

Boundary Element Analysis of Bimaterials Using Anisotropic Elastic Green's Functions

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Abstract

The boundary integral equations incorporating the Green's function for anisotropic solids containing planar interfaces are presented. The fundamental displacement and traction solutions are determined from the displacement Green's function of Tewary, Wagoner, and Hirth [*Journal of Materials Research*, Vol. 4, pp. 113-123]. The fundamental solutions are shown to numerically degenerate to the Kelvin solution in the homogeneous, isotropic limit. The boundary integral equations are formulated with the use of constant boundary elements. The constant elements allow for analytic evaluation of the boundary integrals. The application of the method is demonstrated by analyzing a copper-nickel system subjected to mechanical load.

1. Introduction

The analysis of deformation near interfaces in solids has received renewed interest due to reliability issues in electronic packaging. Here we present a boundary element formulation for anisotropic interface problems incorporating the fundamental solution of Tewary, Wagoner, and Hirth [1]. This fundamental solution for the general anisotropic interface problem allows us to model the behavior of stresses and displacements near the interface exactly. No discretization of the interface is required since this behavior is explicitly incorporated in the fundamental solution. A similar approach was taken in [10, 11] for the isotropic interface problem.

The work presented here follows the work of Cruse [4] and Snyder and Cruse [8, 9] who implemented a complex-variable fundamental solution for the crack problem in a homogeneous, anisotropic plate. They found such a formulation to be computationally efficient as well as providing exact modeling of the singular fields near the crack tips. The use of enriched elements [14-16] has received a good deal of attention in the finite element literature for use in modeling singular fields; however, the use of special Green's functions for singular fields has been limited in boundary element formulations. The formulation presented here is useful for boundary element analysis of interface problems where the interface is flaw-free. Future work will focus on the special Green's functions associated with interface cracks in anisotropic solids [13].

In this paper the fundamental displacement and traction solutions are presented as determined from the Green's function given in [1]. We note that four solutions are actually needed depending on the relative location of the source and field points in the two anisotropic solids. The fundamental displacement solutions \mathbf{U} are of the general form

$$\mathbf{U} = \sum_{\mathbf{a}} \mathbf{g}_{\mathbf{a}}^{A,B} \log(z_{\mathbf{a}} - z'_{\mathbf{a}}) \quad (1.1)$$

where the $\mathbf{g}_{\mathbf{a}}^{A,B}$ matrix is a function of the elastic constants in either material A or B and the α roots of the Stroh determinant [2], $\alpha = 1, 2, 3$. The complex coordinates z, z' of the field and source points are defined in the classical Lekhnitskii form [3]. The traction fundamental solution is then derived from the displacement fundamental solution. We compare the degenerate isotropic form of these fundamental displacement and traction solutions with the Kelvin isotropic solution.

The details of implementing the new fundamental solutions into the boundary integral equation

$$c_{ij}u_j + \int_{\mathcal{I}\Omega} T_{ij}u_j d\Gamma = \int_{\mathcal{I}\Omega} U_{ij}t_j d\Gamma \quad (1.2)$$

are given where the integrals are evaluated in the complex plane. We focus on the constant element case to allow for analytic evaluation of the integrals.

Throughout the paper matrices and vectors are indicated either by a bold quantity or by subscript notation. Unless otherwise stated summation is implied over repeated (dummy) indices. We shall use the rectangular Cartesian coordinates x_1, x_2, x_3 and the elastic state is assumed to be independent of x_3 . A superscript asterisk (*) on a complex quantity indicates a complex conjugate, and a primed coordinate will indicate the coordinate system associated with a point source.

2. Anisotropic Fundamental Solution

The problem under consideration is shown in Fig. 1. In [1], Tewary, Wagoner, and Hirth determine the Green's function for a composite solid with a planar interface by taking Fourier transforms of the elastic equations. The Fourier transform method avoids the problems associated with solving a complex eigenvalue problem as arises in the analyses of Eshelby [12] and Stroh [2]. The presence of an interface in the composite solid necessitates solving the elastic equilibrium equations

$$c_{ikjl} \frac{\mathcal{I}^2 u_j}{\mathcal{I}x_k \mathcal{I}x_l} = -f_i \quad (2.1)$$

in both the upper half plane (material A) and the lower half plane (material B). In eq. (2.1), c_{ikjl} are the elastic constants, u_i are the displacements, and f_i are the forces at (x_1, x_2) . By definition, the Green's function G_{ij} solves the equation

$$c_{ikjl} \frac{\mathcal{I}^2}{\mathcal{I}x_k \mathcal{I}x_l} G_{jk} = -\mathbf{d}_{ik} \mathbf{d}(x_1 - x_1^{A,B}) \mathbf{d}(x_2 - x_2^{A,B}), \quad (2.2)$$

where $\delta(x)$ is the Dirac delta function, \mathbf{d}_{ik} is the Kronecker delta, the primed coordinates represent the location of the point source, and the superscripts are for either material A or B.

Eq. (2.2) is solved assuming perfect conditions at the interface between materials A and B,

$$u_i^A|_{x_2=0} = u_i^B|_{x_2=0}, \quad (2.3)$$

$$\mathbf{s}_{i2}^A \Big|_{x_2=0} = \mathbf{s}_{i2}^B \Big|_{x_2=0} . \quad (2.4)$$

The Green's function determined through the Fourier transforms of eq. (2.2) subject to the boundary conditions of eqs. (2.3) and (2.4) is found in [1]. Since the point force and the field point may be located in either material A or B there are in fact four parts to the Green's function. The displacement field is given by the real part of the Green's functions given in [1],

$$\mathbf{U}^{11T} = - \frac{1}{\mathbf{p}} \operatorname{Re} \left\{ \sum_{\mathbf{a}} \mathbf{g}(p_{\mathbf{a}}^A) \log(z_{\mathbf{a}}^A - z'_{\mathbf{a}}) + \sum_{\mathbf{ab}} \mathbf{g}(p_{\mathbf{a}}^A) \mathbf{Q}_{\mathbf{b}}^I \log(z_{\mathbf{a}}^A - z'_{\mathbf{b}}^{A*}) \right\}, \quad (2.5)$$

$$\mathbf{U}^{12T} = - \frac{1}{\mathbf{p}} \operatorname{Re} \left\{ \sum_{\mathbf{ab}} \mathbf{g}^*(p_{\mathbf{a}}^B) \mathbf{Q}_{\mathbf{b}}^{II} \log(z_{\mathbf{a}}^{B*} - z'_{\mathbf{b}}^{A*}) \right\}, \quad (2.6)$$

$$\mathbf{U}^{21T} = \frac{1}{\mathbf{p}} \operatorname{Re} \left\{ \sum_{\mathbf{ab}} \mathbf{g}(p_{\mathbf{a}}^A) \mathbf{Q}_{\mathbf{b}}^{III} \log(z_{\mathbf{a}}^A - z'_{\mathbf{b}}^B) \right\}, \quad (2.7)$$

$$\mathbf{U}^{22T} = - \frac{1}{\mathbf{p}} \operatorname{Re} \left\{ \sum_{\mathbf{a}} \mathbf{g}^*(p_{\mathbf{a}}^B) \log(z_{\mathbf{a}}^{B*} - z'_{\mathbf{a}}^{B*}) - \sum_{\mathbf{ab}} \mathbf{g}^*(p_{\mathbf{a}}^B) \mathbf{Q}_{\mathbf{b}}^{IV} \log(z_{\mathbf{a}}^{B*} - z'_{\mathbf{b}}^B) \right\}. \quad (2.8)$$

The definitions of the terms appearing in eqs. (2.5)-(2.8) are given below. The superscripts on the left hand side of eqs. (2.5)-(2.8) identify the location of the source and field points, respectively, as being in material A or B. For example, a 12 superscript indicates that the source point (location of the point load) is in the upper half plane (material A) and the field point (calculation point) is in the lower half plane (material B). The fundamental solutions for the boundary element analysis involve the transpose of the Green's functions given in [1] due to the convention taken in the boundary element literature where U_{ij} represents a displacement in the j -direction due to a unit load in the i -direction. The convention taken in [1] for the displacement Green's function is the reverse, that is

$$\operatorname{Re} G_{ij} = U_{ji} . \quad (2.9)$$

This becomes critical for bimaterial problems since the fundamental displacement solution is not symmetric, $U_{ij} \neq U_{ji}$.

The terms appearing in eqs. (2.5)-(2.8) are defined through the roots of the sextic equation

$$\det \Lambda(\mathbf{q}) = 0, \quad (2.10)$$

where the Christoffel matrix Λ_{ij} is defined by

$$\Lambda_{ij} = c_{ikjl} q_k q_l . \quad (2.11)$$

The wave vector \mathbf{q} has components q_1 and q_2 and the roots $p_{\mathbf{a}}$ are obtained from eq. (2.10) such that $q_2 = p_{\mathbf{a}} q_1$. For elastically stable solids the roots $p_{\mathbf{a}}$ are complex [1, 2, 12] and the roots are labeled such that

$$\operatorname{Im} p_{\mathbf{a}} > 0. \quad (2.12)$$

The complex coordinates appearing in eqs.(2.5)-(2.8) are defined as

$$z_{\mathbf{a}}^{A,B} = x_1 + p_{\mathbf{a}}^{A,B} x_2, \quad (2.13)$$

$$z_{\mathbf{a}}^{A,B} = x_1 + p_{\mathbf{a}}^{A,B} x_2 . \quad (2.14)$$

The remaining matrices in eqs. (2.5)-(2.8) are defined as follows:

$$\mathbf{g}(p_{\mathbf{a}}^{A,B}) = \frac{i}{a q_1^4} \frac{\Gamma(q_2 = q_1 p_{\mathbf{a}})}{(p_{\mathbf{a}}^{A,B} - p_{\mathbf{a}}^{*A,B}) \prod_{\mathbf{b} \neq \mathbf{a}} (p_{\mathbf{a}}^{A,B} - p_{\mathbf{b}}^{A,B})(p_{\mathbf{a}}^{A,B} - p_{\mathbf{b}}^{*A,B})} \quad (2.15)$$

$$\Gamma = \text{Cofactor of } \Lambda \quad (2.16)$$

$$\mathbf{s}(p_{\mathbf{a}}^{A,B}) = \mathbf{L}(p_{\mathbf{a}}^{A,B}) \mathbf{g}(p_{\mathbf{a}}^{A,B}) \quad (2.17)$$

$$L_{ik}(p_{\mathbf{a}}^{A,B}) = c_{i2k1} + p_{\mathbf{a}}^{A,B} c_{i2k2} \quad (2.18)$$

$$\mathbf{Q}_{\mathbf{b}}^I = \mathbf{M} \left\{ \mathbf{s}^*(p_{\mathbf{b}}^A) - \mathbf{s}_s^{*B} \mathbf{g}_s^{*B-1} \mathbf{g}^*(p_{\mathbf{b}}^A) \right\} \quad (2.19)$$

$$\mathbf{Q}_{\mathbf{b}}^{II} = \mathbf{N} \left\{ \mathbf{s}^*(p_{\mathbf{b}}^A) - \mathbf{s}_s^A \mathbf{g}_s^{A-1} \mathbf{g}^*(p_{\mathbf{b}}^A) \right\} \quad (2.20)$$

$$\mathbf{Q}_{\mathbf{b}}^{III} = \mathbf{M} \left\{ \mathbf{s}(p_{\mathbf{b}}^B) - \mathbf{s}_s^{*B} \mathbf{g}_s^{*B-1} \mathbf{g}(p_{\mathbf{b}}^B) \right\} \quad (2.21)$$

$$\mathbf{Q}_{\mathbf{b}}^{IV} = \mathbf{N} \left\{ \mathbf{s}(p_{\mathbf{b}}^B) - \mathbf{s}_s^A \mathbf{g}_s^{A-1} \mathbf{g}(p_{\mathbf{b}}^B) \right\} \quad (2.22)$$

$$\mathbf{M} = \mathbf{g}_s^{A-1} \left\{ \mathbf{s}_s^{*B} \mathbf{g}_s^{*B-1} - \mathbf{s}_s^A \mathbf{g}_s^{A-1} \right\}^{-1} \quad (2.23)$$

$$\mathbf{N} = \mathbf{g}_s^{*B-1} \left\{ \mathbf{s}_s^{*B} \mathbf{g}_s^{*B-1} - \mathbf{s}_s^A \mathbf{g}_s^{A-1} \right\}^{-1} \quad (2.24)$$

$$\mathbf{g}_s^{A,B} = \sum_{\mathbf{a}} \mathbf{g}(p_{\mathbf{a}}^{A,B}) \quad (2.25)$$

$$\mathbf{s}_s^{A,B} = \sum_{\mathbf{a}} \mathbf{s}(p_{\mathbf{a}}^{A,B}) \quad (2.26)$$

In eqs. (2.15)-(2.26), the index α generally takes values of 1 to 3. There are several useful relations for checking the computation of the various matrices given in [13]. These relations are most helpful in checking the Green's function computation in the boundary element code.

The traction fundamental solution may now be computed from the displacement fundamental solution by substituting eqs. (2.5)-(2.8) into

$$t_i = \left(c_{ijk1} + p_{\mathbf{a}}^{A,B} c_{ijk2} \right) \frac{du_k}{dz_{\mathbf{a}}} n_j , \quad (2.27)$$

where n_j is the j -th component of the local normal vector. Here, we consider the case of cubic material symmetry where the stiffness tensor is defined in terms of three elastic constants c_{11} , c_{12} , c_{44} as [6]

$$c_{ijkl} = c_{44} (\mathbf{d}_{ik} \mathbf{d}_{jl} + \mathbf{d}_{il} \mathbf{d}_{jk}) + c_{12} \mathbf{d}_{ij} \mathbf{d}_{kl} - H \mathbf{d}_{ij} \mathbf{d}_{kl} \mathbf{d}_{ik} , \quad (2.28)$$

(no sum on k or i) where the anisotropy ratio H is defined as

$$H = 2c_{44} + c_{12} - c_{11} . \quad (2.29)$$

In eq. (2.27) we have used the relation

$$\frac{\mathcal{I}}{\mathcal{I}x_2} = p_{\mathbf{a}}^{A,B} \frac{d}{dz_{\mathbf{a}}} = p_{\mathbf{a}}^{*A,B} \frac{d}{dz_{\mathbf{a}}^*}. \quad (2.30)$$

Note in eqs. (2.5)-(2.8) that each of the equations is a function of z or z^* only. Therefore, we will use the notation of eq. (2.27) where it is understood that if $u = u(z^*)$, the derivative is taken with respect to z^* .

The traction fundamental solution can be written as

$$\mathbf{T} = \frac{\mathcal{I}\mathbf{U}}{\mathcal{I}z_{\mathbf{a}}} \mathbf{E}^{\mathbf{a}}, \quad (2.31)$$

where

$$\mathbf{E}^{\mathbf{a}} = \begin{bmatrix} c_{11}n_1 + p_{\mathbf{a}}^{A,B}c_{44}n_2 & p_{\mathbf{a}}^{A,B}c_{44}n_1 + c_{12}n_2 \\ p_{\mathbf{a}}^{A,B}c_{12}n_1 + c_{44}n_2 & c_{44}n_1 + p_{\mathbf{a}}^{A,B}c_{11}n_2 \end{bmatrix}. \quad (2.32)$$

The post-multiplication of the displacement derivative matrix arises from the aforementioned convention taken in [1] for U_{ij} . The matrix of derivatives in eq. (2.31) can be determined directly from eqs. (2.5)-(2.8). The derivatives are given in Appendix A. We note again that for the interface problem we have four traction fundamental solutions to consider depending on the relative location of the source and field points.

A natural question to consider is the equivalence of the displacement and traction fundamental solutions given above with the Kelvin solution (see, for example, [5]) for the degenerate isotropic, homogeneous case. The homogeneous case can be modeled simply by specifying identical elastic constants for materials A and B. The isotropic case must be considered through a limiting process for the anisotropic solution given here; however, we can consider a "near" isotropic, homogeneous case and compare the Tewary displacement and traction solutions with the Kelvin solution. We consider the case of tungsten where the anisotropy ratio $H = 0$. From [6] the elastic constants are $c_{11} = 521.0$ Gpa, $c_{12} = 201.0$ Gpa, and $c_{44} = 160.0$ Gpa. These constants give us a Poisson ration $\nu = 0.28$ and a shear modulus $\mu = 160.0$ Gpa. For the anisotropic fundamental solution we use $c_{11} = 521.0$ Gpa, $c_{12} = 201.1$ Gpa, and $c_{44} = 160$ Gpa. These values provide an anisotropy ratio of $H = 0.001$.

The displacement fundamental solution for both the Kelvin and Tewary solutions is shown in Fig. 2. The solution is for a homogeneous solid so $U_{21} = U_{12}$ in the figure. Note that there is essentially no difference in the U_{21} , U_{12} , and U_{22} components; however, the U_{11} components differ by a constant. This constant depends on the elastic constants being used but is spatially invariant. In the boundary integral equations such a constant can be treated separately in the contour integrals and can be shown to be inconsequential to the solution of the equations. Indeed, a constant in the displacement fundamental solution must not have any effect since this is simply related to rigid body motion of the solid.

The traction fundamental solution for the Kelvin and Tewary solution is shown in Fig. 3. The agreement is excellent with no discernible difference between the two solutions. For purposes of comparison, the traction fundamental solution for two cubic crystals (copper-nickel) is shown in Fig. 4. Note the difference in the solutions as the field point crosses the bimaterial interface and that U_{21} and U_{12} are not equal as they are for the homogeneous case.

3. Boundary Integral Equations

We now consider the use of the anisotropic fundamental solution for the interface problem shown in Fig. 1. For the region Ω in the figure, let the boundary of material A be Γ_1 , the boundary of material B be Γ_2 , and $\Gamma_1 + \Gamma_2 = \partial\Omega$. The standard boundary integral equation written for the region is

$$c_{ij}u_j(P) + \int_{\partial\Omega} T_{ij}(P, Q)u_j(Q)d\Gamma = \int_{\partial\Omega} U_{ij}(P, Q)t_j(Q)d\Gamma, \quad (3.1)$$

where P and Q are points on the boundary at the source and field points, respectively. As noted above, we must consider the relative locations of the field and source points in writing the kernels of the integrals. Following [10, 11], we can write eq. (3.1) for the source point P in material A as

$$c_{ij}u_j(P_1) + \int_{\Gamma_1} T_{ij}^{11}(P_1, Q_1)u_j(Q_1)d\Gamma_1 + \int_{\Gamma_2} T_{ij}^{12}(P_1, Q_2)u_j(Q_2)d\Gamma_2 = \int_{\Gamma_1} U_{ij}^{11}(P_1, Q_1)t_j(Q_1)d\Gamma_1 + \int_{\Gamma_2} U_{ij}^{12}(P_1, Q_2)t_j(Q_2)d\Gamma_2, \quad (3.2)$$

where the superscripts on the fundamental solutions terms and the subscripts on P and Q identify the location of the source and field points. We can write a similar equation to eq. (3.2) for the case when P is in material B, or we can write in general

$$c_{ij}u_j(P_{\beta}) + \int_{\Gamma_1} T_{ij}^{\beta 1}(P_{\beta}, Q_1)u_j(Q_1)d\Gamma_1 + \int_{\Gamma_2} T_{ij}^{\beta 2}(P_{\beta}, Q_2)u_j(Q_2)d\Gamma_2 = \int_{\Gamma_1} U_{ij}^{\beta 1}(P_{\beta}, Q_1)t_j(Q_1)d\Gamma_1 + \int_{\Gamma_2} U_{ij}^{\beta 2}(P_{\beta}, Q_2)t_j(Q_2)d\Gamma_2, \quad (3.3)$$

where $\beta = 1, 2$ and no summation on β is implied. The continuity of displacement and traction fields across the interface is insured through the use of the fundamental solution from [1]. Discretizing each boundary into N elements eq. (3.3) becomes

$$c_{ij}u_j(P_{\beta}) + \sum_{n=1}^N \int_{\Gamma_1} T_{ij}^{\beta 1}(P_{\beta}, Q_1)u_j(Q_1)d\Gamma_1 + \sum_{n=1}^N \int_{\Gamma_2} T_{ij}^{\beta 2}(P_{\beta}, Q_2)u_j(Q_2)d\Gamma_2 = \sum_{n=1}^N \int_{\Gamma_1} U_{ij}^{\beta 1}(P_{\beta}, Q_1)t_j(Q_1)d\Gamma_1 + \sum_{n=1}^N \int_{\Gamma_2} U_{ij}^{\beta 2}(P_{\beta}, Q_2)t_j(Q_2)d\Gamma_2. \quad (3.4)$$

At this point we assume that the boundary traction and displacement are constant over each individual boundary element to allow us to analytically evaluate the integrals. Factoring out the constants from the integrals in eq. (3.4) we have

$$c_{ij}u_j(P_{\beta}) + \sum_{n=1}^N u_j(Q_1) \int_{\Gamma_1} T_{ij}^{\beta 1}(P_{\beta}, Q_1)d\Gamma_1 + \sum_{n=1}^N u_j(Q_2) \int_{\Gamma_2} T_{ij}^{\beta 2}(P_{\beta}, Q_2)d\Gamma_2 = \sum_{n=1}^N t_j(Q_1) \int_{\Gamma_1} U_{ij}^{\beta 1}(P_{\beta}, Q_1)d\Gamma_1 + \sum_{n=1}^N t_j(Q_2) \int_{\Gamma_2} U_{ij}^{\beta 2}(P_{\beta}, Q_2)d\Gamma_2, \quad (3.5)$$

where the u_j and t_j are element midpoint values. Note that eq. (3.5) is applied at $2N$ points around $\partial\Omega$, N points each on Γ_1 and Γ_2 . We therefore need to analytically evaluate the following eight integrals:

$$\int_{\Gamma} T_{ij}^{\mathbf{b}1}(P_{\mathbf{b}}, Q_1) d\Gamma, \quad (3.6)$$

$$\int_{\Gamma} T_{ij}^{\mathbf{b}2}(P_{\mathbf{b}}, Q_2) d\Gamma, \quad (3.6)$$

$$\int_{\Gamma} U_{ij}^{\mathbf{b}1}(P_{\mathbf{b}}, Q_1) d\Gamma, \quad (3.6)$$

$$\int_{\Gamma} U_{ij}^{\mathbf{b}2}(P_{\mathbf{b}}, Q_2) d\Gamma, \quad (3.6)$$

where $\beta = 1, 2$. To analytically evaluate these integrals, we use the mapping of Cruse [4] to transform these contour integrals to definite integrals in the complex plane. The differential contour element $d\Gamma$ is mapped as

$$d\Gamma = \frac{1}{B_{\mathbf{a}}} dz_{\mathbf{a}}, \quad (3.7)$$

where

$$B_{\mathbf{a}} = p_{\mathbf{a}}^{A,B} \hat{n}_1 - \hat{n}_2, \quad (3.8)$$

and \hat{n}_1 and \hat{n}_2 are unit vector components of the local normal vector for $d\Gamma$. With this mapping the displacement kernel integrals simply involve evaluating integrals of the form

$$\int_{z_1}^{z_2} \log(z-b) dz, \quad (3.9)$$

and the traction kernel integrals involve evaluating

$$\int_{z_1}^{z_2} \frac{dz}{z-b}, \quad (3.10)$$

where b is a complex constant. These integrals are simple to evaluate; however, special care must be taken when integrating across the branch cut of the principal value logarithm function on the negative real axis. The location of the real axis does not necessarily coincide with the location of the physical interface because of the definition of the complex variables used here (see eqs. (2.13)-(2.14)).

4. Internal Displacement and Stress Calculation

To calculate the displacement at an internal point we use the standard form of Somigliana's identity,

$$u_k(p) = \int_{\mathcal{I}\Omega} U_{kl}(p, Q) t_l(Q) d\Gamma - \int_{\mathcal{I}\Omega} T_{kl}(p, Q) u_l(Q) d\Gamma, \quad (4.1)$$

and the corresponding form for stresses,

$$\mathbf{s}_{ij}(p) = \int_{\mathcal{I}\Omega} D_{kij}(p, Q) t_k(Q) d\Gamma - \int_{\mathcal{I}\Omega} S_{kij}(p, Q) u_k(Q) d\Gamma, \quad (4.2)$$

where p is the internal computation point and Q is on the boundary. The kernels appearing in eq. (4.2) are determined from the fundamental displacement and traction solutions and the anisotropic constitutive law,

$$\mathbf{s}_{ij} = \left(c_{ijk1} + p_{\mathbf{a}}^{A,B} c_{ijk2} \right) \frac{du_k}{dz_{\mathbf{a}}}. \quad (4.3)$$

We then have,

$$D_{lij} = (c_{ijk1} + p_{\mathbf{a}}^{A,B} c_{ijk2}) \frac{dU_{kl}}{dz_{\mathbf{a}}} , \quad (4.4)$$

$$S_{lij} = \left(c_{ijk1} + p_{\mathbf{a}}^{A,B} c_{ijk2} \right) \frac{dT_{kl}}{dz_{\mathbf{a}}} . \quad (4.5)$$

Recall that for a cubic solid the elastic constants are given by eq. (2.28). The derivatives appearing in eq. (4.4) have been computed previously in deriving the fundamental traction solution, eq. (2.31), and are given in Appendix A. The derivatives appearing in eq. (4.5) can be written as

$$\frac{dT_{kl}}{dz_{\mathbf{a}}} = \frac{d^2 U_{km}}{dz_{\mathbf{a}}^2} E_{ml}^{\mathbf{a}} . \quad (3.16)$$

Therefore, we only need the second derivatives of the fundamental displacement solution. These are summarized in Appendix A. Finally, we note again that the $\mathbf{E}^{\mathbf{a}}$ matrix appearing in eq. (4.6) depends on material symmetry. We focus here on cubic material symmetry but others are easily implemented as well.

5. Example Problem

To demonstrate the application of the method we analyze the field distribution in a copper-nickel multilayer. Multilayer materials are fabricated by depositing alternating layers of thin-film materials such as Cu-Ni, Co-Cr, and Fe-GaAs. Here, we demonstrate the usefulness of the analysis method detailed above in analyzing multilayers. For the example problem we use a portion of a Cu-Ni multilayer subjected to mechanical loading, Fig. 6. The specimen shown in the figure is under shear loading with traction free surfaces along the remaining boundaries. We take the elastic constants as [6]:

	c_{11}	c_{12}	c_{44}
Cu	168.4	121.4	75.4
Ni	246.5	147.3	124.7

where all values are in GPa.

The elastic displacements along $x = 0$ are shown in Fig. 6. The displacements exhibit qualitatively what we expect from the shear loading. Note the continuity in the displacements as the interface is crossed and the lack of perfect asymmetry in u_1 and u_2 due to the differing elastic constants in the two materials.

6. Summary

We have presented an efficient, accurate method for analyzing deformation near interfaces in anisotropic solids. The use of a special Green's function for the anisotropic interface problem allows us to discretize only the boundary of the problem. The interface itself does not have to be discretized since its behavior is explicitly incorporated into the Green's function. Embedding the Green's function in a boundary element approach then provides us with a general tool for analyzing a variety of bimaterial problems.

A large portion of the computation time for a complete analysis is used by the computation of the Green's function. The computation involves calculating roots of the

Stroh determinant for use in the matrix functions of the Green's function; however, once calculated the matrices may be stored for subsequent analysis of the same material system.

Acknowledgments

The author acknowledges the support received from the National Academy of Sciences under the National Research Council Postdoctoral Associateships Program at the National Institute of Standards and Technology. The assistance and encouragement of Dr. Vinod Tewary at NIST is gratefully acknowledged.

References

- [1] Tewary, V.K., Wagoner, R.H., and Hirth, J.P., "Elastic Green's Function for a Composite Solid with a Planar Interface," *Journal of Materials Research*, Vol. 4, p.113-123 (1989).
- [2] Stroh, A.N., "Steady State Problems in Anisotropic Elasticity," *Journal of Mathematical Physics*, Vol. 41, p. 77-103 (1962).
- [3] Lekhnitskii, S.G., *Theory of Elasticity of an Anisotropic Elastic Body*, Holden-Day, Inc., (1963).
- [4] Cruse, T. A., *Boundary Element Analysis in Computational Fracture Mechanics*, Kluwer Academic Publishers (1988).
- [5] Brebbia, C. A. and Dominguez, J., *Boundary Elements, An Introductory Course*, Computational Mechanics Publications/McGraw-Hill Book Company (1989).
- [6] Hirth, J. P. and Lothe, J., *Theory of Dislocations*, Second Edition, Krieger Publishing Co. (1992).
- [7] Cruse, T. A., *Mathematical Foundations of the Boundary-Integral Equation Method in Solid Mechanics*, Air Force Office of Scientific Research Report, AFOSR-TR-77-1002 (1977).
- [8] Snyder, M. D. and Cruse, T. A., *Crack Tip Stress Intensity Factors in Finite Anisotropic Plates*, Air Force Material Laboratory Technical Report, AFML-TR-73-209 (1973).
- [9] Snyder, M. D. and Cruse, T. A., "Boundary Integral Equation Analysis of Cracked Anisotropic Plates," *International Journal of Fracture*, Vol. 11, No. 2, pp. 315-328 (1975).
- [10] Yuuki, R., Cho, S.-B., Matsumoto, T., "Usefulness of Hetenyi's Solution for Boundary Element Analysis of Crack Problems in Dissimilar Materials," *Role of Fracture Mechanics in Modern Technology*, G. C. Sih, H. Nisitani, and T. Ishihara (Eds.), Elsevier Science Publishers, pp. 823-834 (1987).

- [11] Yuuki, R. and Cho, S.-B., "Efficient Boundary Element Analysis of Stress Intensity Factors for Interface Cracks in Dissimilar Materials," *Engineering Fracture Mechanics*, Vol. 34, No. 1, pp. 179-188 (1989).
- [12] Eshelby, E., Read, W. T. , and Shockley, W., "Anisotropic Elasticity with Applications to Dislocation Theory," *Acta Metallurgica*, Vol. 1, pp. 251-259 (1953).
- [13] Tewary, V. K., Wagoner, R. H., and Hirth, J. P., "Elastic Green's Function for a Composite Solid with a Planar Crack in the Interface," *Journal of Materials Research*, Vol. 4, No. 1 (1989).
- [14] Benzley, S. P., "Representation of Singularities with Isoparametric Finite Elements," *International Journal for Numerical Methods in Engineering*, Vol. 8, pp. 537-545 (1974).
- [15] Gifford, L. N., and Hilton, P. D., "Stress Intensity Factors by Enriched Finite Elements," *Engineering Fracture Mechanics*, Vol. 10, pp. 485-496 (1978).
- [16] Heyliger, P. R., and Kriz, R. D., "Stress Intensity Factors by Enriched Mixed Finite Elements", *International Journal for Numerical Methods in Engineering*, Vol. 28, pp. 1461-1473 (1989).

Appendix A - Derivatives of the Fundamental Displacement Solution

Case (i): $x_2 > 0, x_2' > 0$

$$\frac{d\mathbf{U}^{11}}{dz_{\mathbf{a}}} = \text{Re} \frac{1}{\mathbf{P}} \left\{ -\sum_{\mathbf{a}} \frac{\mathbf{g}(p_{\mathbf{a}}^A)^T}{z_{\mathbf{a}}^A - z_{\mathbf{a}}'^A} - \sum_{\mathbf{ab}} \frac{\mathbf{Q}_{\mathbf{b}}^{I^T} \mathbf{g}(p_{\mathbf{a}}^A)^T}{z_{\mathbf{a}}^A - z_{\mathbf{b}}'^{A^*}} \right\} \quad (\text{A.1})$$

$$\frac{d^2\mathbf{U}^{11}}{dz_{\mathbf{a}}^2} = \text{Re} \frac{1}{\mathbf{P}} \left\{ \sum_{\mathbf{a}} \frac{\mathbf{g}(p_{\mathbf{a}}^A)^T}{(z_{\mathbf{a}}^A - z_{\mathbf{a}}'^A)^2} + \sum_{\mathbf{ab}} \frac{\mathbf{Q}_{\mathbf{b}}^{I^T} \mathbf{g}(p_{\mathbf{a}}^A)^T}{(z_{\mathbf{a}}^A - z_{\mathbf{b}}'^{A^*})^2} \right\} \quad (\text{A.2})$$

Case (ii): $x_2 < 0, x_2' > 0$

$$\frac{d\mathbf{U}^{12}}{dz_{\mathbf{a}}^*} = \text{Re} \frac{1}{\mathbf{P}} \left\{ -\sum_{\mathbf{ab}} \frac{\mathbf{Q}_{\mathbf{b}}^{II^T} \mathbf{g}^*(p_{\mathbf{a}}^B)^T}{z_{\mathbf{a}}^{B^*} - z_{\mathbf{b}}'^{A^*}} \right\} \quad (\text{A.3})$$

$$\frac{d^2\mathbf{U}^{12}}{dz_{\mathbf{a}}^{*2}} = \text{Re} \frac{1}{\mathbf{P}} \left\{ \sum_{\mathbf{ab}} \frac{\mathbf{Q}_{\mathbf{b}}^{II^T} \mathbf{g}^*(p_{\mathbf{a}}^B)^T}{(z_{\mathbf{a}}^{B^*} - z_{\mathbf{b}}'^{A^*})^2} \right\} \quad (\text{A.4})$$

Case (iii): $x_2 > 0, x_2' < 0$

$$\frac{d\mathbf{U}^{21}}{dz_{\mathbf{a}}} = \text{Re} \frac{1}{\mathbf{P}} \left\{ -\sum_{\mathbf{ab}} \frac{\mathbf{Q}_{\mathbf{b}}^{III^T} \mathbf{g}(p_{\mathbf{a}}^A)^T}{z_{\mathbf{a}}^A - z_{\mathbf{b}}'^B} \right\} \quad (\text{A.5})$$

$$\frac{d^2\mathbf{U}^{21}}{dz_{\mathbf{a}}^2} = \text{Re} \frac{1}{\mathbf{P}} \left\{ -\sum_{\mathbf{ab}} \frac{\mathbf{Q}_{\mathbf{b}}^{III^T} \mathbf{g}(p_{\mathbf{a}}^A)^T}{(z_{\mathbf{a}}^A - z_{\mathbf{b}}'^B)^2} \right\} \quad (\text{A.4})$$

Case (iii): $x_2 < 0, x_2' < 0$

$$\frac{d\mathbf{U}^{22}}{dz_{\mathbf{a}}^*} = \text{Re} \frac{1}{\mathbf{P}} \left\{ -\sum_{\mathbf{a}} \frac{\mathbf{g}^*(p_{\mathbf{a}}^B)^T}{z_{\mathbf{a}}^{B^*} - z_{\mathbf{a}}'^{B^*}} + \sum_{\mathbf{ab}} \frac{\mathbf{Q}_{\mathbf{b}}^{IV^T} \mathbf{g}^*(p_{\mathbf{a}}^B)^T}{z_{\mathbf{a}}^{B^*} - z_{\mathbf{b}}'^B} \right\} \quad (\text{A.5})$$

$$\frac{d^2\mathbf{U}^{22}}{dz_{\mathbf{a}}^{*2}} = \text{Re} \frac{1}{\mathbf{P}} \left\{ \sum_{\mathbf{a}} \frac{\mathbf{g}^*(p_{\mathbf{a}}^B)^T}{(z_{\mathbf{a}}^{B^*} - z_{\mathbf{a}}'^{B^*})^2} - \sum_{\mathbf{ab}} \frac{\mathbf{Q}_{\mathbf{b}}^{IV^T} \mathbf{g}^*(p_{\mathbf{a}}^B)^T}{(z_{\mathbf{a}}^{B^*} - z_{\mathbf{b}}'^B)^2} \right\} \quad (\text{A.6})$$