Dynamical equations for high-order structure functions, and a comparison of a mean-field theory with experiments in three-dimensional turbulence

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Two recent papers [V. Yakhot, Phys. Rev. E **63**, 026307, (2001) and R. J. Hill, J. Fluid Mech. **434**, 379, (2001)] derive, through two different approaches that have the Navier-Stokes equations as the common starting point, a set of dynamic equations for structure functions of arbitrary order in turbulence. These equations are not closed. Yakhot proposed a ''mean-field theory'' to close the equations for locally isotropic turbulence, and obtained scaling exponents of structure functions and expressions for the peak in the probability density function of transverse velocity increments, and for its behavior for intermediate amplitudes. At high Reynolds numbers, some relevant experimental data on pressure gradient and dissipation terms are presented that are needed to provide closure, as well as on other aspects predicted by the theory. Comparison between the theory and the data shows varying levels of agreement, and reveals gaps inherent to the implementation of the theory.

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I. INTRODUCTION

It is well known that the Navier-Stokes (NS) equations of fluid motion can be used to derive equations for statistical quantities such as the moments of turbulent velocity at several simultaneous spatial points. This statistical reformulation of the dynamic equations introduces extra variables, giving rise to the familiar ''closure'' problem in turbulence. If the interest is in small-scale properties of turbulence, it is more appropriate to obtain equations for quantities that involve only small scales, $r \ll L$, where L corresponds to the energetic large-scale motion of turbulence. The simplest such quantities involve two-point statistics and are called structure functions. These are the moments of velocity increments $\Delta \boldsymbol{u}_r = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})$ over a separation vector $\boldsymbol{r}$, and $|\boldsymbol{r}| \ll L$ is the range of interest. The equation governing the third-order structure function is known from Kolmogorov's pioneering work [1]. In the so-called inertial range defined by $\eta \ll r \ll L$, where η is the Kolmogorov scale representative of the dissipative motion, the equation takes a particularly simple form if the turbulence is locally homogeneous and locally isotropic. For the longitudinal velocity increment $\Delta u_r = u(x+r) - u(x)$, where u is the velocity component along the direction of the separation vector, it takes the form

$$\langle \Delta u_r^3 \rangle = -\tfrac{4}{5} \langle \varepsilon \rangle r, \tag{1}$$

where $\langle \varepsilon \rangle$ is the average energy dissipation rate. This is Kolmogorov's 4/5 law. For the third-order structure function of the transverse velocity increment $\Delta v_r = v(x+r) - v(x)$, where v is the velocity component transverse to the separation vector, we also have the result

$$\langle \Delta v_r^3 \rangle = 0. \tag{2}$$

Recently, Yakhot [2] and Hill [3] have derived dynamical equations for structure functions of all higher orders. Yakhot first derived an equation for the generating function Z from which structure functions of all orders can be obtained by simple differentiation. Hill worked directly in the velocity space to generate those same equations. The equations are, on the whole, new. However, unlike Kolmogorov's 4/5 law, they are not closed. Yakhot presents a ''mean-field approach'' to obtain contributions from the pressure gradient and dissipation terms, thereby closing the equations. From these closed equations, it is possible to obtain certain small-scale properties such as the probability density function (PDF) of transverse velocity differences and the scaling exponents of structure functions of all orders.

Our overall goal is to inquire if anything new can be learnt about turbulence through the new equations of Yakhot and Hill. In particular, we wish to clarify the closure assumptions, and assess the value of the mean-field approach by providing experimental comparisons for theoretical predictions. We accomplish our goal in the following specific way. Since the equations and the procedure for deriving them are not yet familiar, we summarize them in Sec. II and, for later use, explicitly write them down for structure functions of several orders. Section III introduces the experimental background needed for our purposes, while Sec. IV examines the approximate balance of the equations without closure assumptions—mostly to set the stage for further discussions. We summarize the mean-field theory in Sec. V and present in Sec. VI comparisons of its predictions with experimental data on the PDFs of transverse velocity increments and their scaling exponents. Section VII deals explicitly with the magnitude of dissipation terms, and our conclusions are summarized in Sec. VIII. We consider the direct numerical simulations data at some high Reynolds number as the best resource for carrying this work forward, and hope that this work will stimulate such efforts.

II. THEORETICAL BACKGROUND

A. Brief review of the relevant equations

Yakhot [2] writes the NS equations in terms of the generating function $Z = \langle e^{\boldsymbol{\lambda} \cdot \Delta \boldsymbol{u}_r} \rangle$, where $\Delta \boldsymbol{u}_r$ is the vector velocity difference between two space points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ separated by the vector distance $\boldsymbol{r}$. The generating function Z is con-

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structed, in the spirit of field theories, so that its Laplace transform gives the PDF of the velocity differences, and obeys the equation

$$\frac{\partial Z}{\partial t}+\frac{\partial^2 Z}{\partial \lambda_\mu \partial r_\mu}=I_f+I_p+D. \tag{3}$$

Here I_f, I_p and D are the (known) forcing, pressure, and dissipative terms, respectively, and are given by

$$I_f=\langle \boldsymbol{\lambda}\cdot\Delta \boldsymbol{f} e^{\boldsymbol{\lambda}\cdot\Delta \boldsymbol{u}_r}\rangle, \tag{4}$$

$$I_p=-\boldsymbol{\lambda}\cdot\langle e^{\boldsymbol{\lambda}\cdot\Delta \boldsymbol{u}_r}[\nabla_2 p(\boldsymbol{x}_2)-\nabla_1 p(\boldsymbol{x}_1)]\rangle,$$

$$D=\nu\boldsymbol{\lambda}\cdot\langle[\nabla_2^2\Delta \boldsymbol{u}_r(\boldsymbol{x}_2)-\nabla_1^2\Delta \boldsymbol{u}_r(\boldsymbol{x}_1)]e^{\boldsymbol{\lambda}\cdot\Delta \boldsymbol{u}_r}\rangle.$$

The forcing term is small in the inertial range and so neglected. The closure of the equation requires a knowledge of I_p and D. In particular, the advection terms are treated exactly. The pressure term I_p contains correlation functions of the form $\langle(\Delta \boldsymbol{u}_r)_i(\Delta \boldsymbol{u}_r)_j\cdots(\Delta \boldsymbol{u}_r)_m\Delta(\nabla p)\rangle$, and so one requires only the knowledge of the correlation of the pressure gradient *increments* and multipoint velocity increments. The multipoint energy dissipation function D has a structure that depends on details such as the order of the moment considered. As we shall see later, its structure resembles the well known refined similarity hypotheses of Kolmogorov [4]. In any case, the terms needed for the closure of Eq. (3) are, in principle, well defined.

In locally homogeneous and isotropic turbulence, the following transformation of variables is justified: $h_1=r$, parallel to the separation vector; $h_2=(\lambda.\boldsymbol{r})/r$, the component of λ along r; $h_3=\sqrt{\lambda^2-h_2^2}$, the component of λ in the direction perpendicular the separation vector. In these new variables, Eq. (3) for the generating function becomes

$$\partial_t Z+\left[\partial_1\partial_2+\frac{d-1}{r}\partial_2+\frac{h_3}{r}\partial_2\partial_3+\frac{(2-d)h_2}{rh_3}\partial_3-\frac{h_2}{r}\partial_3^2\right]Z =I_f+I_p+D, \tag{5}$$

where ∂_i denotes the partial derivative with respect to h_i. In the new variables, the generating function can be written as

$$Z=\langle\exp(h_2\Delta u_r+h_3\Delta v_r)\rangle, \tag{6}$$

where Δu_r and Δv_r are the familiar velocity differences in the longitudinal and transverse directions, respectively. The structure functions are then generated by successive differentiation of Z as

$$S_{n,m}\equiv\langle(\Delta u_r)^n(\Delta v_r)^n\rangle=\partial_2^n\partial_3^m Z(r)\big|_{h_2=h_3=0}. \tag{7}$$

From this point on, we use the symbols Δu_r and Δu interchangeably (also Δv_r and Δv). Let us first focus attention on even-order structure functions for which the dissipation terms are negligible in the inertial range. Multiply Eq. (5) by h_3, perform $\partial_3\partial_2^{2n-1}$ of the resulting equation, and take the limit $h_2=h_3\to 0$. We then have

$$\frac{\partial S_{2n,0}}{\partial r}+\frac{d-1}{r}S_{2n,0}-\frac{(d-1)(2n-1)}{r}S_{2n-2,2} =-(2n-1)\langle\Delta p_x(\Delta u)^{2n-2}\rangle+P(1-\cos r/l_f) \times a_{n,d}S_{2n-3,0}, \tag{8}$$

where d denotes the space dimension, P is the mean energy ''pumping'' rate, l_f is the forcing scale and $a_{n,d}\equiv 2(2n-1)(2n-2)/d$. For $n=1$, we obtain the well-known relationship between the second-order longitudinal and transverse structure functions, as

$$\frac{\partial S_{2,0}}{\partial r}+\frac{d-1}{r}S_{2,0}=\frac{d-1}{r}S_{0,2}. \tag{9}$$

For $n=2$, Eq. (8) yields the relation

$$\frac{\partial S_{4,0}}{\partial r}+\frac{d-1}{r}S_{4,0}=\frac{3(d-1)}{r}S_{2,2}-3\langle(\Delta p_x)(\Delta u)^2\rangle, \tag{10}$$

which is exact in the case of incompressible, isotropic turbulence in the inertial range (where dissipation terms are negligible). This equation was derived in Ref. [5] using Hill's method, and investigated numerically in Ref. [6]. To extract further information from this equation, one has to invoke some closure. We consider only three-dimensional turbulence below unless specified otherwise. If the pressure term is small in the inertial range (as argued in Ref. [6] and will be amplified momentarily), we obtain

$$\frac{\partial S_{4,0}}{\partial r}+\frac{2}{r}S_{4,0}\approx\frac{6}{r}S_{2,2}. \tag{11}$$

Following this same procedure, it is now easy to write down a sixth-order equation with $n=3$ in Eq. (8). Neglecting the pressure term again, we obtain

$$\frac{\partial S_{6,0}}{\partial r}+\frac{2}{r}S_{6,0}\approx\frac{10}{r}S_{4,2}. \tag{12}$$

In a similar manner, we can extract two additional relations for fourth and sixth orders. Their corresponding approximate forms (again neglecting pressure contributions) are

$$\frac{\partial S_{2,2}}{\partial r}+\frac{4}{r}S_{2,2}\approx\frac{4}{3r}S_{0,4}, \tag{13}$$

$$\frac{\partial S_{2,4}}{\partial r}+\frac{6}{r}S_{2,4}\approx\frac{6}{5r}S_{0,6}. \tag{14}$$

Equations (11)–(14) were also derived by Hill [3] on the basis of homogeneity or local homogeneity. For the particular case of isotropy, Hill developed a matrix algorithm for solving the system of equations that determined all components of the tensorial structure function of a given order. The resulting equations confirm Yakhot's results. Hill's formulation provides an additional useful fact that there are exactly two equations relating fourth-order structure functions,

namely, Eqs. (11) and (13). For the sixth order, his procedure shows that there are *three* equations relating the nonzero components of structure functions, and the third equation is easily generated. Again without the pressure terms, this remaining sixth-order equation is

$$\frac{\partial S_{4,2}}{\partial r}+\frac{4}{r}S_{4,2}\approx\frac{4}{r}S_{2,4}. \tag{15}$$

Equations (11)–(15) are relationships among different components of structure-function tensor of the same order.

Hill's procedure is convenient for writing down the equations for odd-order structure functions. The equation for the third-order is well known [see Eqs. (1) and (2)] and need not be written again. There are three equations for the fifth order, which, again without pressure and dissipation terms, are

$$\frac{\partial S_{5,0}}{\partial r}+\frac{2}{r}S_{5,0}\approx\frac{8}{r}S_{3,2}; \tag{16}$$

$$\frac{\partial S_{3,2}}{\partial r}+\frac{4}{r}S_{3,2}\approx\frac{8}{3r}S_{1,4}; \tag{17}$$

$$\frac{\partial S_{1,4}}{\partial r}+\frac{6}{r}S_{1,4}\approx 0. \tag{18}$$

The $\approx$ symbol in this set of equations has to be treated with greater caution than for even orders because dissipation terms may not be small for odd orders even in the inertial range, see Ref. [2]. Thus, the approximation implied in Eqs. (16)–(17) may be more suspect than for even orders, for which dissipation terms are indeed small in the inertial range. We shall assess this feature in Sec. VII, but can readily see by inspection that the approximation cannot be correct for at least the last equation above, Eq. (18): It contains only one component of the fifth-order structure function $S_{1,4}$, which, when estimated using Kolmogorov's scaling (1991) [7], generally denoted as K41, shows an imbalance of order unity.

A further discussion of these equations, and of the degree to which they may be reasonable, requires tangible contact with experiments. It is, therefore, necessary to introduce some basic experimental details at this stage before resuming the discussion of the theory.

III. EXPERIMENTAL CONDITIONS

The velocity data were acquired by means of a cross-wire probe mounted at a height of about 35 m above the ground on a meteorological tower at the Brookhaven National Laboratory. The hot wires were about 0.7 mm in length at 5 μm in diameter. They were calibrated just prior to being mounted on the tower, and operated on DISA 55M01 constant-temperature anemometers. The frequency response of the system was typically good up to 20 kHz. The voltages from the anemometers were low-pass filtered and digitized. The low-pass cutoff was never more than half the sampling frequency f_s. The sampling rate was adequate to resolve most of the scales, including dissipative ones. The voltages were converted to velocities in a standard way through the

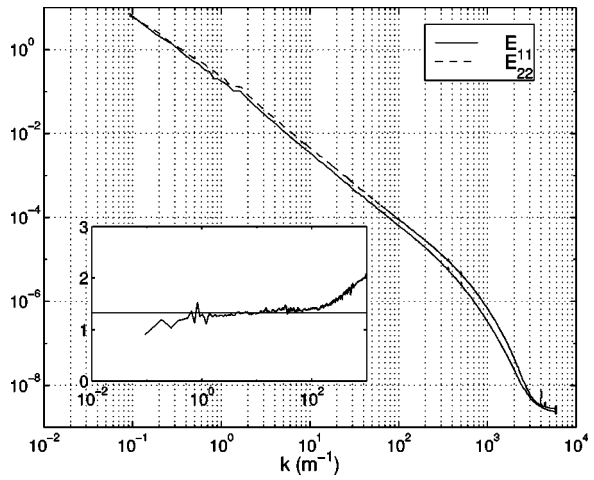

FIG. 1. Longitudinal and transverse spectral densities, Taylor microscale Reynolds number $\approx$19 000. The high-frequency noise is present in only some sets of data considered.

standard nonlinear calibration procedure. The mean wind velocities, roughly constant over the duration of a given data set, ranged between 5 and 10 ms^{-1} in the entire series. The usual procedure of surrogating time for space (''Taylor's hypothesis'') was used to obtain the mean dissipation rate $\langle\varepsilon\rangle$ and an estimate for the Kolmogorov scale η.

The details of the data analyzed here are similar to those given in Refs. [8,9]. Briefly, the Taylor microscale Reynolds number is 10 700, the large scale L is about 42 m, Kolmogorov scale η is 0.44 mm, the scaling range according to the linear part in the third-order structure function is conservatively between 0.01 m and 0.2 m; although both these numbers could be stretched in either direction by factors of order 2, this range will be regarded as the operational definition of the inertial range.

The theoretical development discussed here is meant for locally isotropic turbulence appropriate to asymptotically large Reynolds numbers. The present measurements are indeed at large enough Reynolds numbers, but it is not obvious that the small scales are isotropic. Indeed, we have used similar data before [10,11] to extract anisotropic parts of the structure functions by performing the SO(3) decomposition. Thus, a few explanatory words regarding the degree of small-scale isotropy are appropriately described here in terms of two familiar quantities.

Figure 1 shows the longitudinal spectral density E_{11} and the transverse spectral density E_{22} for one of the data sets. If strict isotropy prevails in the inertial range, the ratio E_{22}/E_{11} should be a constant ($=4/3$ if K41 scaling is correct). The data show that the ratio, while varying slowly in the inertial range from a somewhat smaller value than 4/3 to a somewhat larger value, is not far from being 4/3. Similarly, for third-order structure functions, we have the exact result [12] that the ratio $S_{1,2}/S_{3,0}$ should be 1/3. Measurements show that this ratio, which is indeed reasonably constant in the inertial range, has a magnitude of about 0.42 (see Fig. 2). There is undoubtedly some degree of anisotropy in the inertial range, and so the conclusions are to some degree affected by this feature. To pursue this issue further, we note that $S_{3,0}$ has the

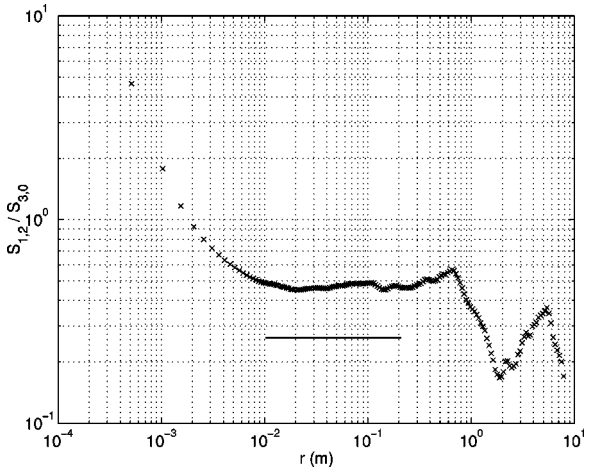

FIG. 2. The ratio of $S_{1,2}$ to $S_{3,0}$ showing some deviations from the isotropic value of 1/3 in the inertial range. In this figure and elsewhere, the horizontal bar in the figure represents a conservative extent of the scaling region determined from Kolmogorov's 4/5 law.

expected value of 4/5, and $S_{0,3}$, which should be zero exactly for the isotropic case, is small, no more than about a tenth of $S_{3,0}$. The anisotropy is thus not large for present purposes, and the transverse component is perhaps the one more affected. This conclusion is consistent with Refs. [11,13].

In summary, then, we regard for present purposes the departures from isotropy to be benign in the inertial range, especially for even orders, and expect that the results will not be qualitatively affected by their presence. A completely satisfactory demonstration of this statement is, however, not easy.

IV. THE BALANCE OF THE APPROXIMATE EQUATIONS

Using the experimental data just described, we calculate the left-hand side (LHS) and right-hand side (RHS) of each approximate equation of Sec. II, and obtain the relative size of the difference [(LHS)-(RHS)]/(LHS); this difference is an

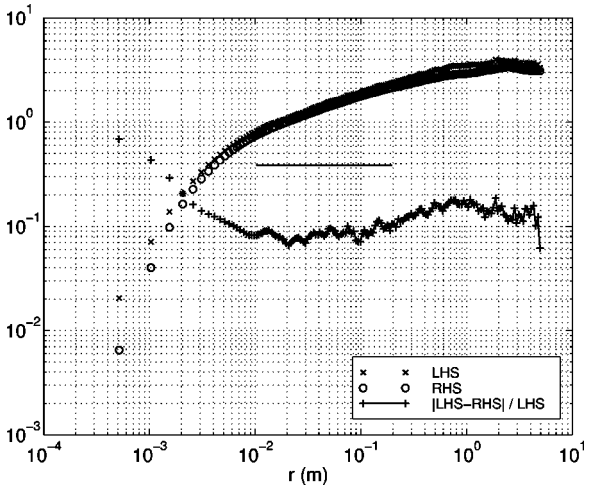

FIG. 3. The terms of Eq. (11) with their relative difference.

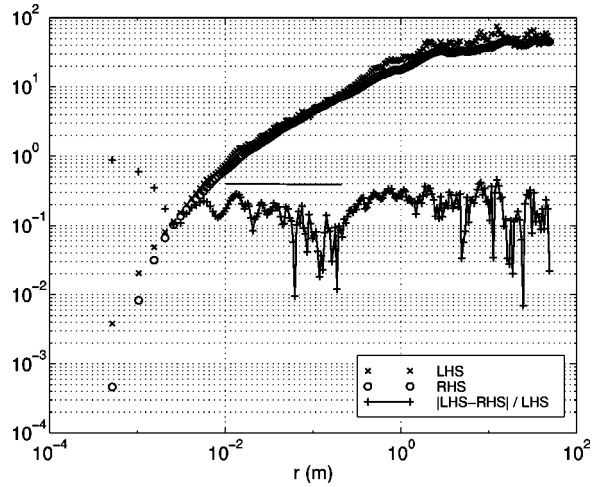

FIG. 4. The terms of Eq. (12) with their relative difference.

estimate of the magnitude of the neglected terms. Since, for even orders, the dissipation terms are small in the inertial range, the imbalance is mostly due to the neglected pressure terms. Figures 3, 4, 5, 6, and 7 display the results for Eqs. (11), (12), (13), (14), and (15), respectively. The absolute value of the relative difference is also shown in each case. In the inertial range, all four equations seem to balance within about 10% of the LHS.

This smallness of pressure terms in the inertial range suggests that the intercomponent energy transfer (for which they are responsible) is probably small. Thus, if the forcing is strongly anisotropic, there is a suggestion that the anisotropy does not tend to diminish rapidly as the scale size decreases through the inertial range. There is growing evidence that this might indeed be so [11,16]. This same smallness may also suggest that the operating physics in the inertial range might have some connection to the forced Burgers equation. This thought is worth some consideration. For smaller r, where some of the imbalance in Eqs. (11)–(15) is no doubt due to increasing dissipation effects, it is likely that some of it is due the neglected pressure terms; if so, one may conclude that the pressure-gradient effects that are negligible in

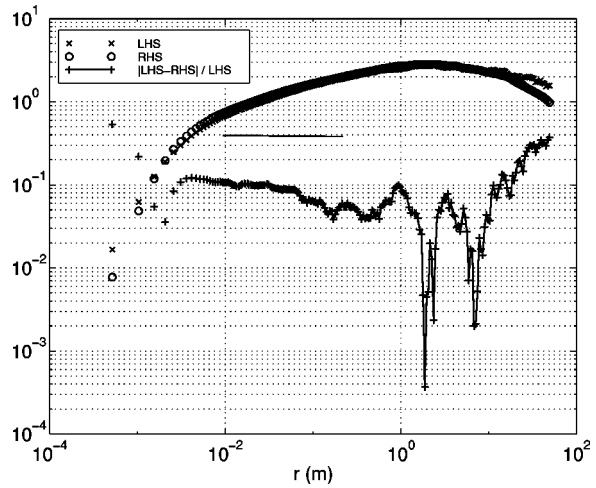

FIG. 5. The terms of Eq. (13) with their relative difference.

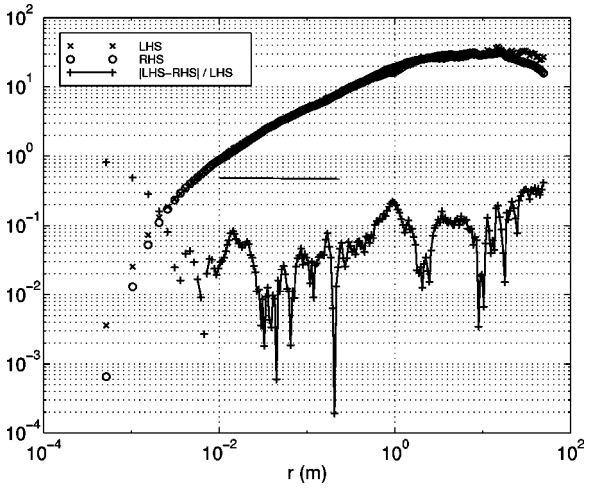

FIG. 6. The terms of Eq. (14) with their relative difference.

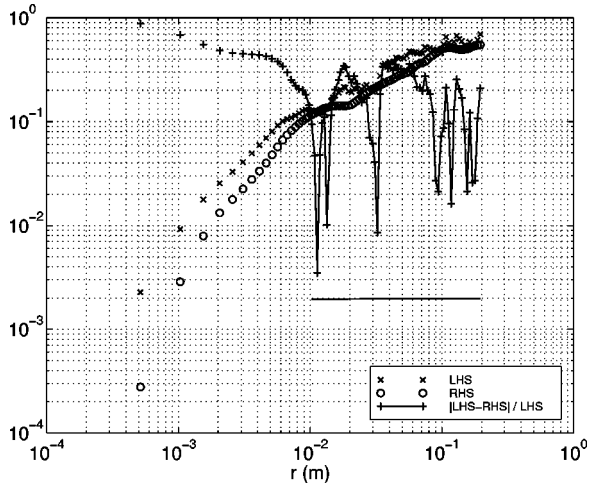

FIG. 8. The terms of Eq. (16) with their relative difference.

the inertial range reenter the balance as one approaches the small-scale end of the inertial range. It is usually the case (for example towards the tips of aircraft wings) that vortices are generated when pressure-gradient terms are activated. Perhaps the small-scale vortex tubes (or worms, see Refs. [14,15]) are a result of this effect.

Despite this intriguing suggestion, it is difficult to state that pressureless physics has serious relevance to the inertial range of three-dimensional turbulence. One reason is that it is not clear if the pressure terms are similarly small in equations for odd-order structure functions. Unfortunately, examining the balance of Eqs. (16)–(18) will not immediately give us the needed information. This is so because, at this stage, there is no means of estimating dissipation terms that may also be important in the inertial range for odd-orders. Even so, it is instructive to assess the balance of these equations. This is done in Figs. 8–10. Since odd moments do not converge as well as even moments, there is a significantly more scatter in the plots. Despite this scatter, it is reasonably clear that the relative difference of LHS and RHS in Fig. 8 is of the order of 20-30% in the inertial range. In Fig. 9, the relative difference is perhaps larger, up to 50% for some r. For Eq. (18), as was expected and remarked upon earlier, the imbalance is quite large (see Fig. 10).

The discussion of how one might ascertain the separate contributions to this imbalance by pressure and dissipation terms is relegated to Sec. VII.

V. A MEAN-FIELD THEORY

The previous section has shown that the imbalance in the approximate equations for even-order structure functions, caused entirely by the neglect of pressure terms, is of the order 10%. The imbalance is larger for odd orders, for which it is to be remembered that dissipation terms might also contribute. Thus, ignoring pressure and dissipation terms is not an option in general, and it is clear that one must make a plausible theory for them. This has been attempted in the mean-field theory of Ref. [2]. Though our interest and contributions are primarily in the experimental assessment of the theory, it is helpful to provide here a summary of the theory itself—if only to clarify the motivation for the experimental tests performed. We shall focus on the essence of the physical arguments, rather than on analytical details.

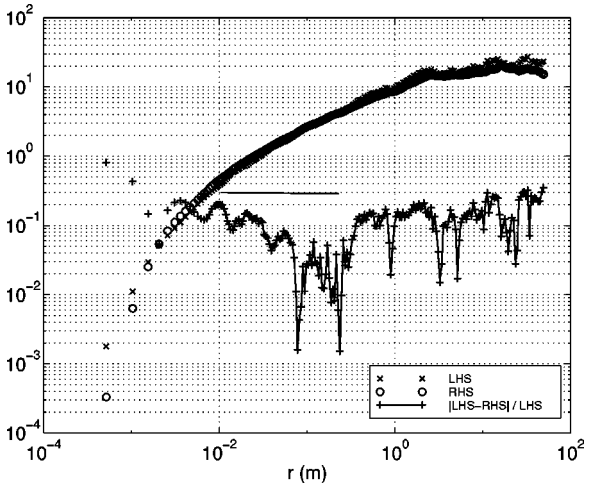

FIG. 7. The terms of Eq. (15) with their relative difference.

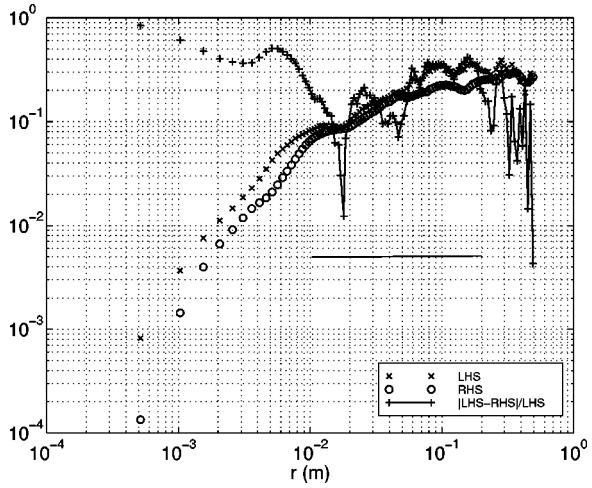

FIG. 9. The terms of Eq. (17) with their relative difference.

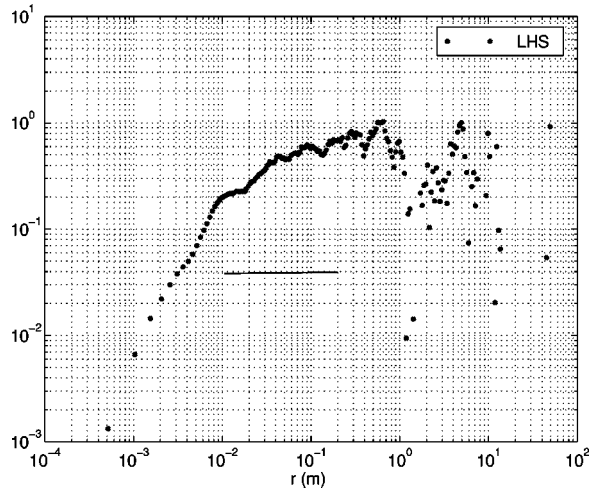

FIG. 10. The terms of Eq. (18) with their relative difference.

A. General remarks

Mean-field theories provide approximate means of describing a thermodynamic system by supposing that each ''particle'' in a many-body system moves in the ''mean'' field of all other particles in the system. This is opposite to the situation in which only nearest neighbor interactions matter. More formally, attribute to the system an order parameter ϕ that is zero when the system is ordered and becomes increasingly nonzero with increasing disorder. If the fluctuations in the order parameter are small, then it may be replaced by a spatially uniform average value. The mean-field approximation implies infinite range interactions; while this cannot be realized in practice, the order parameter in many thermodynamic systems could become arbitrarily small as the temperature approaches a phase transition value T_c. The Ginzburg-Landau theory makes use of this feature to propose a description of the free energy and to derive critical exponents at phase transitions. In general terms, the free energy $F(\phi,T)$ is expanded in powers of ϕ as

$$F=F_0+A\phi+B\phi^2+C\phi^3+\cdots, \tag{19}$$

where A, B, and C are functions of T. Near the critical point in the T space, where $T\rightarrow T_c$, the expansion can be truncated at the lowest-order terms in ϕ. The expansion then provides a qualitative description of the thermodynamic processes; in practice, this mean-field approach may work even far from the critical point.

Strictly speaking, a mean-field theory may not apply to turbulence where quantities such as the free energy and order parameter cannot be defined unambiguously. In Yakhot's theory, the idea is carried over qualitatively by identifying a small parameter in some regime and expanding other dependent quantities around that small parameter. The ''phase transition'' considered is the change of sign in energy flux that occurs in going from two-dimensional ($2d$) to three-dimensional ($3d$) turbulence. It is understood from Kolmogorov's equation for the third-order structure function that the energy transfer is from the small to the large scale in $2d$ turbulence, and vice versa in $3d$ turbulence. It is assumed that this change is continuous, changing sign at some critical dimension d_c—analogous to the critical temperature T_c in thermodynamic phase transitions. In $2d$ turbulence, the dissipation is negligible for high Reynolds numbers (because the energy ultimately concentrates in the large scale). In $3d$ turbulence, on the other hand, the dissipation is the key to energy transfer from large to small scales. The hope, then, is that both pressure gradient and dissipation can be expanded in terms of a small parameter in the vicinity of d_c.

B. The pressure terms

The energy transfer in turbulence dynamics is usually discussed in terms of longitudinal structure functions (for example, via Kolmogorov's 4/5 law), and transverse structure functions are not assigned a direct role. Yakhot, therefore, regards the fluctuations in the transverse velocity increment Δv_r as small—in effect, if not in actual fact. It is known from numerical simulations [17] as well as experiments [18] of the inverse cascade in $2d$ turbulence that Δv_r is almost exactly gaussian. The absence of intermittency makes it plausible to regard the fluctuations as ''small.'' We shall, therefore, consider the $2d$ case briefly.

The key step for further analysis is the introduction of a conditional expectation of the pressure gradient increment for a fixed value of Δu_r, Δv_r, and r as

$$\langle \partial_y p(x+r)-\partial_y p(x)|\Delta u_r,\Delta v_r,r\rangle \approx \sum_{m,n} \kappa_{m,n}(r)(\Delta u_r)^m(\Delta v_r)^n. \tag{20}$$

This is related to the needed correlations in I_p, which is of the form

$$\langle[\partial_y p(x+r)-\partial_y p(x)](\Delta u_r)^p(\Delta v_r)^q\rangle = \int \langle \partial_y p(x+r)-\partial_y p(x)|\Delta u_r,\Delta v_r,r\rangle \Delta u_r^p \Delta v_r^q \times P(\Delta u_r,\Delta v_r,r)d(\Delta u_r)d(\Delta v_r). \tag{21}$$

The use of the conditional expectation provides a tool for expanding the pressure terms in terms of the ''small quantity'' Δv_r. Now, in the spirit of the Ginzburg-Landau expansion, only the lowest-order terms in Δv_r are retained (corresponding to $\Delta u_r,\Delta v_r$, and Δv_r). The prefactors of the expansion are constrained by the incompressibility condition and by the dimensionality of space.

By substituting in Eqs. (9)–(11) the pressure term derived from the conditional expectation value, and assuming the exponents to be given by from Kolmogorov's K41 scaling arguments [7], Yakhot concludes that the high-order even moments are consistent with gaussianity. The argument is circular but internally consistent. The gaussianity of the transverse increment Δv_r is then deduced from Eqs. (13)–(15). This is in excellent agreement with the results of numerical simulations of Ref. [17]. Thus, we might conclude that a plausible mean-field expression for the pressure contribution exists for $2d$.

The next crucial assumption of the theory is that the same form of the mean-field approximation is applicable also for $3d$ turbulence. The rationale is not easy to articulate, especially because, unlike in $2d$ turbulence, the PDFs of Δv_r possess stretched-exponential tails in $3d$ turbulence [19]. We shall provide some statements of mild justification subsequently, but emphasize that the validity or otherwise of this assumption must be based on the agreement, or lack thereof, with experiments.

C. The small parameter and the dissipation term

We need to consider the dissipation term before returning to experiments. In the inverse cascade range in $2d$ turbulence the dissipation term D can be set to zero because the flow evolution is towards larger and larger scales. However, D is central in $3d$ turbulence, and it is known that dissipation fluctuations are immense at high Reynolds numbers [20]. The objective in a mean-field approach is to locally smooth out the fluctuations, through some procedure such as Obukhov's [21]. For closure, there is a need to relate this coarse-grained dissipation field to velocity fluctuations, analogous to that employed in the refined similarity hypotheses [4,22]. Yakhot's theory is similar in spirit but the details are different, as we shall illustrate.

Let us denote a coarse-grained velocity field for a given spatial scale r by V_r. This will be assumed to be the same as Δv_r. Certain one-loop calculations of Yakhot and Orszag [23] give the effective viscosity as

$$\nu_r \approx (d-d_c)^{1/3} N (\varepsilon_r r^4)^{1/3} \approx V_r^2 \tau_r + (\text{higher-order nonlinear terms}), \tag{22}$$

where N is a constant that depends weakly on the space dimensionality d and ε_r is the dissipation rate coarse grained on the scale r. If we ignore nonlinear terms, this equation provides a natural definition of τ_r, the characteristic time for the field V_r.

There is no obvious justification for ignoring the higher-order nonlinear terms in $3d$ turbulence, which are typically $O(1)$, nor in assuming that τ_r is small compared to V_r/r. However, if we assume that the theory can be analytically continued into noninteger dimensions between 2 and 3, a suitable small parameter can be generated as follows near d_c. The time scale characterizing the interaction of a scale r with all other scales less than r is the so-called eddy turnover time, or the time taken for energy transfer to occur between r and the Kolmogorov scale η. One may use K41 to estimate this time scale. The process of energy transfer can be thought to consist of two distinct steps, one involving nonlinear transfer across scales without any pressure effects, and another involving the relaxation due to pressure effects. In $3d$ turbulence, these two steps are part of the same inseparable process, so the time scales associated with them cannot be separated. But, if, as one approaches d_c, it is increasingly true that the pressure effects are small except when scales of the order η are reached, the two time scales involved could become disparate, and the relaxation due to pressure terms enters the picture only at the smallest scale and can, therefore, be assumed to be fast. Then the dimensionless ratio $q_r \equiv \tau_r/\theta_r$, where τ_r is the time scale for relaxation effects and θ_r is the time scale for energy transfer, would be a small parameter.

Using this basic idea and his one-loop calculations [23], Yakhot deduces the following results:

$$d_c = 2.56, \quad q_r \approx (d-d_c)^{1/2}, \quad V_r \approx (d-d_c)^{-1/6},$$

and so forth. The notion of a critical dimension is not new (see Ref. [24]), though the estimates for it in Refs. [2] and [24] are substantially different. The precise numbers and powers in the above equation depend in detail on the approximation made to compute them, and are presumably not final; they cannot, in any case, be verified experimentally near the critical dimension. Here, we merely wish to draw upon the general idea of a critical dimension near which a small parameter can be defined, and in whose vicinity the energy piles up (as shown by the last of the three relations above: the energy is being pumped at a constant rate but is being transferred neither upscale as in $2d$ nor downscale as in $3d$). These ideas allow Yakhot to truncate the effective viscosity and write the dissipation ε_r in terms of the lowest-order terms in terms of the coarse-grained velocity field *in the vicinity of* d_c, as

$$\varepsilon_r \approx -\frac{1}{2}\frac{\partial}{\partial r_i}\{V_{ri} V_{rj}^2 [1 + O(d-d_c)]\}. \tag{23}$$

Perhaps two additional remarks might be usefully made. First, the coarse-grained velocity fluctuations become very large as the critical dimension is approached, yet it may seem that the mean-field approximation proposed for pressure terms assumes that fluctuations are small. To avoid confusion, it is important to keep in mind the distinction between fluctuations in longitudinal and transverse velocity increments. The velocity scale that blows up is related to energy transfer, and hence the longitudinal velocity component, but the component whose fluctuations are supposed to remain small is the transverse velocity. The sense in which those fluctuations are small is unclear (because they too are intermittent in $3d$, see Ref. [19]), but the fact remains that it takes no part in energy transfer and so its *effects* are thought to be ''small'' in some rough sense. Since the pressure effects are small, the intercomponent energy transfer is inhibited, and so, once fluctuations in Δv_r are small at some scale, they will presumably remain small at others as well. Secondly, in order to be able to truncate the energy dissipation, the higher-order viscosity terms have to decay faster than the rate of blowup of velocity fluctuations. This is indeed the case above.

Now, keeping in the mind the symmetries of the NS equations, the simplest form for the V_r contributions to the dissipation rate is

$$\varepsilon \approx c(d)\Delta u_r \Delta v_r \frac{\partial \Delta v_r}{\partial r}. \tag{24}$$

The coefficient $c(d)$ must reflect the change in going from $2d$ to $3d$ (zero dissipation to finite dissipation). This may not

be a smooth change (as in second-order phase transitions) because $c(d_c)$ could well be singular (as in first-order phase transitions). Yakhot assumes, however, that it is $O(1)$ for $d-d_c>0$. Then D takes on a form similar to Kolmogorov's refined similarity hypothesis, relating $\langle \varepsilon_r \rangle$ with the third-order longitudinal structure function

$$D \approx c(d) h_3 \partial_{h_2} \partial_{h_3} \partial_r Z$$
$$\approx c(d) h_3^3 \left\langle \Delta u_r \Delta v_r \frac{\partial \Delta v_r}{\partial r} \exp(h_2 \Delta u_r + h_3 \Delta v_r) \right\rangle$$
$$+ \text{terms neglected.} \tag{25}$$

This enables the closure to be complete.

A nontrivial difficulty is the testing of the theory in noninteger dimensions near d_c. At present, the consequences of the theory can only be tested in $2d$ or $3d$. The extrapolation to noninteger dimensions is not an intrinsic limitation of the theory, but reflects the lack of experimental ingenuity at present. Simulations offer a better opportunity. It must, however, be noted that in shell models where an interaction parameter can be tuned to change the direction of energy transfer, one can make more reasonable contact with the theory. Such comparisons have been attempted recently [25] and the results are encouraging. We have already noted that the conclusions of the theory are consistent with experiments and simulations of $2d$ turbulence. We shall examine in the rest of the paper the extent to which the predictions of the theory are applicable also to $3d$.

VI. COMPARISON OF THE THEORY WITH MEASUREMENTS IN THREE-DIMENSIONAL TURBULENCE

A. Probability density function of transverse velocity increments

When the forms of I_p and D from previous sections are substituted into the full structure function equations, one can generate the following equation for the PDF of transverse velocity increments $P(\Delta v_r, r) \equiv P(V,r)$:

$$\frac{\partial P}{\partial r} + \frac{1+3\beta}{3r} \frac{\partial}{\partial V} VP - \beta \frac{\partial}{\partial V} V \frac{\partial P}{\partial r} = 0. \tag{26}$$

Here, $\beta \propto c(d)$. This equation is linear and can be solved in principle, but we have found no simple analytic form of the solution. (For some discussion of this aspect, see Ref. [26].) For small V, however, the equation admits a solution of the type $r^{\kappa} F(V/r^{\kappa})$ with

$$P(V=0,r) \propto r^{-\kappa}, \tag{27}$$

where, from Yakhot's theory,

$$\kappa \equiv \frac{1+3\beta}{3(1-\beta)} \approx 0.4 \tag{28}$$

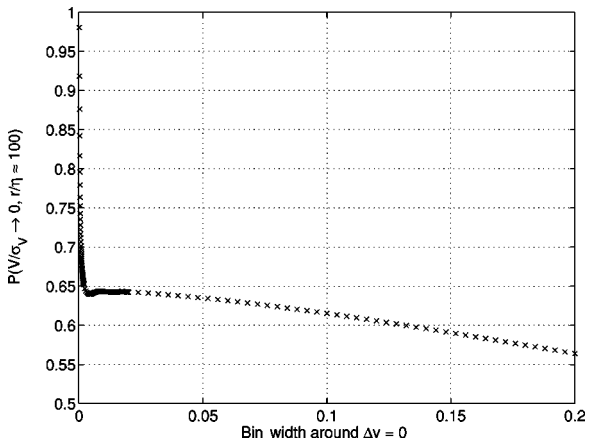

FIG. 11. The value of the PDF as $V/\sigma_V \to 0$ for $r \approx 100\eta$ for different binwidths.

and $\beta \approx 0.05$, a semiempirical estimate given in Ref. [2]. We estimate the constant independently from our data. Using the fact that the variance σ_V of V is expected to vary as $r^{0.35}$ (see Ref. [8]), we have the result

$$P\left(\frac{V}{\sigma_V} \to 0, r\right) \equiv \sigma_V P(V \to 0, r) \approx r^{-0.052}. \tag{29}$$

We shall now test these predictions.

The precise measurement of the peak value of the PDFs from the data must be done carefully because it is sensitive to the bin width chosen around $V=0$. In our measurements, the bin width around $V=0$ was gradually refined until the PDF value at the origin no longer depended on the bin size. Figure 11 shows that $P(V \to 0)$ at $r \approx 100\eta$ asymptotes to a value of 0.64. The sharp ascent of the numbers for very small values of the bin width is an artifact of the extreme narrowness of the bin width, which results in falsely large values due to normalization. This is to be ignored. The procedure was repeated for several values of r. Figure 12 shows the properly normalized PDF values for $V=0$ for different scales r ranging from the Kolmogorov scale η to the large scale L. The scaling exponent for this quantity is ≈ -0.065

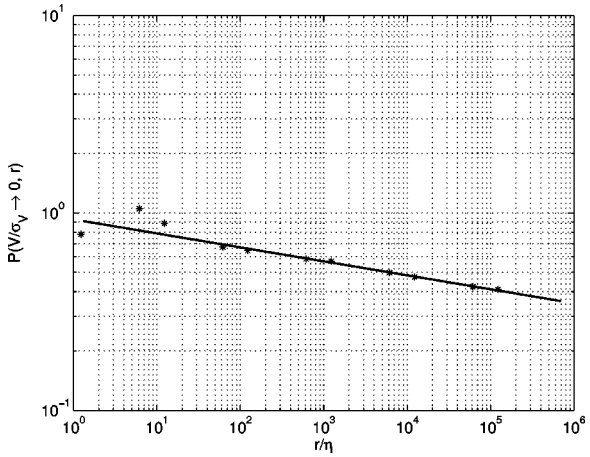

FIG. 12. Log-log plot of the peak values of the PDFs of transverse velocity increments. The line indicates a slope of -0.065.

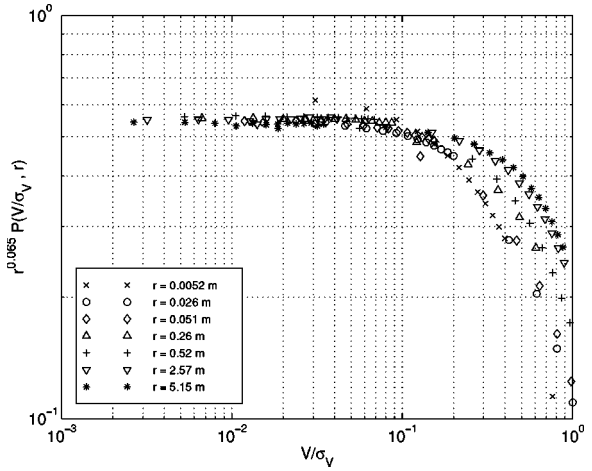

FIG. 13. Log-log plot of the near-peak values of the PDFs of transverse velocity increments. The collapse of the data occurs for the normalization $r^{\kappa}P(V/r^{\kappa})$, where $\kappa=0.065$.

in the inertial range, numerically about 25% larger than the theoretical value of -0.052. With this experimentally derived scaling exponent, we can evaluate that $\beta\approx 0.058$ compared to the estimate in Ref. [2] of 0.05, a 16% difference. Figure 13 also shows that the form $r^{0.065}P(V/\sigma_V)$ is essentially constant for small V.

One can obtain the form of F for large V by a steepest descent approximation (see Ref. [26] for more details). The result from Ref. [2] is

$$F\propto\frac{1}{\sqrt{\Omega(r)}}\exp\left(-\frac{(\ln\xi)^2}{\Omega(r)}\right), \tag{30}$$

where $\xi=VL^{\beta\kappa/\beta-1}/r^{\kappa(1-2\beta)/1-\beta}$, and $\Omega(r)=\beta\kappa/(1-\beta)^2|\ln(r/L)|$. (The corresponding expressions in Ref. [2] are printed incorrectly.) The prefactor of Eq. (30) is possibly r dependent. Equation (30) can be rewritten as

$$[-\Omega(r)\ln\{P(V,r)r^{\kappa}\sqrt{\Omega(r)}\}]^{0.5}\propto\ln(\xi). \tag{31}$$

Figure 14 shows plausible linear behaviors for the intermediate range (between peaks and tails) of the PDF in the proposed logarithmic units of Eq. (31). There is, however, evidently still some r dependence that precludes their collapse. We recall that corrections to steepest descent approximations are often logarithmic, but are difficult to calculate here analytically. We assume a dependence of the form $[\ln(r/L)]^{\gamma}$ for the proportionality factor. Figure 15 shows a replot of the data with the additional factor of $[\ln(r/L)]^2$ multiplying the PDF. The exponent 2 was chosen because it collapses the data best in the inertial range. (The one separation distance that does not collapse belongs to the dissipation range.)

Our main conclusion so far is that the mean-field models for pressure and dissipation terms provide a means for closing the PDF equation, and for solving it for the limiting situations. The prediction is that, to first approximation, the intermediate range estimates of the PDF of Δv_r are lognormal for moments $n<20$. The experimental data suggest that this might be so, but that an r-dependent contribution is

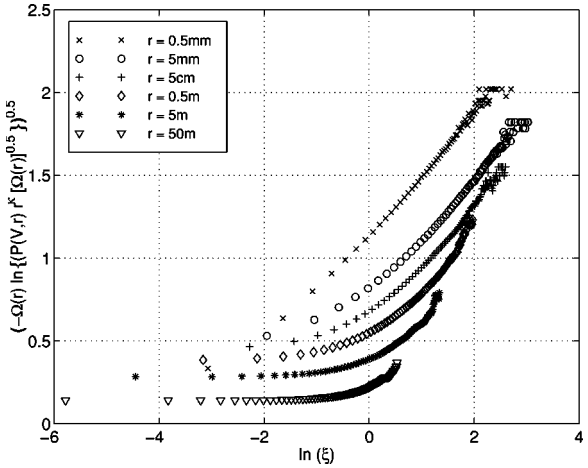

FIG. 14. The tails and the intermediate regions of the PDFs in the form required by the theory [see Eq. (31)]. Lognormality of the intermediate range (between peaks and tails) requires that the data follow straight lines.

missing. It is at present not clear whether this missing aspect is merely a correction to asymptotics, or corresponds to additional terms in the mean-field expansion, or is even more fundamental.

B. The scaling exponents and the prospect of their saturation

Seeking the solution to Eq. (26) under the K41 constraint for the third-order structure functions and assuming $S_{0,n}\propto r^{\zeta_n}$, Yakhot obtained the following formula for the structure function exponents:

$$\zeta_n=\frac{n(1+3\beta)}{3(1+\beta n)}. \tag{32}$$

Table I and Fig. 16 show the calculated exponents and compare them with those obtained from the direct numerical simulations (DNS), data [27] as well as experiments. The agreement is good for all orders, perhaps slightly better for the DNS data for high-order exponents. The formula indi-

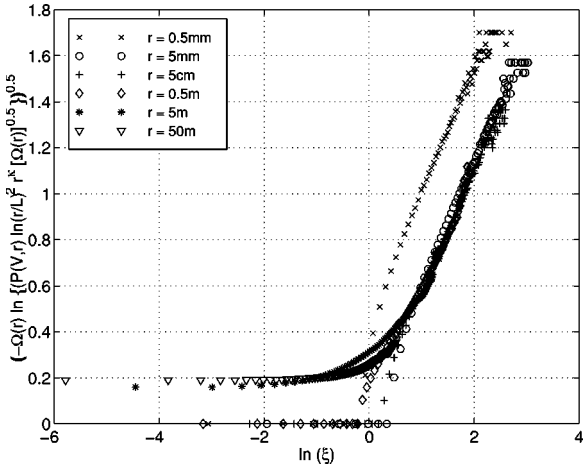

FIG. 15. The collapsed PDF from the previous plot, taking into account an *ad hoc* prefactor of $[\ln(r/L)]^2$.

TABLE I. Comparison of exponents from the DNS data and the experiment (both using ESS) and Yakhot's formula, Eq. (32).

Order	DNS	Experiment	From Eq. (32)
−0.80	−0.317	−0.313	−0.328
−0.20	−0.077	−0.078	−0.079
0.10	0.036	0.039	0.039
0.20	0.073	0.076	0.077
0.30	0.112	0.113	0.115
0.40	0.150	0.150	0.153
0.50	0.187	0.190	0.190
0.60	0.223	0.221	0.227
0.70	0.260	0.265	0.263
0.80	0.296	0.292	0.299
0.90	0.332	0.333	0.335
1.00	0.366	0.372	0.370
1.25	0.452	0.458	0.456
1.50	0.536	0.542	0.540
1.75	0.619	0.628	0.622
2	0.699	0.708	0.701
3	1	1	1
4	1.279	1.26	1.271
5	1.536	1.56	1.517
6	1.772	1.71	1.742
7	1.989	1.97	1.948
8	2.188	2.05	2.138
9	2.320	2.20	2.314
10	2.451	2.38	2.477

cates saturation for large n, beyond about 20 or so, but this range is inaccessible at present to experiment as well as simulations.

Using probability density functions to define the statistical quantities, we have (putting $U=\Delta u_r$) the conditional expectation value of V^2 for a fixed value of U, $Q_2(U)$, as

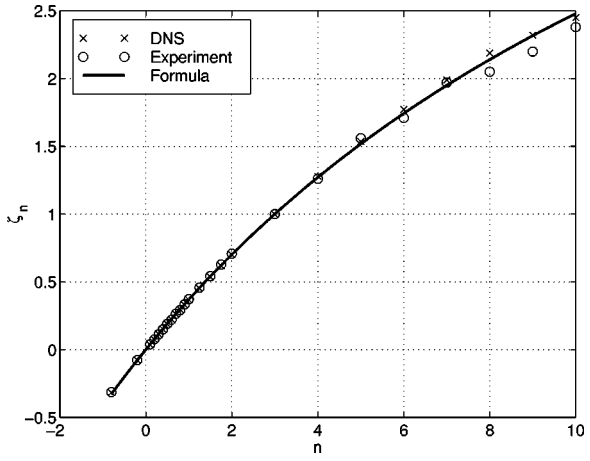

FIG. 16. Comparison of the DNS and experimental values of ζ_n with those from Eq. (32). The numbers in the plot are for longitudinal structure functions, but these are identical to those for transverse structure functions in the isotropic sector see Refs. [10,11].

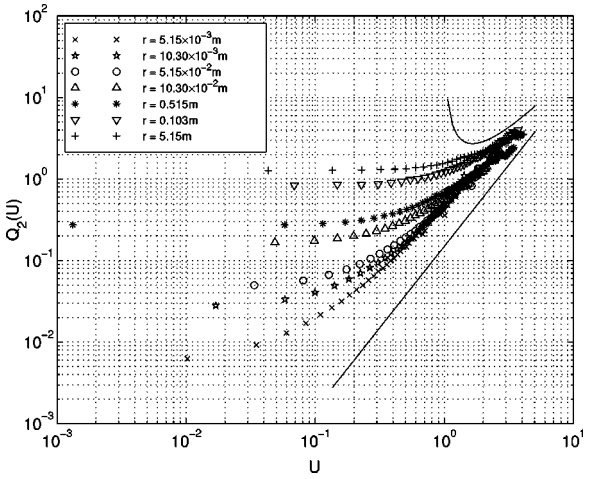

FIG. 17. $Q_2(U)$, the conditional expectation value of V^2 on U for various different r, on a log-log scale. The upper solid line indicates $U^2/\log(U^2)$, the lower solid line indicates the U^2 scaling slope.

$$S_{2n-2,2}=\int P(U)U^{2n-2}Q_2(U)dU. \tag{33}$$

See Sec. IV of Ref. [2]. The Kolmogorov scaling will hold (by dimensional arguments) for $Q_2(U)\propto U^2$. On the other hand, saturation of exponents, $\zeta_{2n}\rightarrow$ constant as $n\rightarrow\infty$, is possible for $Q_2(U)\propto U^\delta$ for $\delta<2$ and U large. We present the conditional statistical quantity $Q_2(U)$ as a function of r in Fig. 17. It is not clear if the trend for large U is in agreement with the saturation condition. There is a very small range of U towards the tails that seems to vary as $U^2/\ln(U^2)$ but this is not conclusive. There might also be the influence of anisotropy in the PDFs, as is evident, for example, in the asymmetry of the joint PDFs, which in turn could change the nature of the tails of conditional statistics.

VII. REMARKS ON THE MAGNITUDE OF DISSIPATION TERMS

It is helpful to recall that the equation relating even order transverse moments to mixed moments of the same order in $3d$, without dissipation terms, is

$$\frac{\partial S_{2,2n}}{\partial r}+\frac{2+2n}{r}S_{2,2n}=\frac{2+2n}{2n+1}\frac{S_{0,2n+2}}{r}-2n\langle\mathcal{P}_y\Delta u(\Delta v)^{2n-1}\rangle - \langle\mathcal{P}_x(\Delta v)^{2n}\rangle, \tag{34}$$

where $\mathcal{P}_y\equiv\partial_y p(x+r)-\partial_y p(x)$ and $\mathcal{P}_x\equiv\partial_x p(x+r)-\partial_x p(x)$. Note that the last term in this equation is inadvertently omitted in Ref. [2]. The subscripts x,y denote the component of the pressure gradient. In the inertial range, the dissipation term is small for even orders and the forcing term negligible. In certain equations, only the $\mathcal{P}_x$ term of the pressure effects appears, and in certain others only $\mathcal{P}_y$ does. Since all other terms can be measured in such equations, we can use those equations and the mean-field pressure model to estimate $\mathcal{P}_x$ and $\mathcal{P}_y$, and substitute the estimates in equations

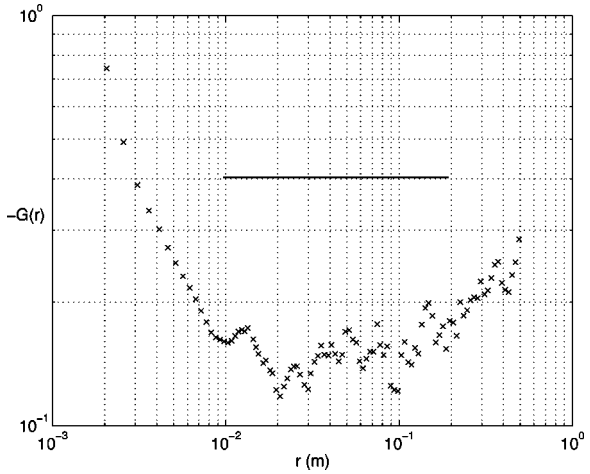

FIG. 18. The coefficient G of the pressure term computed according to Eq. (38.)

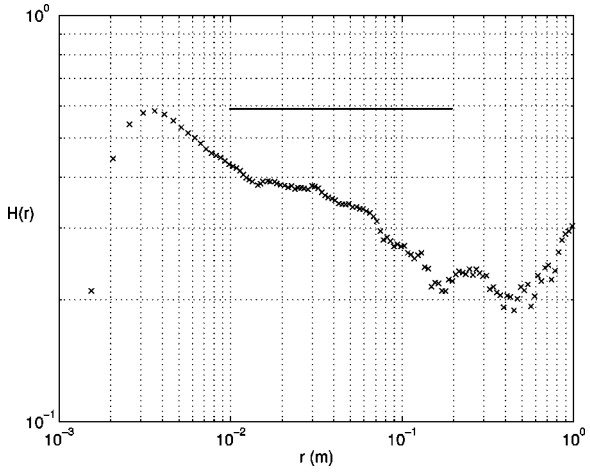

FIG. 19. The coefficient H of the pressure term computed according to Eq. (42).

for the odd order that contain the form of the pressure terms. From these steps, the unknown terms in the odd-order equations, namely, the dissipation terms, can be estimated. This is the strategy here.

Retaining the pressure term in Eq. (8), we have

$$\frac{\partial S_{4,0}}{\partial r}+\frac{2}{r}S_{4,0}=\frac{6}{r}S_{2,2}-3\langle \mathcal{P}_x(\Delta u)^2\rangle, \tag{35}$$

in which, according to the mean-field approximation,

$$\mathcal{P}_x=\frac{G(\Delta u)^2+A(\Delta v)^2}{r}, \tag{36}$$

and G and A are unknown constants constrained by the relation $A=-G(S_{2,0}/S_{0,2})$ [Eq. (8) with $n=1$]. This model for the pressure term may be used to close Eq. (35) giving

$$\frac{\partial S_{4,0}}{\partial r}+\frac{2}{r}S_{4,0}=\frac{6}{r}S_{2,2}-\frac{3}{r}G\left(S_{4,0}-\frac{S_{2,0}}{S_{0,2}}S_{2,2}\right). \tag{37}$$

All the terms in this equation may be calculated explicitly from the data and the constant G determined via

$$G=\frac{\dfrac{\partial S_{4,0}}{\partial r}+\dfrac{2}{r}S_{4,0}-\dfrac{6}{r}S_{2,2}}{-\dfrac{3}{r}\left(S_{4,0}-\dfrac{S_{2,0}}{S_{0,2}}S_{2,2}\right)}. \tag{38}$$

Figure 18 shows G as a function of r. Despite some scatter, we estimate an inertial range value of $G=-0.15$. This provides the $\mathcal{P}_x$ part of the pressure term in Eq. (34). The $\mathcal{P}_y$ part in Eq. (34) may be modeled by

$$\mathcal{P}_y=-H\frac{\Delta u_r\Delta v_r}{r}-B\frac{\Delta v_r}{(Pr)^{2/3}}, \tag{39}$$

where H and B are unknown constants, and P, as before, is the rate of forcing. The second term is chosen in order that the pressure term in the third-order equation $\overline{\mathcal{P}_y\Delta v_r}=0$ (see Ref. [28] for a proof of this result). The two constants H and B are then related through

$$B=-H\frac{S_{1,2}}{S_{0,2}}\frac{P^{2/3}}{r^{1/3}}. \tag{40}$$

These models for $\mathcal{P}_x$ and $\mathcal{P}_y$ provide closure for Eq. (34), which now becomes

$$\begin{aligned}\frac{\partial S_{2,2n}}{\partial r}+\frac{2+2n}{r}S_{2,2n}=&\frac{2+2n}{2n+1}\frac{S_{0,2n+2}}{r}-2n\frac{H}{r}\\&\times\left(S_{2,2n}-\frac{S_{1,2}}{S_{0,2}}S_{1,2n}\right)-\frac{G}{r}\\&\times\left(S_{2,2n}-\frac{S_{2,0}}{S_{0,2}}S_{0,2n+2}\right).\end{aligned} \tag{41}$$

All the terms in this equation may be computed from the data and, with $G=-0.15$, the only remaining constant H may be obtained. For $n=1$,

$$H=\frac{\dfrac{\partial S_{2,2}}{\partial r}+\dfrac{4}{r}S_{2,2}-\dfrac{4}{3}\dfrac{S_{0,4}}{r}+\dfrac{G}{r}\left(S_{2,2}-\dfrac{S_{2,0}}{S_{0,2}}S_{0,4}\right)}{-\dfrac{2}{r}\left(S_{2,2}-\dfrac{S_{1,2}}{S_{0,2}}S_{1,2}\right)}. \tag{42}$$

Figure 19 shows H as a function of r calculated from the data using Eq. (42). There is a more or less steady decrease of H as r increases. In the inertial range, the value decreases from about 0.42 to about 0.29. The uncertainty in this estimate does not allow us to be definitive but we shall proceed with an average value of $H=0.37$ in the inertial range of in what follows. Once the pressure terms are accounted for by the model just evaluated, the remaining imbalance in equations for odd orders must come from dissipative contributions alone.

Consider one such odd-order structure function equation

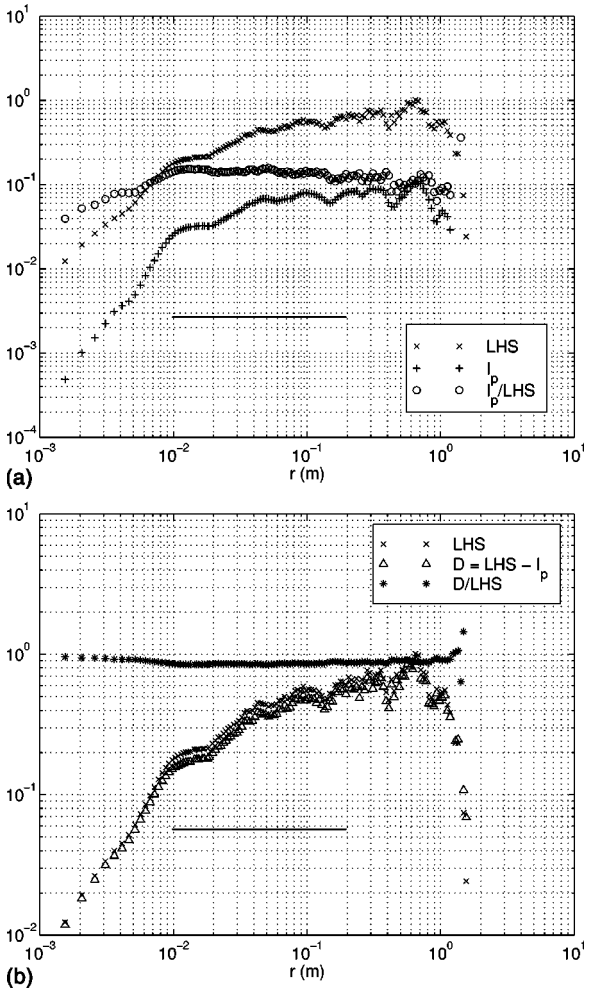

FIG. 20. (a) The LHS and pressure terms of Eq. (43) with $n=2$ and $H=0.37$ in the pressure model; the ratio indicates that the pressure term as computed from the model is about 10–15% of the balance. (b) The terms of Eq. (43) with $n=2$, $H=0.37$; the difference LHS$-I_p$ [where I_p is calculated as in (a)], indicates the magnitude of the dissipation term.

$$\frac{\partial S_{1,2n}}{\partial r}+\frac{2+2n}{r}S_{1,2n}=-2n\langle \mathcal{P}_y(\Delta v_r)^{2n-1}\rangle+D. \quad (43)$$

We now substitute the pressure model of Eq. (39) into Eq. (43) with $n=2$ [i.e., Eq. (18) with pressure and dissipation terms included], in order to estimate the only unknown term D; we use $H=0.37$ and B given by Eq. (40) in the substitution. Figure 20(a) shows that the pressure term I_p computed in this manner contributes between 10 and 15% to the balance. Figure 20(b) shows that the dissipation term, being the remainder ($D=$LHS$-$RHS$-I_p$), dominates the right-hand side of the equation, and is much larger than the pressure term; it alone balances about 85 to 90% of the LHS of Eq. (43).

There is another equation, Eq. (17), for fifth-order structure functions that contains both kinds of contributions $\mathcal{P}_x$ and $\mathcal{P}_y$ in the pressure term. In full, it has the form

$$\frac{\partial S_{3,2}}{\partial r}+\frac{4}{r}S_{3,2}=\frac{8}{3r}S_{1,4}-2\langle \mathcal{P}_x\Delta u(\Delta v)^2\rangle -2\langle \mathcal{P}_y(\Delta u_r)^2\Delta v_r\rangle+D. \quad (44)$$

We now follow a similar procedure as before. With known values for G and H in the mean-field model, the pressure terms are

$$-2\langle \mathcal{P}_x(\Delta u_r)(\Delta v_r)^2\rangle=-2\frac{G}{r}\left(S_{3,2}-\frac{S_{2,0}}{S_{0,2}}S_{1,4}\right),$$

$$-2\langle \mathcal{P}_y(\Delta u_r)^2\Delta v_r\rangle=2\frac{H}{r}\left(S_{3,2}-\frac{S_{1,2}}{S_{0,2}}S_{2,2}\right). \quad (45)$$

We have already seen in Fig. 9 that the RHS of Eq. (17) balances the LHS up to about 80% in the inertial range. The imbalance of about 20% is due to a possible mix of pressure and dissipation terms, both of which are now included in Eq. (44), thus completing Eq. (17). The pressure term computed from Eqs. (45) is shown in Fig. 21(a). It makes a 10% contribution in the inertial range. The dissipation term from the remainder is plotted in Fig. 21(b); while it shows significant scatter, it is clearly small in the inertial range (of the order of 15% or less) while increasing, as it must, toward dissipative scales. On the whole, it appears that both pressure and dissipation are relatively small compared to the RHS of Eq. (17), with dissipation terms taking over towards the small scales.

From the above two examples it appears that the mere knowledge of the overall order of the structure function (in this case the fifth) is not enough to prescribe the importance of the pressure and dissipation terms. The equations that relate different *components* of the fifth-order structure function tensor have different structures. While equation Eq. (44) seems to balance more or less without pressure and dissipation terms, in Eq. (43) the dissipation term is overwhelmingly large.

VIII. CONCLUDING REMARKS

Our experimental results are assessed in the context of a mean-field model due to Yakhot, using as framework the new equations derived for structure functions of all orders. The model allows us to write the pressure terms that we cannot measure directly, in terms of the velocity structure functions that we can measure. (The pressure terms appear here in a different form from those used in turbulence modeling, and so the value of the present work to that endeavor is unclear.) Among the assumptions made, the most drastic one is the use of the same pressure model for $2d$ and $3d$ turbulence.

Nevertheless, if we adopt the pressure model in Eq. (34) in which the dissipation terms are thought to be negligible (see Refs. [2] and [3] for symmetry and asymptotic arguments as to why this might be so), the coefficients H and G can be obtained, and thus the pressure terms can be modeled. We can now proceed to analyze odd-order equations that have the same structure for pressure terms. Since the pres-

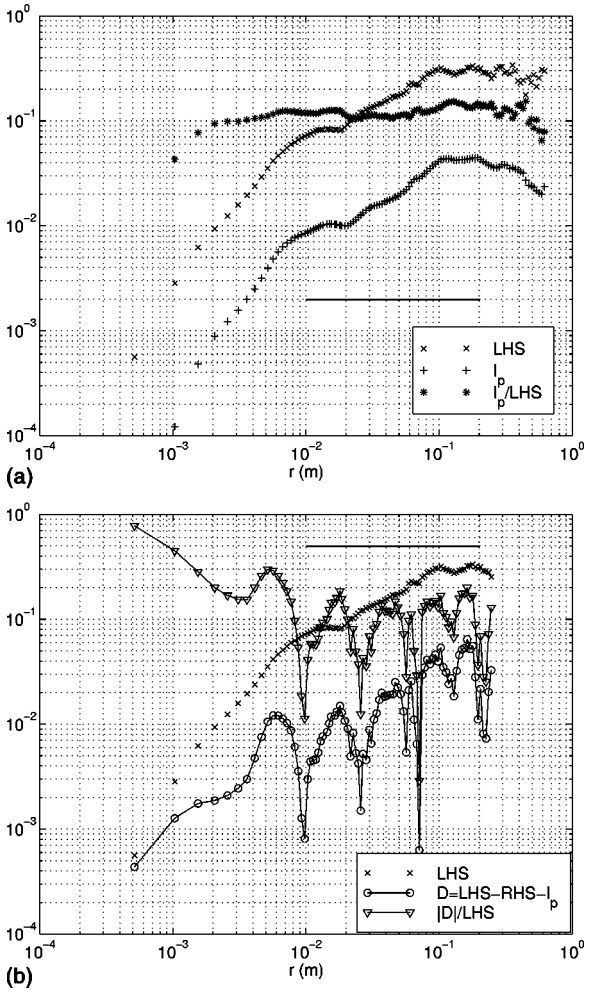

FIG. 21. (a) The LHS and pressure terms I_p of Eq. (44) with their ratio showing that the pressure only accounts for about 10–12% of the balance, less as one approaches the dissipative scales and large scales. (b) The LHS and dissipation terms D of Eq. (44) with the dissipation computed by subtracting the RHS and the pressure [see (a)] from LHS; the ratio of D to LHS indicates a large scatter but shows that the dissipation contributes in the inertial range only about 10%, but that it increases significantly towards the dissipative scales.

sure term is known, we can deduce the only remaining term, namely, the dissipation. For one equation (43), the dissipation term is of the order of 80% of the balance. Another dynamical equation (44) for the same order of the structure function has a different structure, and there, the dissipation term is relatively small. This is a new and interesting statement about the inertial range dynamics, but its validity depends on the pressure model used. At least one outcome of the calculations is tautologically correct: in all the cases considered here, the dissipation range is always dominated by the dissipation term D.

Yakhot's theory postulates the existence of a critical dimension, d_c. This, in itself, is not implausible [24]. However, the precise numerical value of d_c, the analytic structure of the NS equations in the neighborhood of d_c and the extent of its ''neighborhood'' remain unclear. The theory yields certain exponents for the vicinity of d_c, but the details on which they are based need closer scrutiny; at least to us, some of the steps remain unclear. Thus, while the numerical values of the exponents, as well as that of d_c itself, are probably not to be taken literally, we should be interested in drawing some qualitative conclusions.

Such conclusions come from a few independent sources. First, the prediction of the theory for the PDF of Δv_r for $2d$ turbulence is in good agreement with simulations and experiments [17,18]. Second, the conditional expectation of the pressure terms in $3d$ simulations [29] appear to follow the mean-field theory, at least for modest values of the velocity increments. Third, shell model calculations [25] show that the behavior expected near the critical dimension can be observed as one varies a coupling parameter. Finally, the present comparisons with experimental data at high Reynolds numbers reveal that the scaling of the PDF of Δv_r for small and large Δv_r are in some measure of agreement with the theory. All these are positive developments. However, since many details are unclear, it remains to be seen as to whether the theory will evolve into a rational framework. For now, we find it to be both interesting and worthy of some attention.

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