Chaotic Processes
In Astronomical Data

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Abstract: This paper reviews the techniques that have been developed to detect and model chaotic processes in time series data, summarizes applications of these methods in astronomy, and makes some general recommendations for an approach to chaotic time series analysis.

Contents:

I. Introduction: The Nature of Chaos

Chaos is a new paradigm for the evolution of physical systems: deterministic yet disordered behavior. Chaos is a paradox. A chaotic physical system is deterministic but evolves in a way that seems unpredictable and random. The paradox is that determinism normally implies predictability.

How can a perfectly deterministic physical system be unpredictable? The answer lies in a kind of instability, called sensitivity to initial conditions (SIC). That is, a chaotic system has SIC in the sense that the distance between the evolutionary paths starting from two nearby but distinct initial conditions grows exponentially in time. This leads to an amplification of errors or uncertainties in the initial state which forestalls prediction of the future. This instability cannot arise in a linear system; chaotic phenomena are possible only in nonlinear systems.

I.1 An Example of Chaos: the Logistic Map

Let's consider an example of a chaotic process. Perhaps the best studied chaotic system is the *logistic map*:

$$X_{n+1} = \lambda X_n (1 - X_n), \tag{1}$$

so called because this relation maps the interval $[0, 1]$ onto itself.

This map is a prescription for generating a time series, X_n. That is, it defines a stochastic process. The behavior of the series depends in interesting ways on both the initial value and the value of the parameter λ. For $\lambda = 4$ and an initial value X_0 chosen randomly from $[0, 1]$, the realization of the process is a time series which looks quite random. However, for some initial values the time series is far from random, namely periodic. The simplest such case is $X_0 = \frac{3}{4}$, for which the X_n is periodic as a function of n, with period one. But the case $X_0 = \frac{3}{4} + \epsilon$, where ϵ is an arbitrarily small, positive, irrational number, generates a chaotic, random looking data stream. This process is uncorrelated.

The enormous literature on the logistic map (e.g. Schuster 1988, May 1976, Grebogi, Ott and Yorke 1982) centers on the extraordinary dependence of the behavior of the time series on the parameter λ. The only feature of this dependence that we have space for here is the phenomenon of *period doubling.* As mentioned above, there are values of λ for which X_n is periodic. If λ is gradually increased above such a value, the period remains fixed until suddenly it increases (at a discrete value of λ) by a factor of two – hence *period doubling.* Then, if λ is increased further the same phenomenon occurs over and over. The doubling continues until the period becomes infinite, which happens at a specific value too. At this point the data stream is chaotic, with no periodicity present at all. The values of λ at which successive period doubling occur have fascinating regularity (Feigenbaum 1978). We will see that the phenomenon of period doubling may be a useful diagnostic for the presence of chaos (see §II.3).

There is also a large literature dealing with many similar discrete-time systems. Essentially any nonlinear map or transformation with the "stretch-and-fold" property (think of kneading bread dough or taffy) yields chaos. Other common maps that are used to simulate artificial chaotic data to test analysis algorithms include: the Bernoulli shift ("doubling map"), the Henon map, the Ikeda map, and the "standard" map. Similarly, there are a set of standard sets of differential equations that are known to produce chaotic behavior for some parameter values, including the Lorentz, Rossler and Brusselator system. These are described in any of the standard textbooks, as well as specialized articles.

I.2 Astronomical Chaos?

Since the universe is largely nonlinear, many astronomical systems are potential sites of chaotic behavior. Recognizing this fact, astronomers hope to reveal the nature of the order behind chaos, in order to understand the underlying physical processes.

There are two arenas for the application of chaos in astronomy: (1) the study of dynamical systems, to find the conditions under which they have chaotic behavior, and (2) analysis of astronomical time series data for the presence of chaotic processes. To a large extent, of course, these two endeavors are related: good models for the data, consistent with the physics appropriate to the system, often play an important role in the data analysis. And the analyst needs to be prepared for the kind of phenomena that arise in data from chaotic dynamical systems.

There are specific reasons why some astronomers have dared to analyze their data for the presence of chaos without any physical justification. As we will see in the discussion of the embedding procedure in §II.2, it is possible to get dynamical information from observations of variables that are not even dynamical variables. This remarkable mathematical result means that observations of subsidiary quantities that are not directly involved in the dynamics can provide clues to underlying chaotic dynamical processes. An example is that observations of brightness variations in stars can yield understanding of the dynamics of the stellar interior.

In addition, a viable approach is to carry out a preliminary analysis of the data without regard to any physical model. The idea is to ask the simple question: Is chaos present? This can work because there is a characterization of chaos that is independent of the specifics of physical models. If chaos is found to be present, then one can go on to make specific physical models.

The subject of this review is data analysis, and we shall concentrate on methods for analysis of chaotic time series. But for the reasons mentioned, we shall touch briefly on dynamical systems in astronomy.

I.3 Chaos *vs.* Randomness

A key problem facing the analyst is to distinguish between chaotic and random processes. Unfortunately this issue is easily clouded by important but elusive philosophical issues, which we will eschew. Here I attempt to set up a pragmatic scheme for differentiating between the two kinds of processes.

An important element of the physical difference between chaos and randomness is that in chaos the successive observations are strongly dependent – knowledge of the dynamical laws plus previous values of the observables leads to an estimated value for the next observation that is highly accurate. (It is an important detail that SIC prevents this kind of "prediction" from extending very far into the future.)

In contrast, if the physical process is random, there is an intrinsic uncertainty of future values due, say, to complexity. The system may be strictly deterministic but too complex to predict because there are too many variables to measure. Another source of intrinsic randomness is quantum mechanics effects.

For practical purposes, then, short-term predictability is the key: chaotic processes may be predicted accurately over short periods of time, and random ones cannot.

But, you object, to some extent random noise can be predicted. For example, even white noise can be predicted by guessing that the next value will be the mean. More germaine is the fact that nonwhite noise can be predicted even more effectively. The correlations present in such a process can be used to generate predictions far superior to the mean value. Such partial predictability needs to be distinguished from the more accurate, but short term, predictability of a chaotic process.

II. Signatures of Chaos

This section reviews the aspects of time series that in one way or another are connected with the presence of chaotic processes. Some of these aspects are merely consistent with chaotic dynamics, and do not really prove that chaos is present. Others are properties that of necessity must be present if the underlying process is chaotic.

II.0 The Time Series

Does the appearance of the time series indicate the presence of chaos? Unfortunately not. Generally speaking visual inspection of time series data can distinguish *stochastic* processes from linearly deterministic ones, such as periodic oscillations or polynomial trends. [I use the term *stochastic* for the quality of disorder that is common to random and chaotic processes.] But the distinction between chaos and randomness is more subtle. Plots of chaotic and random data with the same degree of correlation (e.g. white noise and the output of the logistic map for $\lambda = 4$) are nearly indistinguishable. I encourage the reader to make such plots using equation (1) for white chaos and a random number generator for white random noise.

The eye seems good at distinguishing order from disorder. Why do we have so much difficulty visualizing the difference between chaos and randomness? The problem lies not with the abilities of the human eye but with the way the data are presented. The distinction between the kinds of disorder (in chaotic and a random processes) reside in a different space than the standard time series plot ... as we will see in the next section.

II.1 The Return Map (Phase Portrait)

Part of the above problem with visual recognition is that the dependence of successive values of the observable X is not revealed very well by plotting them, next to each other, on a time axis. This standard time series plot places too much emphasis on the time at which the measurements were made. What matters are the values of the relationships between observations at successive times, not the absolute times of the observations. This suggests plotting X_n against X_{n+1}, as opposed to X_n against n.

This simple idea has been around in many guises for a long time, and probably cannot be credited to any one person. A charming introduction to the resulting plot, sometimes called a *return map*, was given by Shaw (1984). He introduced a slight twist, by plotting successive time intervals between discrete events (the falling of a water drop from a drippy faucet) against each other, rather than the values of an observable.

This simple idea has grown into the most powerful tool of chaotic time series analysis. To see why this is so, we make what seems like a diversion – but we will return soon.

This diversion is *state space*. It is generally useful to study a physical system in this abstract space, the coordinates of which are a complete set of independent dynamical variables. The use of this space lies in the fact that any configuration of the system corresponds to a unique point of this space, and vice versa. The system's complete evolution similarly corresponds to a trajectory or *orbit* in state space. This surprisingly useful geometrical view is old (Poincaré) and well described in the modern literature of dynamical systems (Abraham and Shaw 1983, Wiggins 1988, 1990, etc.).

Unfortunately, it is rare that an experimentalist can measure all the physical coordinates. Practical considerations often permit measurement of only a few state variables or, even worse, only peripheral quantities that depend indirectly on the state. For example, astronomers have studied the chaotic tumbling of Saturn's satellite Hyperion through space, using observations of its changing brightness (see §III.1).

Remarkably, extensive measurements of just one variable can reveal the structure of the orbits in the full multivariate state space. Packard, Crutchfield, Farmer, and Shaw (1980) and Takens (1981) show that under certain conditions on the dynamical system there is a multidimensional *embedding space*, the coordinates of which can be derived from a single observed variable. The time series data generate trajectories in this space that are simply related to the system's state-space orbits.

(To underscore the difference between state space and embedding space, we use the terms *orbit* and *trajectory* for the trace of the system's evolution in the respective spaces.) This relation is in the form of a smooth map, from the state space to the embedding space, that preserves the topology of the evolutionary paths. Thus the essential features of the unobservable state-space orbits can be understood by studying the accessible trajectories in the embedding space. In this analysis, one does not necessarily need to know the dimension of the state space – or even the identity of its coordinates.

Of the many choices of embedding space coordinates the most common is the observed variable evaluated at a set of lagged times:

$$\mathbf{X} = (X_n, X_{n+k}, X_{n+2k}, \ldots, X_{n+(M-1)k}). \tag{2}$$

The lag k and the *embedding dimension* M are positive integers. In principle the lags need not be equal, but in practice they are almost always chosen to be so. As time goes by, points defined by equation (2) fall on embedding-space trajectories topologically equivalent to the system's state-space orbits. The trajectories can be traced out using time series data, albeit crudely if the noise level is large. When the system satisfies the requisite mathematical conditions (and M is high enough – see below) we say that we have constructed a *suitable embedding space*.

We thus arrive at a most important tool for the analysis of chaos, the phase portrait. [As discussed above *state-space portrait* might be a better term. However, we follow standard usage and refer to the embedding-space picture of dynamical trajectories depicted by time series data as a phase space portrait.] We now see that the phase portrait is just an extension of the return map to a space of higher dimension.

The phase portrait plays a key role in the search for order in chaos. It can be constructed directly from the time series data. In simplest form it is just a plot of X_{n+1} against X_n. In other cases the process must be described in a space of higher dimensions, and we add coordinates $X_{n+2}, X_{n+3}, \ldots$. Also, the lag k need not be one unit. A caution: the example described by Harding, Shinbrot and Cordes (1989) shows that one can be fooled by the appearance of the phase portrait. They exhibit purely random, nonchaotic models for their pulsar data which produce phase portraits that show interesting structure that is sometimes thought to identify chaos.

It may be noted that "projection pursuit" (Friedman and Tukey 1974, Jones and Sibson 1987, Walden 1989) provides a way to deal with multidimensional data of this kind. While most workers have used it only to search for clustering in multidimensional data, projection pursuit may prove to be useful for the sheet-like structures relevant here.

In practice, with real data, there are further complications. Observational errors always produce noise that somewhat blurs the phase portrait. Yet with a sufficiently large signal-to-noise ratio this plot will reveal the nature of the dynamics in spite of noise. One does not know a priori the value of the lag k. In the theorems justifying the embedding procedure – namely infinite, noise-free data streams – the value of k does not matter. In practice k does matter, and one must figure out a good value from the data. One does not know a priori the value of the dimension M of the embedding space. In theory M must be larger than $1 + 2d$ (d is the dimension of the physical state space), but in practice considerably smaller values provide suitable embeddings. However, it may well be that the correct dimension for viewing the underlying process is higher than 3, in which case there are problems displaying and visualizing the data. We will deal with dimension considerations further in the next section.

II.2 Attractor Dimension

The *attractor* of a physical system is, simply put, the part of state space toward which the system tends to evolve. It ranges from simple closed curves in the case of periodic motion to highly complex fractal sets for chaotic motion (Abraham and Shaw 1983).

Another view of the embedding procedure described in the previous section is that it gives the experimentalist a view of the attractor of the system. The transformation described above (mapping state space into the embedding space) maps the attractor into a subset of the embedding space. The experimentalist cannot deal with the attractor itself, and must learn about it by studying this image in the embedding space. The dimension of this image (which clearly cannot be larger than the embedding dimension) is thus an important quantity. The relation between the attractor dimension, the embedding dimension, the dimension of the embedding-space image of the attractor, and a host of other dimension quantities that have been introduced is a complex subject (Takens 1981, Farmer, Ott and Yorke 1983).

It is possible to estimate the dimension of the attractor from the time series data. There are many different kinds of dimensions that can be defined, and most of them can be estimated from data. There are many different techniques for estimating a given dimension. And a single number does not always characterize fully the dimension of an attractor. Generalized dimensions and a function which represents the distribution of dimensions have been introduced (e.g., Mandelbrot 1989, 1990, Theiler 1990b).

The literature on this subject is too large to review here. A recent overview (Theiler 1990b) contains 143 references. The classic paper by the Santa Cruz group (Froehling, Crutchfield, Farmer, Packard and Shaw 1981; see also Farmer, Ott and Yorke 1983) give a very nice physical introduction. The book (Mayer-Kress 1986) contains many articles on dimension and dimension estimation.

The statistics of dimension estimates (e.g. Holzfuss and Meyer-Kress 1986, Hunt 1988, Theiler 1990a, Smith 1990, Ramsey and Yuan 1990) is of interest to experimentalists, for they need to assess the reliability of their results. Unfortunately the uncertainties due to other effects (Caswell and Yorke 1986), some of which we will soon meet, may often overwhelm purely statistical uncertainties.

The technique most commonly used by experimentalists provides an estimate of the correlation dimension (Grassberger and Procaccia 1983a,b). This method implements the notion that the rate at which the volume contained in a hypersphere embedded in M-dimensional space grows with radius to the M-th power.

Let r denote the distance between data points in the embedding space (i.e., the Euclidean distance between two vectors given by equation (2). Let $N(r)$ be the number of pairs of data points separated by a distance $< r$. If one finds a linear relation between $logN(r)$ and $logr$, then the slope is an estimate of the dimension. One does this for a sequence of increasing values of the embedding dimension, M. At first the slope changes with M, but then above some value it may level off and remain constant; if it does so, this constant is taken as the estimate of the dimension. This "saturation" may not happen, indicating that the data are not chaotic, or that the actual dimension is too large to be measured with the data at hand.

There is a relatively small range of the $logN - logr$ plot over which the slope is well defined. Below the smallest separation, r_{min}, defined by $N(r_{min}) = 1$, $N(r)$ does not change with r. And in practice statistical fluctuations connected with small values of N mean that r must considerably exceed r_{min} in order that the measured slope be meaningful. Similarly r must be less than the largest separation, r_{max}, in order that a meaningful linear relation be defined. As discussed by Takens (1989), this imposes an upper limit on the dimension that can be determined this way. Since the slope of interest is

$$\frac{logN(r'') - logN(r')}{log(r'') - log(r')} \tag{3}$$

where r'' and r'' are some numbers satisfying

$$r_{min} \leq r' < r \leq r_{max} \tag{4}$$

, and because clearly $N(r') \geq 1$ and $N(r'') \leq \frac{1}{2}N(N-1) < N^2$, we have a limit on the slope:

$$D \leq \frac{2logN}{\Delta logr}, \tag{5}$$

where $\Delta logr = log(r'') - log(r')$. Takens arives at $D \leq 2log_{10}N$ by assuming that the smallest meaningful range of r values over which the slope can be well-defined is a decade (i.e. $log_{10}(r'') - log_{10}(r') \geq log_{10}(10)$), but we will use the slightly more general form in equation (5) for two reasons.

Unfortunately **there are severe problems with the use of dimension estimates to diagnose chaotic dynamics**. It is fair to say that the technique is essentially useless for detecting the presence of chaos in most astronomical applications. To start with, large amounts of data are necessary to determine dimensions with any reliability. Next, even modest noise can affect the results (Norris and Matilsky 1989). Smoothing of the data, due to either experimental effects (e.g., finite-time-constant instrumentation) or physical effects (e.g., radiative transfer effects in the case of astronomical objects of variable luminosity) distorts the dimension (Badii and Politi 1986, Badii, Broggi, Derighetti, Ravani, Cilberto, Politi and Rubio 1988, Paoli, Politi, Broggi, Ravani and Badii 1988, Mitschke, Moeller and Lange 1988, Chennaoui, Pawelzik, Liebert, Schuster and Pfister 1988, Scargle 1989a,b, 1990a,b, 1991). Even more important, **determination of a low**

dimension, fractional or otherwise, does not necessarily mean that a chaotic process is operating. Osborne and Provenzale (1989) have shown that purely nonchaotic colored noise has a low dimension when it is determined with the Grassberger-Procaccia algorithm. Thieler (1986) discusses an anomaly in the behavior of the correlation integral when small amounts of data are available that can lead to errors in the estimate of the dimension; he proposes a modification of the standard algorithm that eliminates the difficulty.

In summary, if the underlying dynamics is chaotic with an attractor of low (say fractional) dimension, then the dimension determined from the data will also be of low dimension. But low dimension is not a sufficient condition for the presence of a chaotic process, and thus unfortunately does not provide a litmus test for chaos.

II.3 Period Doubling

Period doubling was mentioned in §I.1 in connection with the example of the logistic map. In many situations, when a *control parameter* such as the rate of fluid flow, is slowly increased the physical system exhibits a sequence of behaviors: constant → periodic → periodic with period multiplied by 2 → ...chaos. This sequence, or "route to chaos," has received a lot of attention, especially with regard to laboratory or computer situations in which it is possible to vary control parameters and study the sequence. In observational sciences, such as astronomy, it is generally not possible to manipulate the control parameters. More typically one has data for one particular value of such parameters (and the value itself may be unknown). Therefore considerations of what happens when one adjusts (sometimes with infinite fineness) parameters are not relevant.

Nevertheless, there is at least one feature of time series data that has been taken as indicative of possible period doubling, namely systematic amplitude changes of an otherwise periodic process. For example, in the light curve of a variable star, "the alternation of deep and shallow minima" is some indication of period doubling (Buchler and Regev 1990). Clearly such a conclusion is speculative, and would need to be supported by further observations, e.g., of the light curve as conditions in the stellar interior or atmosphere undergo secular changes.

II.4 Lyapunov Exponents

It was remarked earlier that SIC is a key feature to chaos. The Lyapunov exponents (e.g., Eckmann, Kamphorst, Ruelle and Ciliberto 1986) of a dynamical system are measures of this sensitivity.

Consider two very nearby initial states of the system, say $\mathbf{X_1(0)}$ and $\mathbf{X_2(0)}$, where the symbol $\mathbf{X}$ refers to the vector of co-ordinates in the system's state space. Compute the evolution forward in time from these states, and compare the resulting orbits. Simply put, the Lyapunov exponents measure the rate of divergence of the two orbits. One of the characteristics of chaos is that this divergence grows exponentially in time:

$$\delta\mathbf{X(t)} \sim \delta\mathbf{X(0)}e^{\lambda t}, \tag{3}$$

where $\delta\mathbf{X}(\mathbf{t}) = \mathbf{X_1}(\mathbf{t}) - \mathbf{X_2}(\mathbf{t})$, and $\delta\mathbf{X}(\mathbf{0}) = \mathbf{X_1}(\mathbf{0}) - \mathbf{X_2}(\mathbf{0})$. By definition, the exponential coefficient λ is the *Lyapunov exponent.*

A more complete discussion than we have space for here takes into account the fact that the rate of divergence depends on the direction between the two initial states, and that the Lyapunov exponent is not defined at a specific point in state space, but computed by averaging over the trajectories. Also, in an M-dimensional state space there are M exponents corresponding to the growth in different directions. Of most importance is the largest, λ_{max}: if it is > 0 the system is chaotic; if $\lambda_{max} \leq 0$ there is no exponential growth and therefore no sensitivity to initial conditions. Most textbooks discuss the definition and meaning of *Lyapunov characteristic exponents*, and (*e.g.*, Bergé, P., Pomeau, Y., Vidal, C. 1984; the best recent introduction I have seen is §6.4 of Hao Bai-lin (1989).

There are several algorithms to estimate the Lyapunov exponents from time series data (e.g., Eckmann and Ruelle 1985, Wolf, Swift, Swinney, and Vastano 1985, Tongue and Smith 1989). These methods rely on finding near state-space neighbors of a given point. How can one find near neighbors to a point in state space? This is hard, because often one has only one trajectory. The only hope is that one has enough data that the system state returns close to the initial location; then one can assess how rapidly this orbit moves away.

There are several more practical problems with estimating Lyapunov exponents. One must first of all have enough data that many points in state space will indeed have near neighbors. For a space of high dimensions this leads to insatiable demands for data. Additionally, the noise cannot be very large, because neighbors closer together than the noise level are practically useless. Another pitfall is that there are often places in state space where the trajectories arrive at critical points where some go left and others – a hair away – go right. If X' and X'' take different paths, the result is a catastrophically wrong contribution to the estimate of the Lyapunov exponent. The paper (Brown, Bryant and Abarbanel 1990) discusses many of the practical problems arising in these computations.

Thus, although the algorithms can provide accurate results (judged by comparison with exact values known for a few systems), their practical application is far from being trouble-free. Experimentalists (Frank, Lookman, Nerenberg and Essex 1990, to pick one recent example) are wrestling with these problems, and if their data are clean enough, they may get reasonable results. Alan Wolf (personal communication) feels that in cases of short data segments and/or moderate to large amounts of noise, the estimation of lyapunov exponents is a bit easier than that of dimension.

As a final note, Alan Wolf has made his code for computation of Lyapunov exponents available, in both FORTRAN and C versions. References describing the methodology are (Wolf, Swift, Swinney and Vastano 1985), Wolf 1986, and Wolf and Bessoir 1991). To obtain C or Fortran source code, including PC executables, should contact Alan by e-mail (Internet address: wolf sleepy.cooper.edu) or at the following postal address: Professor Alan Wolf, Department of Physics, the Cooper Union, 51 Astor Place, New York, NY , 10003). Specify disk size and hardware environment. A users manual that discusses parameter selection, etc., in in preparation.

II.5 Non Linear Dynamics: Prediction and Noise Reduction

Prediction of a process from observations of its past is as important in the analysis of chaotic time series as it is for random ones. Accordingly a large literature has developed (e.g. Abarbanel, Brown and Kadtke 1989, Brock 1990, Casdagli 1989, Crutchfield 1989, Crutchfield and McNamara 1987, Farmer and Sidorowich 1987, Lewis and Stevens 1990, Townshend 1990, and Weigend, Huberman and Rumelhart 1990). Various authors have pursued the related topic *noise reduction* (Farmer and Sidorowich 1988a,b, Kostelich and Yorke 1988, Marteau and Abarbanel 1990, Sugihara and May 1990).

Chaotic processes are deterministic and therefore predictable in principle. In practice **SIC** greatly limits predictability. Even a tiny uncertainty in the initial state grows rapidly with the passage of time. Nevertheless there is much research in this area, mainly because nonlinear prediction amounts to a determination of aspects of the system evolution which are physically important. (The terminology in vogue is that one "captures the dynamics" by determining the structure of a map that approximately describes the evolution of the system in time.) The main differences between the works referenced here is the mathematical forms that are used in such approximate maps, and the procedures by which parameters in the forms are determined.

II.6 Kolmogorov Entropy

To be added: discussion of the concept; computation from time series data discussed by Grassberger and Procaccia (1983b).

III. The Search for Chaos in Astronomical Data

This section summarizes all attempts to detect chaotic processes in astronomical data that I am aware of. It is organized around the part of the universe involved, namely the sun, the solar system, planetary atmospheres, stars (variability) and motions, and other regions of the cosmos.

While the focus of this review is data analysis, in each section I give a few references to the dynamical systems work in order to set the context in which the data analysis is carried out. An appreciation of the results and methods of dynamical systems research in astronomy is helpful in modeling and analysis of time series data.

There are three general collections of papers dealing with chaotic phenomena in astrophysics (Buchler, Perdang and Spiegel 1985, Buchler and Eichhorn 1987, and Buchler and Gottesman 1991).

III.1 The Sun

The complex variability and activity of the sun on various time scales is evidence for the existence of nonlinear magnetohydrodynamic phenomena. This has led to several searches for chaos in solar data.

Romanelli, Figliola, Hirsch and Radicella (1987) studied the radiofrequency flux from the sun, at a wavelength of 10.7 cm. Using time series of daily values, over intervales ranging from 3 to 10 years, they computed the correlation dimension and the Kolmogorov entropy using the Grassberger-Procaccia method. They found a correlation dimension of 3.3 for intervals of high solar activity and 4.5 for times of low activity. They took their positive estimates of the entropy (0.02 and 0.04 for high and low solar activity, respectively) as showing "that chaos is indeed present," and the correlation dimensions as "indicating that a low-dimension chaotic attractor is present in the time series analyzed." In view of the results discussion in §II.2, especially the results of Osborne and Provenzale (1989), an alternative interpretation is that the underlying process is colored noise, with a different power spectrum during times of high and low solar activity, and that the estimated entropies are not (statistically) significantly greater than zero.

At about the same time Kurths and Herzel (1987) analyzed a type-IV radio burst from a solar flare. The data was a time series of 640 points over a 40-second interval, at a radiofrequency of 778 MHz. They found that the power spectrum appeared broad-band, with no clear evidence for the periodic components expected from some models. The Grassberger-Procaccia method yielded a correlation dimension of about 3.7 ± 0.3, a Kolmogorov entropy of 0.04, and a maximum Lyapunov exponent of 0.08. They state that their "analysis of observed solar radio pulsations suggests that there must be a low-dimensional attractor," but point out that the result is not well-established. This work includes a nice study of the results of application of their time series analysis techniques to synthetic data for a periodic signal and for the Lorentz model. They state "the properties of the attractor deduced from the data cannot be explained by ... a stochastic AR model," by which one presumes they mean the positive entropy and Lyapunov exponent – for a stochastic AR model can certainly give dimension results like those reported by the authors, as discussed in §II.2.

A similar analysis of a flare event with similar properties was given by Kurths and Karlicky (1989). In this case the authors found that a "sudden transition from a regular to a chaotic structure is identified as part of the Ruell-Takens-Newhouse route to chaos."

Perhaps the most obvious candidate for solar chaos lies in the sunspot cycle. The overall behavior of the sunspot activity is at least approximately periodic, so it is clear that the process is not like the white chaos from the logistic map. What has been investigated is whether the departures from strict periodicity might be due to some kind of nonlinear, chaotic phenomenon.

Spiegel and Wolf (19??) studied this problem, using three-dimensional state space reconstructions of Zurich sunspot data using a lag of 1200 days. They estimated a fractal dimension of about 5, but concluded that there is not enough data for this estimate, or that of the maximum Lyapunov exponent to be reliable, for "38,000 daily sunspot counts only define 10 oscillations of length 11 years, or 5 oscillations of length 22 years" (Wolf, private communication).

Spiegel and Wolf make the important and often ignored point that the data must make a reasonable number of circuits around the putative attractor in order to make a reasonable case for the presence of chaos, and in order that the quantitative methods be meaningful. Furthermore, the noise must be reasonably small too. It is often sadly true that astronomical time series data do not meet either of these requirements.

Feynman and Gabriel (1989) studied the solar variability question by making use of 1500 years of sunspot, geomagnetic and auroral activity. These authors did not use any of the standard time series methods, but rather tried to identify subharmonics of the fundamental 11-year solar cycle with the period doubling phenomenon. They also studied the phase of the subharmonics and concluded that there have been losses of phase coherence, for example upon emergence from the Maunder minimum, to be expected of a chaotic process. The authors also discussed the properties of the solar dynamo and its (nonlinear) physics.

To summarize, a few attempts have been made to detect chaotic effects in the variability of the sun's radio emission and its spotting. While chaotic processes certainly abound in this highly active arena, none of the studies published has unambiguously identified such processes.

As a footnote, a number of authors who have developed modeling techniques for nonlinear processes have used the sunspot data as a test case. Examples are the threshold-autoregressive models of Tong (1990), the multivariate adaptive regression splines of Lewis and Stevens (1990), and the interesting approach to modeling time series using neural networks, by Weigend, Huberman and Rumelhart (1990). It is curious that a time series dealing with such an ill-understood system as the sun has been used as a test case for modeling.

III.1 Solar System Dynamics

Beginning with Poincaré (1892), celestial mechanics has been a breeding ground for many discoveries in chaotic dynamics, and it remains so. See Thompson (1989) for an interesting overview of chaotic dynamics with a Newtonian flavor, and (Milani 1989a and Killian 1989) for comments.

Recent results on the motions of asteroids (Wisdom 1982, 1983, 1985a, 1987a,b), the rotation of Hyperion (Wisdom, Peale and Mignard 1984, Peale 1984), and other celestial mechanical systems are among the most secure and interesting chaotic dynamical systems. In particular, the pioneering work on various restricted versions of the three-body problem by Hénon and Petit (Hénon 1969, 1970, Hénon and Petit 1986, and Petit and Hénon 1986) is being applied in attempts to understand the dynamics of comets (Baille and Froeschlé 1990, Duncan, Quinn and Tremaine 19??, Quinn, Tremaine and Duncan 1990, Dvorak and Kribbel 1990 Torbett 1989, Levinson 1990, Chirikov and Vecheslavov 1989), meteorites (Wisdom 1985b), asteroids (above-cited work by Wisdom, Soper, Franklin and Lecar 1990, Chauvineau and Mignard 1990), planetary rings (Goldreich and Tremaine 1978, Petit and Hénon 1987a,b), planetary satellites (Murison 1989, 1990, Broucke 1988) and other solar system bodies (Nakazawa, Ida and Nakagawa 1989a,b, Nishida 1983, Ida and Nakazawa 1989, Hasegawa and Nakazawa 1990 Spirig and Waldvogel 1985, Goldreich and Tremaine 1980, Duncan, Quinn and Tremaine 1989). And there is considerable interest in the problem of stability and possible chaotic behavior of planetary orbits (Franklin, Lecar and Quinn 1990 Quinn, Tremaine and Duncan 1990, Sussman and Wisdom 1988, Quinlan and Tremaine 1990, Quinn and Tremaine 1990 Nobili, Carpino and Milani 1989, Nobili, A.M., and Burns, J.A. 1989 Milani 1989b, Laskar 1989, 1990, 1991).

The chaotic time scales in most celestial mechanics problems, while perhaps short compared to the age of the universe, are very long compared to the intervals for which we have observations. Thus in few cases can the predictions based on study of the relevant dynamical systems be directly compared with time series data. In the case of the asteroids, much can be gathered from observations of the current distributions of their *orbital elements* (see references cited above).

Since the focus of this paper is time series analysis, we consider here only the one case that I know of where the time scale is short and time series data exist. The Voyager images of Saturn's satellite Hyperion (Smith *et al.* 1982, Thomas and Veverka 1985) show that it is a very irregular satellite, with a grossly nonspherical shape. In addition, its orbit around Saturn is moderately eccentric ($e = .1$) so that the gravitational torque on Hyperion due to Saturn is modulated periodically. Analysis of such a periodically driven system (Wisdom, Peale and Mignard 1984, Peale 1984, Wisdom 1987) led to the conclusion that the rotation of Hyperion should be chaotic.

Since Hyperion is so irregular, the chaotic rotation should make the observed brightness vary due to the irregularities of the surface plus the changing cross-sectional area projected toward the earth. Analysis of photometric data indicated that Hyperion's rotation is not in synchronism with its orbital revolution around Saturn (period is about 21.3 days). Rotational periods of about 13 days were determined both from Voyager (Thomas, Veverka, Wenkert, Danielson, and Davies 1984) and ground-based (Goguen, Cruikshank, Hammel and Hartman 1983) photometry.

However it was argued (Peale and Wisdom 1984, Wisdom and Peale 1984) that this was not conclusive disproof of the hypothesis of chaotic tumbling because of insufficient sampling. Binzel, Green and Opal (1986) presented independent photometry and analysis that cast doubt on the presence of a coherent 13-day rotational period. More recently Klavetter (1989a) made a concerted effort to obtain ground-based photometry to clear up this ambiguous situation. He argues that his data are not consistent with a coherent 13-day period, and even attempted to fit a detailed dynamical model to his photometry (Klavetter 1989b, 19??). Because of the sensitive dependence of the model's evolution, there are many severe problems with this fitting procedure. Klavetter deals with such problems, but it is still difficult to assess the significance of fits in such an ill-conditioned parameter estimation context.

III.2 Stellar Variability

Despite their appearance of steadiness, all stars are variable in some degree. The many stars that vary significantly in brightness are called *variable stars*. Some of these variables are quite periodic, with an extremely predictable light curve (i.e. the plot of the brightness of the star against time). Others have a variability which is aptly described as *irregular*.

Recently Regev and Buchler (Regev and Buchler 1989, Buchler and Regev 1990) reviewed both theoretical work on possible chaotic dynamics of stellar pulsations and analysis of variable star data. Numerical work which shows chaotic behavior in stellar models includes (Aikawa 1987, Buchler and Kovács 1987, Buchler, Goupil and Kovács 1987, Kovács and Buchler 1988, Andrews and Baierlein 198?, Buchler 1990, and Moskalik and Buchler 1990).

From the observational point of view, it seems likely that chaos might be involved here, because in some cases the shape of the light curve suggests a period doubling phenomenon, as discussed above. An example is variation characterized by an alternating of large and small peaks, or more objectively as revealed by power spectrum analysis.

A number of papers reporting on searches for chaos in variable star data have been published, and the goal of the rest of this section is to survey this literature. Beginning with rather uncritical applications of dimension estimation techniques without regard to their limitations or real understanding of the meaning of the results, this discipline has evolved to the point where most workers realize that techniques other than dimension estimation are needed to prove the existence of chaos. Thus considerable progress has been made, but we have yet to develop a robust, general purpose set of procedures to detect and model chaotic processes in astronomical data.

Auvergne and Baglin (1986) outlined the possibilities for phase-space reconstruction and dimension computation on variable stars in general, and included suggestions for methodology, and discussions of the effects of noise and gaps in the time series. They gave a preliminary analysis of about 10,000 seconds of data on the variability of the pulsating white dwarf GD 66, just to demonstrate the feasibility of the methods suggested. (They reported a correlation-integral dimension somewhere between 2 and 3.) Somewhat later Auvergne (1988) showed how the technique of singular value decomposition pioneered by Broomhead and King (1986) can be used in the analysis of variable star data, and demonstrated the technique on the pulsating white dwarf PG 1351 +489.

Goupil, Auvergne and Baglin (1988) also discuss PG 1351 +489. Power spectra from high-speed photometry of this star show a dell-defined oscillation at $\nu_0 =$ 2.055 MHz. They present spectra that show evidence for a subharmonic at $\frac{1}{2}\nu_0$, and argue that this indicates that this star is on its way to chaos through a period-doubling scenario, or possibly that it is "already chaotic and is experiencing an inverse cascade of period doubling bifurcations." They discuss a physical model for nonlinear radial pulsations, present a two-dimensional phase portrait for the stellar data, discuss other possible (linear) explanations for their results, and suggest future observations of the white dwarf GD 117-B15A that could produce more definitive results.

Cataclysmic variables form a class of interesting, highly variable stars in which there appears to be continuing but irregular flow of material from one member of a binary system onto the other, a white dwarf. The cataclysmic variable SS Cygni, belonging to the subclass called *dwarf novae*, is well studied, and several authors have investigated the question of whether deterministic chaos can be detected in the photometric data.

Cannizzo and Goodings (1988) analyzed 7770 days of photometric data for SS Cygni, using the Grassberger-Procaccia technique. They found that the convergence of the correlation dimension as the embedding dimension is increased was not to be found, and there was thus no evidence for a chaotic attractor in this system. A more ambitious study of this star is reported by Hemplemann and Kurths (1990). They found ... [to be added]

Cannizzo, Goodings and Mattei (1990) studied three long-period variables using autocorrelation and power spectrum analysis, and searched for chaos using the Grassberger-Procaccia method. By comparing their results on the variable star data to those obtained with simulated random and chaotic data, they concluded that their estimation of finite dimensions did not necessarily indicate the presence of chaos. They discovered that the dimension seemed to change when the data was transformed from magnitudes to intensity, and noted that in authentically chaotic processes the dimension should be, and is, independent of such data transformations.

Serre, Buchler, and Goupil (1991) present a technique for prediction of the future light-curve of a variable star, based on least-squares fitting of parameters in a predictor to past time series data. They demonstrate results where the predictor is linear, noting that nonlinear predictors are also possible. The reason a linear predictor can be useful in prediction of a nonlinear process is that it operates in an embedding space of variable dimension, m, as in prediction schemes like those developed by Farmer and Sidorowich (1987, 1988a,b), Casdagli (1989), and others. They found that the best predictions we obtained for Roessler data with $m = 5$. They then turned to a short run of data for the pulsating white dwarf PG 1351+489, "trained" a 3-dimensional linear predictor on 90% of the data, and then investigated the prediction errors made on the remaining 10% . They were satisfied with the predictions, which were limited by the shortness of the data series available, which yielded errors in the parameters and prevented the investigation of dimensions m larger than 3. In this work it is not clear (a) whether there really is deterministic chaos in the variations of this star (see above discussion of the work by Goupil, Auvergne and Baglin), (b) if there is, how important it is to the structure of the predictor and to the values of the parameters, (c) how well the method would work on a nonchaotic star, and (d) how well other, conventional prediction schemes would work on these data.

Kollath (1990) studied 150 years of data on the light variation of the RV Tauri star R Scuti. He compared phase portraits of smoothed data for R Scuti, the

Rossler system, and a purely random process constructed to have roughly the same power spectrum as R Scuti. The stellar phase portrait has some similarity to that of the Rossler system. An obvious criticism of this result is that it is subjective, as the features of the phase portraits on which the conclusion is based are ill-defined. And from the phase portrait of the random noise (his Figure 10), it is clear that the correlation structure of the data, including that induced by the filtering process, causes much of the "ball of spaghetti" appearance; one could easily imagine that a simple nonstationary noise process could reproduce the main features of the stellar phase portrait – without the need to invoke deterministic chaos. Kollath found a correlation dimension between 4 and 5, but the noisiness of the data prevented an more precise determination.

Hempelmann and Kurths (1990) carried out a novel analysis of data on the outburst series of SS Cygni. They used techniques from symbolic dynamics. [more to be added ...]

III.3 Planetary Atmospheres

To be added:

Meyers, Sommeria and Swinney 19??, work on the chaotic fluid flow in Jupiter's atmosphere.

Polvani, Wisadom, DeJong and Ingersoll (1990), study of Neptune's Great Dark Spot.

III.4 Galactic X-ray Sources

Galactic x-ray sources frequently or observed to exhibit rapid stochastic variability in luminosity. Two such sources have been analyzed for evidence of the presence of chaotic dynamics.

Voges, Atmanspacher and Scheingraber (1987) analyzed data on the variability of the 1-50 keV flux from Hercules X-1. They computed correlation dimensions for many different runs during different "states" of the variability, the runs typically lasting several hours, with a sample time about of about 10 ms., but smoothed with various time scales in the neighborhood of 1 sec. They found that the dimension varied, but was characteristically in the range 2-3.

In a similar study of different data on the same object, by Norris and Matilsky (1989), led to contrary results. These authors carefully considered problems and ambiguities involved in the dimensional analysis carried out by Voges, Atmanspacher and Scheingraber. They concluded that sufficient signal-to-noise is not present in available data to distinguish chaos from randomness.

The other source to come under similar scrutiny is Cygnus X-1. Lochner, Swank and Szymkowiak (1989) constructed phase portraits based on 35,000 data points, and estimated the dimension using the Grassberger-Procaccia method. They found no evidence for a chaotic attractor in the data, and pointed out that the presence of noise severely limits the ability of the dimension analysis to detect an authentically chaotic process.

III.5 Stellar Dynamics and Other Cosmic Chaos

Harding, Shinbrot and Cordes (1990) studied timing data for the Vela pulsars, in order to assess the possible role of chaos in the exceedingly interesting and poorly understood stochastic slowing down of the rotation of neutron stars in pulsars. In my opinion this is a seminal paper, no because of the conclusions reached regarding this pulsar (a correlation dimension of about 1.5). More importantly, the authors developed a set of random process models and studied them with the same techniques that the used on the real data. They concluded that not only can conventionally random models give rise to (spurious) low dimensions (a lá Osborne and Provenzale 1989), but they can even produce interesting fractal-like structure in the return map. In particular, a very plausible physical model for a rotating neutron star with a random walk imposed on the time derivative of the frequency of rotation produced a return map very similar to that derived by the authors for the Vela pulsar data.

To be added:

Miller's early work (1964-1971) on N-body dynamics that presaged some of the more modern developments.

Thieberger, Spiegel and Smith (1990).

Regev (1990).

Atmanspacher's work on galaxy statistics, using a modified correlation approach close to GP: Atmanspacher, Scheingraber and Wiedenmann, Phys Rev A 40, 3954 (1989), Wiedenmann and Atmanspacher, Astronomy and Astrophysics (1990), Wiedenmann, Atmanspacher and Scheingraber, Can J Phys 68, 827 (1990).

V. Discussion

[This section will be amplified and expanded a bit in the final version.]

There are a number of general discussion of chaotic time series analysis that may be of use to astronomers; (Casdagli, Des Jardins, Eubank, Farmer, Gibson, Hunter and Theiler 1991, Fraser and Swinney 1986, Ruelle 1989, Tong 1990, and Scargle 1989a,b, 1990a,b, 1991) are among them.

The following points were made in various places in the above discussion: (1) typical astronomical time series are relatively short and noisy compared to what is needed for a good detection and analysis of chaotic processes; (2) this problem is compounded by the fact that the most easily computed numbers – dimension in particular – are unfortunately not sufficient to assure the presence of chaos. The deconvolution that I discuss in the above references may hold at least a partial answer. The analysis by Osborne and Provenzale (1989) shows that at least part of the corruption of dimension estimates is due simply to the coloring of the spectrum, due to extrinsic or intrinsic smoothing (filtering) effects. My work has shown that at least some of the filtering can be removed from the data, and the prospect is that this will yield more meaningful dimension determinations.

Phase portraits need to be studied more carefully. In particular study the time sequence of points moving around the "attractor."

Almost all astronomical discoveries are preceded by a period of mere theoretical possibilities. Neutron stars and gravitational lenses are examples, as someday black holes may be. Perhaps a future addition to this list may be chaotic processes pervading the universe.

["Truth knows but a brief moment of glory, between when it is condemned as heresy and then belittled as trivial." – Schopenhauer. (more exact quotation to be supplied)

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