1. BIG-BANG COSMOLOGY

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1.1. Introduction to Standard Big-Bang Model

The observed expansion of the Universe [1,2,3] is a natural (almost inevitable) result of any homogeneous and isotropic cosmological model based on general relativity. In order to account for the possibility that the abundances of the elements had a cosmological origin, Alpher and Herman proposed that the early Universe which was once very hot and dense (enough so as to allow for the nucleosynthetic processing of hydrogen), and has expanded and cooled to its present state [4,5]. In 1948, Alpher and Herman predicted that a direct consequence of this model is the presence of a relic background radiation with a temperature of order a few K [6,7]. It was the observation of the 3 K background radiation that singled out the Big-Bang model as the prime candidate to describe our Universe. Subsequent work on Big-Bang nucleosynthesis further confirmed the necessity of our hot and dense past. These relativistic cosmological models face severe problems with their initial conditions, to which the best modern solution is inflationary cosmology.

1.1.1. *The Robertson-Walker Universe*:

The observed homogeneity and isotropy enable us to describe the overall geometry and evolution of the Universe in terms of two cosmological parameters accounting for the spatial curvature and the overall expansion (or contraction) of the Universe. These two quantities appear in the most general expression for a space-time metric which has a (3D) maximally symmetric subspace of a 4D space-time, known as the Robertson-Walker metric:

$$ds^2 = dt^2 - R^2\,(t)\left[\frac{dr^2}{1-kr^2} + r^2\,\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] \ . \tag{1.1}$$

Note that we adopt $c = 1$ throughout. By rescaling the radial coordinate, we can choose the curvature constant k to take only the discrete values $+1$, -1, or 0 corresponding to closed, open, or spatially flat geometries.

1.1.2. *The redshift*:

The cosmological redshift is a direct consequence of the Hubble expansion, determined by $R(t)$. A local observer detecting light from a distant emitter sees a redshift in frequency. We can define the redshift as

$$z \equiv \frac{\nu_1 - \nu_2}{\nu_2} \simeq \frac{v_{12}}{c} \ , \tag{1.3}$$

where ν_1 is the frequency of the emitted light, ν_2 is the observed frequency and v_{12} is the relative velocity between the emitter and the observer. While the definition, $z = (\nu_1 - \nu_2)/\nu_2$ is valid on all distance scales, relating the redshift to the relative velocity in this simple way is only true on small scales (*i.e.*, less than cosmological scales) such that the expansion velocity is non-relativistic. For light signals, we can use the metric given by Eq. (1.1) and $ds^2 = 0$ to write

$$1 + z = \frac{\nu_1}{\nu_2} = \frac{R_2}{R_1} \; . \tag{1.5}$$

This result does not depend on the non-relativistic approximation.

1.1.3. *The Friedmann-Lemaître equations of motion*:

The cosmological equations of motion are derived from Einstein's equations

$$\mathcal{R}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G_{\mathrm{N}} T_{\mu\nu} + \Lambda g_{\mu\nu} \; . \tag{1.6}$$

Gliner [15] and Zeldovich [16] seem to have pioneered the modern view, in which the Λ term is taken to the rhs and interpreted as particle-physics processes yielding an effective energy–momentum tensor $T_{\mu\nu}$ for the vacuum of $\Lambda g_{\mu\nu}/8\pi G_{\mathrm{N}}$. It is common to assume that the matter content of the Universe is a perfect fluid, for which

$$T_{\mu\nu} = -p g_{\mu\nu} + (p + \rho)\, u_\mu u_\nu \; , \tag{1.7}$$

where $g_{\mu\nu}$ is the space-time metric described by Eq. (1.1), p is the isotropic pressure, ρ is the energy density and $u = (1, 0, 0, 0)$ is the velocity vector for the isotropic fluid in co-moving coordinates. With the perfect fluid source, Einstein's equations lead to the Friedmann-Lemaître equations

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi\, G_{\mathrm{N}}\, \rho}{3} - \frac{k}{R^2} + \frac{\Lambda}{3} \; , \tag{1.8}$$

and

$$\frac{\ddot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G_{\mathrm{N}}}{3} \;\; (\rho + 3p) \; , \tag{1.9}$$

where $H(t)$ is the Hubble parameter and Λ is the cosmological constant. The first of these is sometimes called the Friedmann equation. Energy conservation via $T^{\mu\nu}{}_{;\mu} = 0$, leads to a third useful equation

$$\dot{\rho} = -3H\,(\rho + p) \; . \tag{1.10}$$

Eq. (1.10) can also be simply derived as a consequence of the first law of thermodynamics. For $\Lambda = 0$, it is clear that the Universe must be expanding or contracting.

1.1.4. *Definition of cosmological parameters*:

The Friedmann equation can be used to define a critical density such that $k = 0$ when $\Lambda = 0$,

$$\begin{aligned} \rho_c \equiv \frac{3H^2}{8\pi\, G_{\mathrm{N}}} &= 1.88 \times 10^{-26}\, h^2 \ \mathrm{kg\ m^{-3}} \\ &= 1.05 \times 10^{-5}\, h^2 \ \mathrm{GeV\ cm^{-3}} \; , \end{aligned} \tag{1.11}$$

where the scaled Hubble parameter, h, is defined by

$$\begin{aligned} H &\equiv 100\,h \text{ km s}^{-1} \text{ Mpc}^{-1} \\ \Rightarrow H^{-1} &= 9.78\,h^{-1} \text{ Gyr} \\ &= 2998\,h^{-1} \text{ Mpc} . \end{aligned} \tag{1.12}$$

The cosmological density parameter $\Omega_{\rm tot}$ is defined as the energy density relative to the critical density,

$$\Omega_{\rm tot} = \rho/\rho_c . \tag{1.13}$$

Note that one can now rewrite the Friedmann equation as

$$k/R^2 = H^2 \left(\Omega_{\rm tot} - 1\right) . \tag{1.14}$$

From Eq. (1.14), one can see that when $\Omega_{\rm tot} > 1$, $k = +1$ and the Universe is closed, when $\Omega_{\rm tot} < 1$, $k = -1$ and the Universe is open, and when $\Omega_{\rm tot} = 1$, $k = 0$, and the Universe is spatially flat.

It is often necessary to distinguish different contributions to the density. It is therefore convenient to define present-day density parameters for pressureless matter (Ω_m) and relativistic particles (Ω_r), plus the quantity $\Omega_\Lambda = \Lambda/3H^2$. In more general models, we may wish to drop the assumption that the vacuum energy density is constant, and we therefore denote the present-day density parameter of the vacuum by Ω_v. The Friedmann equation then becomes

$$k/R_0^2 = H_0^2 \left(\Omega_m + \Omega_r + \Omega_v - 1\right) , \tag{1.15}$$

where the subscript 0 indicates present-day values. Thus, it is the sum of the densities in matter, relativistic particles, and vacuum that determines the overall sign of the curvature. Note that the quantity $-k/R_0^2 H_0^2$ is sometimes (unfortunately) referred to as Ω_k.

1.1.5. *Standard Model solutions*:

During inflation or perhaps even today if we are living in an accelerating Universe, domination by a cosmological constant or some other form of dark energy should be considered.

Let us first assume a general equation of state parameter for a single component, $w = p/\rho$ which is constant. In this case, Eq. (1.10) can be written as $\dot\rho = -3(1+w)\rho\dot R/R$ and is easily integrated to yield

$$\rho \propto R^{-3(1+w)} . \tag{1.16}$$

Note that at early times when R is small, k/R^2 in the Friedmann equation can be neglected so long as $w > -1/3$. Curvature domination occurs at rather late times (if a cosmological constant term does not dominate sooner). For $w \neq -1$,

$$R\,(t) \propto t^{2/[3(1+w)]} . \tag{1.17}$$

1.1.5.2. *A Radiation-dominated Universe:*

In the early hot and dense Universe, it is appropriate to assume an equation of state corresponding to a gas of radiation (or relativistic particles) for which $w = 1/3$. In this case, Eq. (1.16) becomes $\rho \propto R^{-4}$. Similarly, one can substitute $w = 1/3$ into Eq. (1.17) to obtain

$$R(t) \propto t^{1/2} \; ; \qquad H = 1/2t \; . \tag{1.18}$$

1.1.5.3. *A Matter-dominated Universe:*

Non-relativistic matter eventually dominates the energy density over radiation. A pressureless gas ($w = 0$) leads to the expected dependence $\rho \propto R^{-3}$, and, if $k = 0$, we get

$$R(t) \propto t^{2/3} \; ; \qquad H = 2/3t \; . \tag{1.19}$$

If there is a dominant source of vacuum energy, acting as a cosmological constant with equation of state $w = -1$. This leads to an exponential expansion of the Universe

$$R(t) \propto e^{\sqrt{\Lambda/3}t} \; . \tag{1.20}$$

The equation of state of the vacuum need not be the $w = -1$ of Λ, and may not even be constant [17,18,19]. There is now much interest in the more general possibility of a dynamically evolving vacuum energy, for which the name 'dark energy' has become commonly used. A variety of techniques exist whereby the vacuum density as a function of time may be measured, usually expressed as the value of w as a function of epoch [20,21]. The best current measurement for the equation of state (assumed constant) is $w = -0.967^{+0.073}_{-0.072}$ [22]. Unless stated otherwise, we will assume that the vacuum energy is a cosmological constant with $w = -1$ exactly.

The presence of vacuum energy can dramatically alter the fate of the Universe. For example, if $\Lambda < 0$, the Universe will eventually recollapse independent of the sign of k. For large values of Λ (larger than the Einstein static value needed to halt any cosmological expansion or contraction), even a closed Universe will expand forever. One way to quantify this is the deceleration parameter, q_0, defined as

$$q_0 = - \left. \frac{R\ddot{R}}{\dot{R}^2} \right|_0 = \frac{1}{2}\Omega_m + \Omega_r + \frac{(1+3w)}{2}\Omega_v \; . \tag{1.21}$$

This equation shows us that $w < -1/3$ for the vacuum may lead to an accelerating expansion. Current data indicate that vacuum energy is indeed the largest contributor to the cosmological density budget, with $\Omega_v = 0.759 \pm 0.034$ and $\Omega_m = 0.241 \pm 0.034$ if $k = 0$ is assumed (3-year mean WMAP figure) [22].

1.2. Introduction to Observational Cosmology

1.2.1. *Fluxes, luminosities, and distances*:

The key quantities for observational cosmology can be deduced quite directly from the metric.

(1) The *proper* transverse size of an object seen by us to subtend an angle $d\psi$ is its comoving size $d\psi\, r$ times the scale factor at the time of emission:

$$d\ell = d\psi \; R_0 r / \left(1+z\right) \ . \tag{1.22}$$

(2) The apparent flux density of an object is deduced by allowing its photons to flow through a sphere of current radius $R_0 r$; but photon energies and arrival rates are redshifted, and the bandwidth $d\nu$ is reduced. These relations lead to the following common definitions:

$$\begin{aligned} \text{angular-diameter distance:} \quad & D_{\rm A} = (1+z)^{-1}\, R_0 r \\ \text{luminosity distance:} \quad & D_{\rm L} = (1+z)\ \ R_0 r \ . \end{aligned} \tag{1.24}$$

These distance-redshift relations are expressed in terms of observables by using the equation of a null radial geodesic plus the Friedmann equation:

$$\begin{aligned} \frac{R_0}{R(t)} dt = & \frac{1}{H\,(z)}\, dz = \frac{1}{H_0} \Big[(1 - \Omega_m - \Omega_v - \Omega_r)\,(1+z)^2 \\ & + \ \Omega_v\,(1+z)^{3+3w} \ \ + \ \ \Omega_m\,(1+z)^3 + \Omega_r\,(1+z)^4 \Big]^{-1/2} \ dz \ . \end{aligned} \tag{1.25}$$

The main scale for the distance here is the Hubble length, $1/H_0$.

In combination with Cepheid data from the HST key project on the distance scale, SNe results are the dominant contributor to the best modern value for H_0: $72\,{\rm km\,s^{-1} Mpc^{-1}} \pm 10\%$ [27]. Better still, the analysis of high-z SNe has allowed the first meaningful test of cosmological geometry to be carried out.

1.2.2. *Age of the Universe*:

The dynamical result for the age of the Universe may be written as

$$H_0 t_0 = \int_0^\infty \frac{dz}{(1+z)\ \Big[(1+z)^2\,(1+\Omega_m z) - z\,(2+z)\,\Omega_v\Big]^{1/2}} \ , \tag{1.28}$$

where we have neglected Ω_r and chosen $w = -1$. Over the range of interest ($0.1 \lesssim \Omega_m \lesssim 1$, $|\Omega_v| \lesssim 1$), this exact answer may be approximated to a few % accuracy by

$$H_0 t_0 \simeq \tfrac{2}{3}\ (0.7\Omega_m + 0.3 - 0.3\Omega_v)^{-0.3} \ . \tag{1.29}$$

For the special case that $\Omega_m + \Omega_v = 1$, the integral in Eq. (1.28) can be expressed analytically as

$$H_0 t_0 = \frac{2}{3\sqrt{\Omega_v}}\ \ln \frac{1+\sqrt{\Omega_v}}{\sqrt{1-\Omega_v}} \quad (\Omega_m < 1) \ . \tag{1.30}$$

The present consensus favors ages for the oldest clusters of about 12 Gyr [30,31].

These methods are all consistent with the age deduced from studies of structure formation, using the microwave background and large-scale structure: $t_0 = 13.73 \pm 0.15$ Gyr [22], where the extra accuracy comes at the price of assuming the Cold Dark Matter model to be true.

1.3. The Hot Thermal Universe

1.3.1. *Thermodynamics of the early Universe*:

Through much of the radiation-dominated period, thermal equilibrium is established by the rapid rate of particle interactions relative to the expansion rate of the Universe. In equilibrium, it is straightforward to compute the thermodynamic quantities, ρ, p, and the entropy density, s.

In the Standard Model, a chemical potential is often associated with baryon number, and since the net baryon density relative to the photon density is known to be very small (of order 10^{-10}), we can neglect any such chemical potential when computing total thermodynamic quantities.

For photons, we have

$$\rho_\gamma = \frac{\pi^2}{15}T^4 \; ; \quad p_\gamma = \frac{1}{3}\rho_\gamma \; ; \quad s_\gamma = \frac{4\rho_\gamma}{3T} \; ; \quad n_\gamma = \frac{2\zeta\left(3\right)}{\pi^2}T^3 \; . \tag{1.39}$$

Eq. (1.10) can be converted into an equation for entropy conservation,

$$d\left(sR^3\right)/dt = 0 \; . \tag{1.40}$$

For radiation, this corresponds to the relationship between expansion and cooling, $T \propto R^{-1}$ in an adiabatically expanding Universe. Note also that both s and n_γ scale as T^3.

1.3.2. *Radiation content of the Early Universe*:

At the very high temperatures associated with the early Universe, massive particles are pair produced, and are part of the thermal bath. If for a given particle species i we have $T \gg m_i$, then we can neglect the mass and the thermodynamic quantities are easily computed. In general, we can approximate the energy density (at high temperatures) by including only those particles with $m_i \ll T$. In this case, we have

$$\rho = \left(\sum_B g_B + \frac{7}{8}\sum_F g_F\right)\frac{\pi^2}{30}T^4 \equiv \frac{\pi^2}{30}\,N\left(T\right)\,T^4 \; , \tag{1.41}$$

where $g_{B(F)}$ is the number of degrees of freedom of each boson (fermion) and the sum runs over all boson and fermion states with $m \ll T$. Eq. (1.41) defines the effective number of degrees of freedom, $N(T)$, by taking into account new particle degrees of freedom as the temperature is raised.

The value of $N(T)$ at any given temperature depends on the particle physics model. In the standard SU(3) $\times$ SU(2) $\times$ U(1) model, we can specify $N(T)$ up to temperatures of O(100) GeV. The change

Temperature	New Particles	$4N(T)$
$T < m_e$	γ's + ν's	29
$m_e < T < m_\mu$	$e^\pm$	43
$m_\mu < T < m_\pi$	$\mu^\pm$	57
$m_\pi < T < T_c{}^\dagger$	π's	69
$T_c < T < m_{\text{strange}}$	π's + $u, \bar{u}, d, \bar{d}$ + gluons	205
$m_s < T < m_{\text{charm}}$	$s, \bar{s}$	247
$m_c < T < m_\tau$	$c, \bar{c}$	289
$m_\tau < T < m_{\text{bottom}}$	$\tau^\pm$	303
$m_b < T < m_{\text{W,Z}}$	$b, \bar{b}$	345
$m_{W,Z} < T < m_{\text{Higgs}}$	$W^\pm, Z$	381
$m_H < T < m_{\text{top}}$	H^0	385
$m_t < T$	$t, \bar{t}$	427

$^\dagger T_c$ corresponds to the confinement-deconfinement transition between quarks and hadrons.

in N (ignoring mass effects) can be seen in the table below. At higher temperatures, $N(T)$ will be model-dependent.

In the radiation-dominated epoch, Eq. (1.10) can be integrated (neglecting the T-dependence of N) giving us a relationship between the age of the Universe and its temperature

$$t = \left(\frac{90}{32\pi^3 G_{\rm N} N\,(T)}\right)^{1/2} T^{-2} \ . \tag{1.42}$$

Put into a more convenient form

$$t\,T_{\rm MeV}^2 = 2.4\,[N\,(T)]^{-1/2} \ , \tag{1.43}$$

where t is measured in seconds and $T_{\rm MeV}$ in units of MeV.

1.3.7. *Nucleosynthesis*:

An essential element of the standard cosmological model is Big-Bang nucleosynthesis (BBN), the theory which predicts the abundances of the light element isotopes D, ^{3}He, ^{4}He, and ^{7}Li. Nucleosynthesis takes place at a temperature scale of order 1 MeV. The nuclear processes lead primarily to ^{4}He, with a primordial mass fraction of about 24%. Lesser amounts of the other light elements are produced: about 10^{-5} of D and ^{3}He and about 10^{-10} of ^{7}Li by number relative to H. The abundances of the light elements depend almost solely on one key parameter, the baryon-to-photon ratio, η. The nucleosynthesis predictions can be compared with observational determinations of the abundances of the light elements. Consistency between theory and observations leads to a conservative range of

$$4.7 \times 10^{-10} < \eta < 6.5 \times 10^{-10} \ . \tag{1.53}$$

η is related to the fraction of Ω contained in baryons, Ω_b

$$\Omega_b = 3.66 \times 10^7 \eta\, h^{-2} \ , \tag{1.54}$$

or $10^{10}\eta = 274\Omega_b h^2$.

1.4. The Universe at late times

We are beginning to inventory the composition of the Universe:

total: $\Omega = 1.0 \pm 0.1$ (from CMB anisotropy)

matter: $\Omega_m = 0.2$–0.5

CDM: $\Omega_{\rm CDM} = \Omega_m - \Omega_{\rm B}$

baryons: $\Omega_{\rm B} = 0.02$–0.05

neutrinos: $0.003 \lesssim \Omega_\nu$

energy: $\Omega_E = 0.5$–0.8

dark energy: $\Omega_v \approx \Omega_E$

photons: $\Omega_\gamma = 2.471 \times 10^{-5}$

Further discussion and all references may be found in the full *Review of Particle Physics*. The numbering of references and equations used here corresponds to that version.