus_{n \text{ odd}} \mathrm{H}^n\,(X; \mathbb{Q})$$

Proof. □

Remark. The above definitions provide contravariant functors K^{-n}. By reformulating in terms of modules over $C(X)$, we can obtain covariant functors from commutative Banach algebras to abelian groups. It is these functors which will be extended to the noncommutative case.

Definition. Let $\mathcal{A}$ be a commutative Banach algebra with unit. Let $V(\mathcal{A})$ be the isomorphism classes of finitely-generated projective $\mathcal{A}$ modules. Define $K_0(\mathcal{A}) \equiv \mathrm{Groth}(V(\mathcal{A}))$.

Remark. Note that $K_0(C(X)) \cong K^0_{\mathbb{C}}(X)$, and $K_0(C_{\mathbb{R}}(X)) \cong K^0_{\mathbb{R}}(X)$.

Definition. If $\mathcal{A}$ does not have a unit, define $K_0(\mathcal{A}) \equiv \ker\,(h) \subset K_0(\mathcal{A}^+)$, where h is the group homomorphism $h : K_0(\mathcal{A}^+) \longrightarrow K_0(\mathbb{Z}) \cong \mathbb{Z}$, and $\mathcal{A}^+$ is $\mathcal{A}$ with unit attached.

Remark. Attaching a unit is the analog of the one-point compactification process in the previously described classical theory.

Definition. Let $\mathcal{A}$ be a commutative Banach algebra. Define the suspension of $\mathcal{A}$ to be the algebra

$$\mathrm{S}\mathcal{A} \equiv \left\{ f : \mathbb{R} \longrightarrow \mathcal{A} \,:\, \lim_{x\to\infty} \|f(x)\| = 0 \right\}.$$

Note that $\mathrm{S}\mathcal{A}$ is also a commutative Banach algebra.

Definition. Let $\mathcal{A}$ be a commutative Banach algebra. Define $K_n(\mathcal{A}) \equiv K_0(\mathrm{S}^n\mathcal{A})$.

8.3 K_0-Theory

Definition. Let $\mathcal{A}$ be a Banach algebra. Then $\mathrm{Proj}(\mathcal{A})$ is the set of algebraic equivalence classes of idempotents in $\mathcal{A}$. Algebraic equivalence is defined by $p \sim q \Longleftrightarrow \exists x, y \in \mathcal{A}$ with $xy = p, yx = q$.

Remark. If $\mathcal{A}$ is a C^* algebra then $\mathrm{Proj}(\mathcal{A})$ can be defined as the set of equivalence classes of projections with equivalence defined by similarity.

Definition. Let $V(\mathcal{A})$ denote $\mathrm{Proj}(\mathrm{Mat}_\infty(\mathcal{A}))$. Then $V(\cdot)$ is a covariant functor from Banach algebras to commutative semigroups with identity.

Remark. One can equivalently define $V(\mathcal{A})$ as the set of isomorphism classes of finitely-generated projective $\mathcal{A}$-modules, as in the commutative case.

Example. $V(\mathbb{C}) \cong V(\mathrm{Mat}_n(\mathbb{C})) \cong V(\mathcal{K}(\mathcal{H})) \cong \mathbb{N} \cup \{0\}$.

Example. $V(\mathcal{B}(\mathcal{H})) \cong \mathbb{N} \cup \{0\} \cup \{\infty\}$.

Example. Let $\mathcal{M}$ be a type II_1. Then $V(\mathcal{M}) \cong (0, \infty]$.

Example. Let $\mathcal{M}$ be a type II_∞. Then $V(\mathcal{M}) \cong [0, \infty]$.

Example. Let X be a compact Hausdorff space. Then $V(C(X)) \cong V_{\mathbb{C}}(X)$.

Example. Let X be a connected, locally compact, noncompact Hausdorff space. Then $V(C_0(X)) \cong \{0\}$.

Example. Fix $x, y \in \mathbb{C}$. Let $\mathcal{A} = \{f : [0,1] \longrightarrow \mathrm{Mat}_2(\mathbb{C}) \,:\, f(0) = \mathrm{diag}(x, 0), f(1) = \mathrm{diag}(y, y)\}$. Then $\mathcal{A}^+ \cong \{f : [0,1] \longrightarrow \mathrm{Mat}_2(\mathbb{C}) \,:\, f(0) = \mathrm{diag}(x, z), f(1) = \mathrm{diag}(y, y), z \in \mathbb{C}\}$. $\mathcal{A}$ contains no nonzero projections, but $\mathrm{Mat}_2(\mathcal{A})$ contains nontrivial projections. We have $V(\mathcal{A}) \cong \mathbb{N} \cup \{0\}$, and $V(\mathcal{A}^+) \cong \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \,:\, m, n \geq 0, m + n \text{even}\}$. This shows why we must consider the matrix algebras of $\mathcal{A}$ as well as $\mathcal{A}$ itself.

Remark. Note that any of the example semigroups which contains $\{\infty\}$ does not have cancellation. This can happen generically. Semigroups without cancellation are apparently somewhat difficult to handle.

Definition. Let $\pi : \mathcal{A}^+ \longrightarrow \mathcal{A}^+ \backslash \mathcal{A}$ be the canonical projection for the unitarization of $\mathcal{A}$; π is trivial if $\mathcal{A}$ has a unit. We define $K_0(\mathcal{A}) = \ker(\pi)_* \subseteq \mathrm{Groth}(V(\mathcal{A}^+))$.

Remark. As in the commutative casewe do not define $K_0(\mathcal{A})$ directly as the given Grothendieck group, but as a subgroup. This is because $V(\mathcal{A}^+)$ is somehow too big; it does not include appropriate "constraints at infinity". And equally perplexingly, $V(\mathcal{A})$ is somehow too small.

Example. $C_0(\mathbb{R}^2)$ does not have a unit.

- $\mathrm{Groth}(V(C_0(\mathbb{R}^2)^+)) = \mathbb{Z}^2$.
- $\mathrm{Groth}(V(C_0(\mathbb{R}^2))) = \{0\}$.
- $K_0(C_0(\mathbb{R}^2)) \cong \mathbb{Z}$.

Definition. An ordered group is a pair (G, G_+) where G is an Abelian group and G_+ is a distinguished sub-semigroup containing the identity and having the properties

- $G_+ + (-G_+) = G$
- $G_+ \cap (-G_+) = \{0\}$

G_+ is called the positive cone of G.

Remark. A positove cone as defined above provides a partial order on G by $y \leq x \Longleftrightarrow x - y \in G_+$.

Definition. A scaled ordered group is an ordered group (G, G_+) together with a distinguished element $u \in G_+$ with the property

$$\forall x \in G, \exists n > 0 \quad \text{with} \quad x \leq nu.$$

u is called an order unit.

Definition. An ordered gruop (G, G_+) is called unperforated if $nx \geq 0$ for some $n > 0$ implies $x \geq 0$.

Remark. An unperforated ordered group is torsion free.

Definition. A Banach algebra $\mathcal{A}$ is called finite if for idempotents p, q with $p \leq q$ and $p \sim q$, we have $p = q$.

Remark. If $\mathcal{A}$ has a unit, finiteness as defined above is equivalent to the property that no proper idempotent has $p \sim 1$.

Remark. A C^* algebra with unit is finite if and only if every isometry is unitary.

Definition. A Banach algebra $\mathcal{A}$ is called stably finite if $\mathrm{Mat}_n(\mathcal{A})$ is finite for all n.

8.6. Theorem. *Let $\mathcal{A}$ be a stably finite Banach algebra with unit. Then $K_0(\mathcal{A})$ is an ordered group with positive cone $K_0(\mathcal{A})_+ = \mathrm{Im}(V(\mathcal{A})) \subset K_0(\mathcal{A})$.*

Proof. See [Bla86]. □

8.4 K_1-Theory

Definition. Let $\mathcal{A}$ be a Banach algebra. Let $GL_\infty(\mathcal{A})_0$ be the identity component in $GL_\infty(\mathcal{A})$. Define $K_1(\mathcal{A}) \equiv GL_\infty(\mathcal{A})/GL_\infty(\mathcal{A})_0$.

8.7. Lemma. *Let $\mathcal{A}$ be a C^* algebra. Then $K_1(\mathcal{A}) \cong U_\infty(\mathcal{A})/U_\infty(\mathcal{A})_0$.*

Proof. By polar decomposition, $U_n(\mathcal{A})$ is a deformation retract of $GL_\infty(\mathcal{A})$. So $U_\infty(\mathcal{A})/U_\infty(\mathcal{A})_0 \cong GL_\infty(\mathcal{A})/GL_\infty(\mathcal{A})_0$. □

Example. $K_1(\mathbb{C}) = \{0\}$.

Example. Let $\mathcal{M}$ be a von Neumann algebra. Then by spectral theory, $U_\infty(\mathcal{M})$ is connected. Therefore $K_1(()\mathcal{M}) \cong \{0\}$.

Example. Let $\mathcal{A} = C(S^1)$. Then $U_1(\mathcal{A})/U_1(\mathcal{A})_0 \cong \mathbb{Z}$, which is the winding number around the circle. Furthermore, $K_1(\mathcal{A}) \cong \mathbb{Z}$.

Definition. Let $\mathcal{A}$ be a Banach algebra. Define the suspension of $\mathcal{A}$,

$$\mathrm{S}\mathcal{A} \equiv \left\{ f : \mathbb{R} \longrightarrow \mathcal{A} \; : \; \lim_{x\to\infty} \|f(x)\| = 0 \right\}.$$

$\mathrm{S}\mathcal{A}$ is a Banach algebra with pointwise multiplication and the sup norm.

8.8. Lemma. *Let $\mathcal{A}$ be a C^* algebra. Then $\mathrm{S}\mathcal{A}$ is a C^* algebra, and $\mathrm{S}\mathcal{A} \cong C_0(\mathbb{R}) \otimes \mathcal{A}$.*

Proof. ? □

8.9. Theorem. *Let $\mathcal{A}$ be a Banach algebra. Then $K_1(\mathcal{A}) \cong K_0(\mathrm{S}\mathcal{A})$.*

Proof. This is dificult. See [Bla86, p. 68]. □

8.10. Theorem. *The sequence*

$$K_1(\mathcal{J}) \to K_1(\mathcal{A}) \to K_1(\mathcal{A}/\mathcal{J}) \to^{\partial} K_0(\mathcal{J}) \to K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{J})$$

is exact.

Proof. See [Bla86, p. 70]. □

Remark. The map ∂ is called the index map. It is related to the Fredholm index of unitary elements of the Calkin algebra. See [Bla86, p. 71].

8.11. Theorem. $K_1(\mathcal{A}) \cong K_1(\mathcal{A}^+)$.

Proof. See [Bla86, p. 72]. □

8.12. Theorem (Bott Periodicity). *Let $\mathcal{A}$ be a Banach algebra. Then $K_1(\mathrm{S}\mathcal{A}) \cong K_0(\mathcal{A})$.*

Proof. See [Bla86, p. 72]. □

8.5 AF Algebras

Definition. Let $\mathcal{A}$ be an inductive limit of a sequence of finite-dimensional C^* algebras. Then $\mathcal{A}$ is a C^* algebra. Any such C^* algebra is called an AF algebra. AF stands for approximately finite.

Example. The algebra of compact operators on a Hilbert space is an AF algebra.

Example. The canonical anti-commutation algebra (CAR) is an AF algebra. Recall the discussion of the fermion algebra, which was a special case of Glimm algebra.

Remark. When $\mathcal{A}$ is an AF algebra, we know that $K_0(\mathcal{A})$ is some inductive limit $\lim_{\rightarrow} \mathbb{Z}^{r_s}$, although we do not a-priori know the inclusions, so we do not know which inductive limit it is. Knowledge of $K_0(\mathcal{A})$ is made critical by the following characterization of AF algebras. For AF algebras $\mathcal{A}$, $K_0(\mathcal{A})$ is called the dimension group.

8.13. Theorem. *Let $\mathcal{A}$, $\mathcal{B}$ be AF algebras. Suppose $K_0(\mathcal{A}) \cong K_0(\mathcal{B})$ as an isomorphism of scaled ordered groups. Then $\mathcal{A} \cong \mathcal{B}$.*

Proof. See [Bla86, p. 55]. □

Definition. Let G be an ordered group. G is said to have the Riesz interpolation property if given $x_1, x_2, y_1, y_2 \in G$, with $x_1, x_2 \leq y_1, y_2$ there is a $z \in G$ with $x_1, x_2 \leq z \leq y_1, y_2$.

8.14. Theorem. *An ordered group is a dimension group if and only if it is countable, unperforated, and it has the Riesz interpolation property.*

Proof. See [Eff81]. □

Example. Let G be a countable dense subgroup of $\mathbb{R}$. Then G is a dimension group.

Example. The Glimm algebras correspond to the dense subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$. These groups are classified by generalized integers $q = 2^{m_2}3^{m_3}5^{m_5}\cdots p_i^{m_i}\cdots$, which entered the Glimm algebra construction directly. The subgroups of $\mathbb{Q}$ corresponding to the generalized integer q is the group of rationals that divide q, denoted $\mathbb{Z}_{(q)}$. The group $\mathbb{Z}_{(2^\infty)}$ corresponds to the CAR algebra.

Example. Let $\mathcal{A}_t = C(S^1) \times_{\alpha_t} \mathbb{Z}$ be the crossed product where α_t is the shift on the circle by $t \in (0,1)$, and $t \notin \mathbb{Q}$. Then $K_0(\mathcal{A}_t) \cong \mathbb{Z} \times \mathbb{Z}$, and $K_1(\mathcal{A}_t) \cong \mathbb{Z} \times \mathbb{Z}$. In fact, we can write $K_0(\mathcal{A}_t) \cong \mathbb{Z} + t\mathbb{Z}$. If u, v are any two unitary elements of a C^* algebra satisfying $uv = e^{2\pi i t} vu$, then u, v generate a C^* algebra isomorphic to $\mathcal{A}_t$. $\mathcal{A}_t$ is sometimes called the (one-dimensional) noncommutative torus. Note that physically this is the algebra of magnetic translation operators for electrons moving in a constant magnetic field.

8.6 Equivariant K-Theory

Remark. Let $(\mathcal{A}, G, \alpha)$ be a C^* dynamical system. Recall the definition of the crossed product $G \times_\alpha \mathcal{A}$. The crossed product provides, roughly speaking, a way to embed $\mathcal{A}$ into a larger C^* algebra in which the automorphisms of $\mathcal{A}$ become inner.

8.15. Theorem (Connes Thom Isomorphism). *Let $\alpha : \mathbb{R} \longrightarrow \mathrm{Aut}\,(\mathcal{A})$ be a continuous homomorphism, with $\mathcal{A}$ a C^* algebra. Then*

$$K_i(\mathcal{A} \times_\alpha \mathbb{R}) \cong K_{1-i}(\mathcal{A}), \quad i = 0, 1.$$

Remark. This is a generalization of Bott periodicity, which is obtained with the case of the trivial action. The surprising point is that the result is independent of the action.

Definition. Let $(\mathcal{A}, G, \alpha)$ be a C^* dynamical system with G compact and $\mathcal{A}$ unital. Let $\mathcal{E}$ be a finitely-generated projective $\mathcal{A}$-module and let λ be a strongly continuous homomorphism from G to the invertible elements in $\mathcal{B}(\mathcal{E})$ satisfying $\lambda_g(ea) = \lambda_g(e)\alpha_g(a)$. Then $\mathcal{E}, \lambda$ is called a finitely-generated projective $(\mathcal{A}, G, \alpha)$-module.

Definition. Let $V^G(\mathcal{A})$ be the set of equivalence classes of finitely-generated projective $(\mathcal{A}, G, \alpha)$-modules. Note that $V^G(\mathcal{A})$ is a commutative semigroup under Whitney sum.

Definition. Let $(\mathcal{A}, G, \alpha)$ be a C^* dynamical system, with $\mathcal{A}$ not necessarily unital. Let $\pi : \mathcal{A}^+ \longrightarrow \mathbb{C}$ be the canonical projection of the unitization of $\mathcal{A}$. so we have $\pi_* : \mathrm{Groth}(V^G(\mathcal{A}^+)) \longrightarrow \mathrm{Groth}(V^G(\mathbb{C}))$, where the action of G on $\mathbb{C}$ is trivial. Define the abelian group

$$K_0^G(\mathcal{A}, \alpha) \equiv \ker(\pi_*) \subseteq \mathrm{Groth}(V^G(\mathcal{A}^+)).$$

Remark. If $\mathcal{A}$ is unital, then we have $K_0^G(\mathcal{A}, \alpha) \cong \mathrm{Groth}(V^G(\mathcal{A}))$

Remark. $K_0^G(\mathbb{C}) \cong \mathrm{Rep}(G)$, the representation ring of G.

8.16. Theorem. *Let $(\mathcal{A}, G, \alpha)$ be a C^* dynamical system with G compact. Then*

$$K_0^G(\mathcal{A}, \alpha) \cong K_0(\mathcal{A} \times_\alpha G)$$

Proof. See [Bla86]. □

Definition. $K_1^G(\mathcal{A}, \alpha) \equiv K_0^G(\mathrm{S}^{\mathcal{A}})$, where we recall that $\mathrm{S}^{\mathcal{A}} \cong C_0(\mathbb{R}) \otimes \mathcal{A}$, and the action of G on $C_0(\mathbb{R})$ is taken to be trivial.

8.17. Theorem (Equivariant Bott Periodicity). *Let $(\mathcal{A}, G, \alpha)$ be a C^* dynamical system, with G compact. Then there exists a natural* $\mathrm{Rep}(G)$*-module isomorphism*

$$K_0^G(\mathcal{A}) \cong K_1^G(\mathrm{S}^{\mathcal{A}}).$$

8.7 Index Theorems

Remark. The Atiyah-Singer index theorem can be formulated in terms of K-theory as follows. Let D be an elliptic pseudo-differential operator on a compact manifold M. The analytic index of D is Ind(D) = $\dim\ker(D)$ - $\dim\mathrm{coker}(D)$. Recall that the topological index of D is defined in terms of the symbol σ_D as

$$\mathrm{Ind}^t(D) = \langle \tau_!(\mathrm{ch}(\sigma_D)) \cup \mathrm{Td}(T^*M \otimes_{\mathbb{R}} \mathbb{C}), [M] \rangle,$$

where $\mathrm{Td}(T^*M \otimes_{\mathbb{R}} \mathbb{C}) \in \mathrm{H}^*(M; \mathbb{Q})$ is the Todd class of the complexified cotangent bundle M, and $\tau_! : \mathrm{H}^*(T^*M; \mathbb{Q}) \longrightarrow \mathrm{H}^*(T^*M; \mathbb{Q})$ is the inverse of the Thom isomorphism. As a map on symbols, Ind^t defines a homomorphism from $K^0(T^*M)$ to $\mathbb{Z}$, since the symbol defines an element of $K^0(T^*M)$. The analytic index, on the other hand, is a composition of the maps $d : K^0(T^*M) \longrightarrow K^0(M)$, sending the equivalence class of the symbol to the equivalence class of the operator, and $p_* : K^0(M) \longrightarrow \mathbb{Z}$ induced by $p : M \longrightarrow *$.

Remark. Roughly speaking, the family index is the following. Let $\{D_y \,:\, y \in Y\}$ be a continuous family of elliptic pseudo-differential operators on a compact manifold. Assume the $\{D_y\}$ are invertible except possibly for a compact subset of Y. Now $\ker(D_y)$ and $\operatorname{coker}(D_y)$ are vector spaces of finite-dimension for each y, and by the appropriate notion of continuity they determine two vector bundles over Y. The analytic index of the family is the difference of the equivalence classes of these bundles in $K_0(Y)$.

Chapter 9

Nuclear C^* Algebras

Definition. A C^* algebra $\mathcal{A}$ is called nuclear if $\pi(\mathcal{A})^{cc}$ is hyperfinite for any representation π of $\mathcal{A}$.

9.1. Theorem. *T.F.A.E.*

1. *$\mathcal{A}$ is a nuclear C^* algebra.*
2. *$id : \mathcal{A} \longrightarrow \mathcal{A}$ can be approximated pointwise in norm by completely positive finite-rank contractions.*
3. *$\mathcal{A}^{cc}$ is hyperfinite.*

Proof. $\square$

9.2. Theorem. *If $\mathcal{A}$ is a type I C^* algebra, then $\mathcal{A}$ is nuclear.*

9.3. Theorem. *Inductive limits of type I C^* algebras are nuclear. In particular, AF algebras are nuclear.*

9.4. Theorem. *Let $\mathcal{A}$ be a separable nuclear C^* algebra and $\mathcal{B}$ be any C^* algebra. Then $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ is a group.*

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