IDS160B - Relativistic Astrophysics and Cosmology
The "NAUGHTS" Notes and Problem Set 9 1995 May

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1 The "NAUGHTS"

The present value X_0 of cosmological parameters such as the Hubble expansion rate H_0, the deceleration parameter q_0, the density of the Universe in units of the critical density Ω_0, the age of the Universe t_0 and its constituents, as well as things like the current cosmological constant Λ_0 are critical to our understanding and ability to make quantative calculations about the Universe and various models. In these notes and this problem set we review the relationships between the "Naughts" for various models and some of the observational techniques/strategies and results discussed in class.

2 Expanion of the Scale Factor

This section reviews the expansion of the scale factor of the universe. The Robertson-Walker metric (the metric for an isotropic homogeneous universe) is

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \tag{1}$$

The Taylor series expansion around t_0 of the scale factor $a(t)$ is

$$\begin{aligned} a(t) &= a(t_0) + \dot{a}(t_0)(t - t_0) + \frac{1}{2}\ddot{a}(t_0)(t - t_0)^2 + \ldots \\ &= a_0 \left(1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + \ldots \right) \end{aligned} \tag{2}$$

where the Hubble $H \equiv \dot{a}/a$ and decelaration $-qH^2 \equiv \ddot{a}/a$ parameters are evaluated at t_0 are H_0 and q_0 respectively.

There are several tests of the cosmological models or parameters that depend directly only on H_0 and q_0. However, there are some that depend upon assuming some kind of dynamical model of the universe such as General Relativity and a restriction on the possible content of the Universe. We deviate here to consider the simplest version of those models.

3 Show $2q_0 = \Omega_0$ in a low pressure model

Most astronomical measurements of the cosmological parameters occur for redshifts less than about 10 where the universe is matter dominated and relatively pressureless.

If one assumes the Friedmann-Einstein equation for acceleration

$$\ddot{a} = -\frac{4\pi}{3} G(\rho + 3P)a \tag{3}$$

show that this implies that $q_0 = \frac{1}{2}\Omega_0$, where $\Omega = \rho/\rho_c \equiv \rho\ (8\pi G)/(3H_0^2)$ provided $P << \rho$. (Or define $\Omega = (\rho + 3P/c^2)/\rho_c$.)

3.1 Pressure in the current universe

How do we know that the intergalactic medium does not produce a high pressure and thus this relationship is not valid? What is the intergalactic medium?

Observations of distant quasars show absorption lines (Lα) that indicate the presence of gas clouds with a comoving space number density that varies roughly as $(1+z)^2$ at a redshift of $z \sim 3$ meaning that the clouds are dissipating or disappearing. The total hydrogen mass in atomic hydrogen is quite small and there is even less in the space between clouds. This observation tells us that structure formation was well advanced by this epoch and that the intergalactic medium was already rare.

3.1.1 The Gunn-Peterson test

The scattering cross section integrated through the resonance in the 1s to 2p atomic hydrogen Lα transition is sufficiently large for observable absorption from relatively small column densities of neutral hydrogen. This can be used for a test of a smoothly distributed intergalactic atomic hydrogen in the range $z \lesssim 2$, where the Lα resonance is redshifted into the optical band observable from the ground (at $\lambda \gtrsim 3200 \mathring{A}$). The optical depth can be calculated to be

$$\begin{aligned} \tau &= \frac{3\Lambda\lambda_\alpha^3 n_1}{8\pi H_0 \Omega^{1/2}}(1+z)^{-3/2}, \\ n_1 &= 2.4 \times 10^{-11}\Omega^{1/2} h(1+z)^{3/2}\tau \text{ cm}^{-3}, \end{aligned} \tag{4}$$

For $z \lesssim 4$, $\tau \lesssim 0.1$ and the smoothly distributed atomic hydrogen number density is less than

$$\begin{aligned} n_1 &= 2 \times 10^{-12}\Omega^{1/2} h(1+z)^{3/2} \text{ cm}^{-3}, \\ \Omega_{n1} &= 2 \times 10^{-7}\Omega^{1/2} h^{-1}(1+z)^{-3/2}. \end{aligned} \tag{5}$$

Note that this is much smaller than the observed matter density in stars/galaxies and that inferred by BBNS.

3.1.2 The 21-cm trough test

Molecular hydrogen (H_2) produces 21-cm wavelength radiation and radio receivers are very sensitive. If there is a significant presence of relatively uniform molecular hydrogen in the universe there would be an absorption trough beginning at 21-cm and extending to longer wavelengths. Many people and myself personally have seen that there is no significant trough leading to a limit of

$$\begin{aligned} n_2 &\leq 4 \times 10^{-9} h \ \mathrm{cm}^{-3}, \\ \Omega_{h2} &= 7 \times 10^{-4} h^{-1} \quad \text{for} \ \ z \lesssim 2. \end{aligned} \tag{6}$$

3.1.3 The COBE Limit on Ionized Medium Pressure

Note that the COBE spectrum limits

$$y = \int \frac{kT}{m_e c^2} d\tau = \int \frac{kT}{m_e c^2} \sigma_T n_e dl < 2 \times 10^{-5} \tag{7}$$

where σ_T is the Thomson cross section, n_e is the electron number density and dl is the path length differential gives a limit on the pressure $P \approx n_e kT \leq y \ m_e c^2/(l\sigma_T)$ in the Universe. Relative to the mass density $\rho \cong \rho_b \approx (m_p + m_e)c^2 \ n_H$, one has

$$P/\rho \approx y \frac{m_e}{m_p + m_e} \frac{1}{n_H \sigma_T} \frac{H_0}{c} \lesssim \frac{2 \times 10^{-12}}{n_H} \ \mathrm{cm}^{-3} \tag{8}$$

or

$$P/\rho \approx y \frac{m_e c^2}{\rho_b} \frac{1}{\sigma_T} \frac{H_0}{c} = y \frac{80 h^{-1} \mathrm{eV}}{\rho_b c^2} \approx y \frac{10^{-2}}{\Omega_b} h^{-3} \lesssim 2 \times 10^{-4} \tag{9}$$

4 Speed of Sound

We digress further to compare to the pressure in the universe before decoupling and thus the speed of sound and the location of the first "Doppler" peak which is one way to measure the curvature of the Universe. Compute the speed of sound for the baryon-photon fluid from the relativistic era until decoupling. Estimate how much it changes the sound horizon at decoupling and thus angle of the "Doppler" peak in the observations via the cosmic background radiation anisotropy.

Suggestion: Use the relationship

$$c_s^2 \equiv \frac{\partial P}{\partial \rho} = \dot{P}/\dot{\rho} \tag{10}$$

after computing the pressure P and density ρ for the baryon-photon fluid as a function of redshift. Show that to first order

$$c_s \simeq \frac{c}{\sqrt{3}} \times \frac{1}{\sqrt{1 + \frac{3}{4}\frac{\rho_b}{\rho_\gamma}}} \tag{11}$$

Make use of the relation $\rho_b = 4.0 \times 10^4 (1+z)^{-1}\ \Omega_b h^2\ \rho_\gamma$ and make a plot of c_s for redshifts greater than 1100 (put time labels as well as redshift tics on the horizontal axis). How much does this alter the relationship

$$\theta_{sh} \simeq \sqrt{\frac{\Omega}{3\ z_{dec}}} \tag{12}$$

you found in problem set 7 assuming that $c_s = c/\sqrt{3}$?

This case (before recominabination) is currently the only case we consider where the matter-dominated pressureless universe approximation is not valid. We return to the calculations.

5 Back to the Naughts

We assume the Friedmann-Einstein equations

$$\ddot{a} = -\frac{4\pi}{3} G(\rho + 3P)a \tag{13}$$

and

$$(\frac{\dot{a}}{a})^2 - \frac{8\pi}{3} G\rho = -\frac{kc^2}{a^2} \tag{14}$$

and derive the relationships. The second (conservation of energy) equation can be expressed as

$$H^2 - \frac{8\pi}{3} G\rho = -\frac{kc^2}{a^2} \quad \text{or as} \quad \frac{8\pi}{3} G\rho = (\frac{\dot{a}}{a})^2 + \frac{kc^2}{a^2} \tag{15}$$

and multiplying the first (acceleration) equation by $2/a$

$$2\frac{\ddot{a}}{a} \equiv -2qH^2 = -\frac{8\pi}{3} G(\rho + 3P) \tag{16}$$

using the definition of the deceleration parameter $\ddot{a}/a \equiv -qH^2$, in the case we assume that $\rho >> 3P$ combining the equations gives

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + kc^2}{a^2} = 0 \tag{17}$$

or using the definition of H and q and regrouping

$$\begin{aligned} (2q-1)H^2 &= \frac{kc^2}{a^2} \\ (2q_0 - 1)H_0^2 &= \frac{kc^2}{a_0^2} \\ a_0^2 &= \frac{kc^2}{(2q_0-1)H_0^2} \quad \text{for later use} \end{aligned} \tag{18}$$

where the second equation comes from evaluating at t_0. This relationship is valid for the pressureless universe. We can regroup to find an expression for the deceleration parameter using both equations

$$\begin{aligned}
q &= \frac{1}{2}\frac{8\pi G}{3H^2}\rho = \frac{1}{2H^2}\left(H^2 + \frac{kc^2}{a^2}\right) \\
q &= \frac{1}{2}\Omega \\
q_0 &= \frac{1}{2}\Omega_0
\end{aligned} \tag{19}$$

This again shows the relation $q = \Omega/2$ is valid for the pressureless universe.

Now we return to the conservation of energy equation and add the assumption that universe is matter dominated so that $\rho = \rho_0 * (a_0/a)^3$. and which we can express as

$$\dot{a}^2 + kc^2 = \frac{8\pi G\rho_0}{3}\frac{a_0^3}{a} = \Omega_0 H_0^2 \frac{a_0^3}{a} = 2q_0 H_0^2 \frac{a_0^3}{a} \tag{20}$$

Now we can use the equations above to get a value for a_0^3 in terms of q_0, H_0, and c and eliminate it from this equation.

$$a_0^3 = \frac{c^3}{H_0^3}\left(\frac{k}{2q_0 - 1}\right)^{3/2} \tag{21}$$

yielding

$$\dot{a}^2 = c^2\left(\frac{\alpha}{a} - k\right) \tag{22}$$

where

$$\alpha \equiv \frac{\Omega}{[k(\Omega - 1)]^{3/2}}\,\frac{c}{H_0} \tag{23}$$

Now we have a simple differential equation for $a(t)$ in a matter-dominated, pressureless universe.

$$\frac{da}{dt} = \dot{a} = c\left(\frac{\alpha}{a} - k\right)^{1/2} \tag{24}$$

We can easily separate variables and integrate

$$\int c\; dt = \int \sqrt{\frac{a}{(\alpha - ka)}}\; da \tag{25}$$

First consider $k = +1$, the closed universe case.

$$\int c\; dt = \int \sqrt{\frac{a}{(\alpha - a)}}\; da \tag{26}$$

and make the substitution

$$a = \alpha sin^2\frac{\Theta}{2} = \frac{1}{2}\alpha(1 - cos\Theta) \tag{27}$$

The integral then becomes

$$ct = \int \alpha sin^2\frac{\Theta}{2}d\Theta = \frac{1}{2}\alpha(\Theta - sin\Theta) \tag{28}$$

where we have set the limits of the integration to $t = 0$ where the scale factor $a(t = 0) = 0$ and developement angle $\Theta = 0$. Show that the delopment angle Θ is related to the conformal time

$$\tau(t) = \int_0^t \frac{dt'}{a(t')} \tag{29}$$

linearly

$$\Theta \propto \tau \tag{30}$$

and find the coefficient of proportionality.

Now we can evaluate this for a number of interesting cases. E.g. t_0

$$\frac{1}{2}\alpha(1 - cos\Theta_0) = \frac{c}{H_0}(2q_0 - 1)^{-1/2} = \frac{(2q_0 - 1)}{2q_0}\alpha, \tag{31}$$

so that

$$cos\Theta_0 = \frac{1 - q_0}{q_0} \qquad sin\Theta_0 = \frac{(2q_0 - 1)^{1/2}}{q_0} \tag{32}$$

Thus we can find a formula for the age of the universe at the present time.

$$t_0 = \frac{\alpha}{2c}(\Theta_0 - sin\Theta_0) = \frac{q_0}{(2q_0 - 1)^{3/2}}\left[arccos\left(\frac{1 - q_0}{q_0}\right) - \frac{(2q_0 - 1)^{1/2}}{q_0}\right]\frac{1}{H_0} \tag{33}$$

Note that the scale factor reaches a maximum at $\Theta = \pi$

$$a_{max} = \alpha = \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}\ \frac{c}{H_0} \tag{34}$$

and by $\Theta = 2\pi$ it is back to zero. The time for the full cycle is

$$t_{crunch} = \frac{\alpha}{2c}\ 2\pi = \pi\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}\ \frac{1}{H_0} \tag{35}$$

Using the same approach work out the case of $k = -1$ using the substitution

$$a = \frac{1}{2}\beta(cosh\Psi - 1) \tag{36}$$

where β is α expressed for $k = -1$. In particular show that

$$t = \frac{1}{2c}(sinh\Psi - \Psi) \tag{37}$$

There is the same relation between conformal time $H_0\tau = \Psi$ and the open development angle.

$$cosh\Psi_0 = \frac{1-q_0}{q_0} \qquad sinh\Psi_0 = \frac{(1-2q_0)^{1/2}}{q_0}. \tag{38}$$

The present age of the universe is

$$t_0 = \frac{\beta}{2c}(sinh\Psi_0 - \Psi_0) \tag{39}$$

$$= \frac{q_0}{(1-2q_0)^{3/2}}\left[\frac{(1-2q_0)^{1/2}}{q_0} - ln\left(\frac{1-q_0+(1-2q_0)^{1/2}}{q_0}\right)\right]\frac{1}{H_0} \tag{40}$$

6 Light Propagation & Luminosity Distance

We consider freely propagating photons meaning that they are not interacting with material from their source (or last scattering) until they reach us and they travel along null geodesics. Conservation of the number of photons per mode (quantum mechanical state) or occupation number

$$n(\nu) = \frac{1}{exp(\frac{h\nu}{kT}) - 1} \tag{41}$$

gives us our usual relation between frequency shift and the apparent temperature shift for that mode. It was what allowed us to calculate the CMB anisotropies for various effects readily. The conservation of photons per mode also provides a relationship between the intensity (I_ν in units of energy per area per unit time per unit solid angle per unit frequency interval) in one location and another which is independent of the temperature. It is

$$\frac{I_\nu}{\nu^3} = \text{constant} \tag{42}$$

This relationship is usually proved by the Liouville theorem. I believe it to be clear from the conservation of photons in each mode and the formula for intensity

$$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{exp(\frac{h\nu}{kT}) - 1} \tag{43}$$

or

$$n_\gamma(\nu) = \frac{1}{exp(\frac{h\nu}{kT}) - 1} = \frac{c^2}{2h}\frac{I_\nu}{\nu^3} \tag{44}$$

If it disturbs you to use the blackbody formula, change the temperature T to be $T(\nu)$ a function of the frequency.

Since we know how the frequency changes with redshift z,

$$\frac{\nu_{received}}{\nu_{emitted}} = 1 + z \tag{45}$$

so that

$$I_{received} = \frac{I_{emitted}}{(1+z)^3} \tag{46}$$

So we can now define an operational luminosity distance in the usual way if we know the intrinsic luminosity (power radiated) of the source and the flux received assuming that the source is radiating isotropically. The flux received should be the energy radiated by the area of a sphere with radius given by the luminosity distance.

$$d_l^2 \equiv \frac{L}{4\pi F} \tag{47}$$

Before we go ahead and calculate the luminosity distance let us digress a moment to look at the question of surface brightness. Some astronomical sources can be resolved and one measures more than the total flux from the source. Galaxies and the CMB are examples. In the case that one observes an extended source, instead of the flux, what is measured is the surface brightness or flux per unit solid angle

$$B = \frac{dF}{d\Omega} = \frac{L}{4\pi dA}\,(1+z)^{-4} \tag{48}$$

The luminosity distance as a function of redshift will clearly depend upon the geometry of the uninverse.

6.1 Einstein-deSitter, $k = 0$ Universe

In a matter-dominated universe the radial distance from us to a source such as a galaxy is

$$\begin{aligned} r_e &= \int_{te}^{t0} \frac{c\ dt}{a(t)} = \frac{c}{a_0}\int_{te}^{t0} t_0^{2/3} t^{-2/3} dt \\ &= \frac{c}{a_0} t_0^{2/3}\ 3(t_0^{1/3} - t_e^{1/3}) \\ &= \frac{3c}{a_0} t_0\ (1 - (\frac{t_e}{t_0})^{1/3}). \end{aligned} \tag{49}$$

Since

$$1 + z = \frac{a(t_0)}{a(t_e)} = (\frac{t_e}{t_0})^{2/3}, \tag{50}$$

$$\begin{aligned} r_e &= \frac{3ct_0}{a_0}\,(1 - (1+z)^{-1/2}) \\ &= \frac{2c}{a_0 H_0}\,(1 - (1+z)^{-1/2}) \end{aligned} \tag{51}$$

where the last part depends on the relationship $t_0 = \frac{2}{3}H_0^{-1}$. The luminosity distance is the coordinate distance times the scale factor or

$$\begin{aligned} D_L &= r_e \times a_0 \times (1+z) \\ &= \frac{2c}{H_0}\,[(1+z) - \sqrt{1+z}] \end{aligned} \tag{52}$$

6.2 Closed, $k = 1$, Universe

We use the same approach The radial distance from us to a source such as a galaxy is

$$\begin{aligned}\int_{r0=0}^{re} \frac{dr}{\sqrt{1-r^2}} &= \int_{te}^{t0} \frac{c\ dt}{a(t)} \\ &= \int \frac{da}{\sqrt{a(\alpha - a)}} \\ arcsin r_e &= \int d\Theta = \Theta_0 - \Theta_e \end{aligned} \tag{53}$$

so that

$$r_e = sin(\Theta_0 - \Theta_e). \tag{54}$$

Since this relates to the redshift

$$1 + z = \frac{a(t_0)}{a(t_e)} = \frac{sin^2 \frac{\Theta_0}{2}}{sin^2 \frac{\Theta_e}{2}}, \tag{55}$$

$$\begin{aligned} sin\Theta_1 &= \frac{2}{1+z}\ sin\frac{\Theta_e}{2}\left(z + cos^2\frac{\Theta_e}{2}\right)^{1/2} \\ cos\Theta_1 &= \frac{z + cos\Theta_0}{1+z}. \end{aligned} \tag{56}$$

Now we also had from before.

$$sin\frac{\Theta_0}{2} = \left(\frac{2q_0 - 1}{2q_0}\right)^{1/2} \quad \text{and} \quad cos\frac{\Theta_0}{2} = \left(\frac{1}{2q_0}\right)^{1/2} \tag{57}$$

Now we leave it to the student to put these into the equation and simply the algebra to get

$$r_e = \frac{(2q_0 - 1)^{1/2}}{q_0^2\ (1+z)}\ \left[q_0 z + (1 - q_0)[1 - (1 + 2zq_0)^{1/2}]\right] \tag{58}$$

So now we have an expression for the luminosity distance in a closed universe

$$\begin{aligned} D_L &= r_e \times a_0 \times (1+z) \\ &= \frac{c}{H_0}\ \frac{1}{q_0^2}\ \left[q_0 z + (1 - q_0)[1 - (1 + 2zq_0)^{1/2}]\right] \end{aligned} \tag{59}$$

6.3 Open, $k = 1$, Universe

We leave this as a full excercise for the student to follow the logic of the closed universe and work out the relations for the open model. The results are

$$r_e = \frac{(1 - 2q_0)^{1/2}}{q_0^2\ (1+z)}\ \left[q_0 z + (1 - q_0)[1 - (1 + 2zq_0)^{1/2}]\right]$$

$$D_L = \frac{c}{H_0}\frac{1}{q_0^2}\left[q_0 z + (1-q_0)[1-(1+2zq_0)^{1/2}]\right] \quad (60)$$

It is interesting to note that the open and closed universe models give the same formula for the luminosity distance. So it is necessary to show that they reduce to the Einstein-deSitter ($k = 0$) case when $q_0 = 1/2$. They all agree as you can easily verify.

It is also important to consider the Hubble diagram that will result when one plots redshift versus luminosity distance. Show that to first order ($z << 1$) that all give a linear Hubble law. Show that for $q_0 = 0$, the empty univers that

$$D_L(q_0 = 0) = \frac{c}{H_0}\; z\;(1 + \frac{z}{2}) \quad (61)$$

So by a redshift of 1, what is the per cent deviation from linear? What is the per cent deviation $q_0 = 0.5$ or $\Omega = 1$? Answer is to second order

$$D_L(q_0 = 1) \cong \frac{c}{H_0}\; z\;(1 + \frac{z}{8}) \quad (62)$$

What is the per cent deviation $q_0 = 1$ or $\Omega = 2$? Answer to last is that it is a linear law.

$$D_L(q_0 = 1) = \frac{c}{H_0}\; z \quad (63)$$

7 Angular Size

Consider an object with fixed physical diameter d at a redshift z. The angle it will subtend is

$$\theta = \frac{d}{r_e a(t_e)} = \frac{d(1+z)^2}{D_L} \quad (64)$$

We can evaluate this to find its dependence on q_0. One interesting result is that for an expanding universe, the angular diameter goes to a minimum and then grows larger as one increase z.

7.1 Einstein-deSitter, $k = 0$ Universe

$$\theta = \frac{dH_0}{c}\;\frac{(1+z)^{3/2}}{(1+z)^{1/2} - 1} \quad (65)$$

The minimum value of the angle is

$$\theta_{min} = 6.75\frac{dH_0}{c}, \quad \text{for} \quad z_{min} = 1.25. \quad (66)$$

7.2 Closed, $k = 1$, Universe

$$\theta = \frac{2d}{\alpha}\frac{1 - cos\Theta_e}{sin(\Theta_0 - \Theta_e)} \tag{67}$$

The minimum angle occurs at

$$sin(\Theta_0 - \frac{3\Theta_e}{2}) = 0 \quad \text{thus} \quad \Theta_e = \frac{2}{3}\Theta_0 \tag{68}$$

This corresponds to a redshift of

$$1 + z_{min} = \frac{1 - cos\Theta_0}{1 - cos(\frac{2}{3}\Theta_0)} \tag{69}$$

We can then calculate the minimum angle

$$\theta_{min} = \frac{(2q_0 - 1)^{3/2}}{q_0}\frac{1}{(1 - cos(\frac{2}{3}\Theta_0))sin\frac{\Theta_0}{3}}\frac{dH_0}{c} \tag{70}$$

Simply this expression to include only z, q_0 and H_0.

7.3 Open, $k = 1$, Universe

The corresponding minimum angle for an open universe is

$$\theta_{min} = \frac{(1 - 2q_0)^{3/2}}{q_0}\frac{1}{(cosh(\frac{2}{3}\Psi_0) - 1)sinh\frac{\Psi_0}{3}}\frac{dH_0}{c} \tag{71}$$

at a minimum redshift of

$$1 + z_{min} = \frac{cosh\Psi_0 - 1}{cosh(\frac{2}{3}\Psi_0) - 1} \tag{72}$$

8 Source Counts

In class I derived a formula for the number of sources per unit volume versus redshift and the number of sources versus luminosity for a number of geometries including Euclidean, curved, and the Friedmann-Lemaitre universes. The primary result was

$$\begin{aligned}\frac{dN}{dz} &= 4\pi n(z)\frac{(2q_0 - 1)^{3/2}}{q_0^4}\frac{[q_0 z + (q_0 - 1)[(1 + 2q_0 z)^{1/2} - 1]]^2}{(1 + 2q_0 z)^{1/2}(1 + z)^3} \\ &= 4\pi\tilde{n}(z)\left(\frac{c}{H_0}\right)^3\frac{1}{q_0^4}\frac{[q_0 z + (q_0 - 1)[(1 + 2q_0 z)^{1/2} - 1]]^2}{(1 + 2q_0 z)^{1/2}(1 + z)^6}\end{aligned} \tag{73}$$

where $n(z)$ is the number of sources per unit coordinate volume and $\tilde{n}(z)$ is the number of sources per unit proper (physical) volume.

9 Radiation Background From Souces

Now consider the flux of radiation form sources distributed throughout the universe. Let us suppose that there are $\tilde{n}(z)dz$ souces per unit proper volume with redshifts between z and $z+dz$. Wwe assume that a typical source at redshift z has a normalized intensity spectrum $J(\nu, z)$ and total luminosity $L(z)$

$$\int_0^\infty J(\nu, z)d\nu = 1. \tag{74}$$

where the total luminosity is the scale factor. Consider sources located in a small solid angle $d\Omega$. The total flux of radiation received from this solid angle from all sourcs in a frequency range ν_0 to $\nu_0 + \Delta\nu_0$ is

$$f(\nu_0)\ \Delta\nu_0\ d\Omega \tag{75}$$

We can estimate $f(\nu_0)$ by multiplying the number of sources at each redshift times their appropriate spectrum and integrating over redshifts

$$f(\nu_0) = \frac{c}{H_)}\frac{1}{4\pi}\int_0^\infty \frac{\tilde{n}(z)L(z)J(\nu_0\cdot(1+z), z)}{(1+z)^5(1+2q_0 z)^{1/2}}\,dz \tag{76}$$

Now we are in a position to consider Olbers' paradox, and various backgrounds such as the CMB, the radio the infrared, the optical (Olbers) and the X-ray.

10 Age of the Universe

Now we consider the age of the Universe

10.1 Age of the Universe including Λ

The current age of the universe, t_0, is given by

$$t_0 = \int dt = \int_0^\infty (aH)^{-1}(z)\left|\frac{da}{dz}\right| dz = \int_0^\infty \frac{dz}{(1+z)H(z)}.$$

Performing this integral for $\lambda_0 = 0$ gives the age of the universe as

$$H_0 t_0 = \{ \begin{array}{ll} \frac{1}{1-\Omega_0} - \frac{\Omega_0}{2(1-\Omega_0)^{3/2}}\cosh^{-1}\left(\frac{2}{\Omega_0}-1\right) & \text{for}\ \ \Omega_0 < 1, \\ \frac{2}{3} & \text{for}\ \ \Omega_0 = 1, \\ \frac{\Omega_0}{2(\Omega_0-1)^{3/2}}\cos^{-1}\left(\frac{2}{\Omega_0}-1\right) - \frac{1}{\Omega_0-1} & \text{for}\ \ \Omega_0 > 1. \end{array}$$

For the flat case where $k = 0$ and $\Omega_0 = 1 - \lambda_0$, the result is

$$H_0 t_0 = \frac{1}{\lambda_0^{1/2}}\ln\left[\frac{1+\lambda_0^{1/2}}{(1-\lambda_0)^{1/2}}\right].$$

11 Dark Matter

See the text for a discussion of dark matter. The primary method for measuring the mass of various bound systems is kinematics. That is the use of the virial relations

$$\frac{1}{2}mv^2 = \frac{GM(R)m}{R} \tag{77}$$

The relation is roughly this with a factor of order unity depending upon the geometry of the system under consideration.

Table 1
Mass Density of Systems

System	Method	Scale	$\frac{M}{L}\frac{L_\odot}{M_\odot}$	Ω
Milky Way	Near Sun	100 pc	5 (?)	$0.003\ h^{-1}$
	Escape Velocity	20 kpc	30	$0.018\ h^{-1}$
	Satellite galaxies	100 kpc	30	$0.018\ h^{-1}$
	Magellanic stream	100 kpc	>80	$>0.05\ h^{-1}$
Ellipticals	Core Fitting	2 kpc	$12h$	0.007
	X-ray halo	100 kpc	> 750	$> 0.46h^{-1}$
Spirals	Rotation Curves	50 kpc	$> 30h$	> 0.018
Gal. Groups	Local Group	800 kpc	100	$> 0.06h^{-1}$
	Other Groups	1 Mpc	$260h$	0.16
Clusters	Coma	2 Mpc	$400h$	0.25
Vel. Flows	IRAS Survey	$100h^{-1}$ Mpc		0.6
Superclusters,	walls, & voids	100 Mpc		~ 1