The Higgs Boson in the Minimal Non-Supersymmetric Standard Model

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1 Introduction

In 1990 the GIF Summer School addressed the question "Où et le Higgs?"[1] The question remains unanswered. In fact, this is THE QUESTION for high energy physics. It dominates all discussions of the future directions of the field. Should we build a TeV scale linear collider? What is the future of the Tevatron collider? What comes after LHC?

Of course "le Higgs" may be 'les Higgs." For many theorists, supersymmetry is an established fact, and thus there are at least five Higgs bosons, three neutral and two charged. My view on this was expressed, perhaps inadvertantly, by a colleague, who in introducing another colleague, Bruno Zumino, co-inventor of supersymmetry said "Supersymmetry has withstood the test of time, though there is no evidence to support it." Indeed it has withstood the test of time because there is so much to like about it. Nonetheless, it is prudent to insist upon it be found in experiment. In this summer school, the Higgs boson and its surrogates will be explored in supersymmetry and in alternatives to supersymmetry. Still, the proper starting point is the Minimal Non-SuperSymmetric Model, the MNSSM, and this is the subject of these lectures.

The Standard Model didn't burst onto the scene. In crept in, almost unnoticed. Though the outlines were present in Glashow's paper [2] of 1961 and nearly the whole story was there in Weinberg's [3, 4] 1967 paper, the cross section for the fundamental process $e^-\nu_e \to e^-\nu_e$ wasn't calculated until a student named 't Hooft did it in 1971 [5]. The model wasn't embraced because it contained neutral weak currents, something known to be absent since there was no decay like $K_L \to \mu^+\mu^-$. Well, that at least showed that there were no strangeness-changing decays. The discovery on non-strangeness changing neutral weak currents by the Gargamelle experiment [6] was circumstantial evidence that the Glashow-Weinberg-Salam model might well be correct.

What the model did was to provide not a model but a theory of heavy W bosons. Fermi in his original paper laying out weak interaction theory already recognized the parallel with electromagnetism. What was needed was not a neutral massless vector particle, but a charged massive vector particle. One could always write down such a model, but with the $V - A$ couplings to fermions of the weak interactions, the model wasn't renormalizable. You couldn't calculate beyond Born (tree) approximation.

The trick of the Standard Model was to fool the model into believing the vector particles were massless when they weren't. This is done by writing down a model with seemingly massless vector (gauge) particles, but then providing them with a mass through their interactions with a ubiquitous and constant field.

This sort of deception is part of a more pervasive phenomenon. When we write down a theory, we prejudice its interpretation by the choice we make for the symbols. If we write m we expect this to represent a mass and if we write ϕ we think it is probably a scalar field, while A_μ must surely be the electromagnetic potential. However, the lesson learned in the renormalization of the archetypical theory, QED, was that you must always look to the physical predictions for interpretation. It is not the bare quantities that we write down that count, but the quantities that ultimately appear in expressions for physical measurements.

2 Spontaneous Breakdown and the Higgs Mechanism

2.1 Scalar Field Models with Spontaneous Breakdown

As a simple example of deception, consider a single scalar field ϕ with the usual sort of Lagrangian

$$\mathcal{L} = \partial_\alpha \phi \partial^\alpha \phi - \frac{1}{2}\mu^2\phi^2 - \frac{1}{2}\lambda\phi^4 \tag{2.1}$$

which appears to describe a neutral scalar field with mass μ, interacting with itself, with a coupling λ. Suppose, however, that $\mu^2 < 0$. What does this theory mean? The potential

$$V = \frac{1}{2}\mu^2\phi^2 + \frac{1}{2}\lambda\phi^4 \tag{2.2}$$

has a minimum when

$$\phi^2 = -\frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2} \tag{2.3}$$

It is therefore appropriate to define a new field that represents the deviation of ϕ from its value when the potential is minimized. Choosing that to be $v/\sqrt{2}$ ($v > 0$) rather than its negative, we write

$$\phi = \frac{v}{\sqrt{2}} + \rho \tag{2.4}$$

In terms of the new field,

$$\mathcal{L} = \frac{1}{2}\partial_\alpha\rho\partial^\alpha\rho - \frac{3}{8}\frac{\mu^4}{\lambda} + \mu^2\rho^2 - \sqrt{\frac{-\mu^2\lambda}{2}}\rho^3 - \frac{\lambda}{2}\rho^4 \tag{2.5}$$

What emerges is a new scalar particle with mass squared equal to $-2\mu^2 > 0$. The constant term is irrelevant. There are both cubic and quartic couplings. The cubic coupling arising from the original quartic coupling, where one of the ϕ fields is replaced by $\langle\phi\rangle = v/\sqrt{2}$.

The original model had a symmetry, $\phi \to -\phi$ that is not enjoyed by the new field ρ. We had to pick either $v/\sqrt{2}$ or $-v/\sqrt{2}$ as the expectation value of ϕ to start our perturbation theory. With that choice, we lost the symmetry.

Consider now an embellishment of the first model. Suppose we have four real scalar fields, each just like the one above. The Lagrangian is

$$\mathcal{L} = \partial_\alpha\phi_i\partial^\alpha\phi_i - \frac{1}{2}\mu^2\phi_i\phi_i - \frac{1}{2}\lambda\phi_i\phi_i \tag{2.6}$$

where we have used the usual summation convention.

It is clear that the minimum of the potential energy occurs for

$$\phi_i\phi_i = -\frac{\mu^2}{2\lambda} \tag{2.7}$$

What is the particle content of this theory? We shall have to pick some direction for the field that gets a vacuum expectation value. Lets take $i = 0$ and introduce new notation:

$$\phi_0 = \sigma; \qquad \phi_i = \pi_i, i \neq 0 \tag{2.8}$$

Now we can write our Lagrangian quite simply as

$$\mathcal{L} = -\frac{\mu^2}{2}(\sigma^2 + \pi_i\pi_i) - \frac{\lambda}{2}(\sigma^2 + \pi_i\pi_i)^2 \tag{2.9}$$

If we write

$$\sigma = v + \rho \tag{2.10}$$

then the terms that do not contain any π_i will be just as before. Thus ρ again represents a scalar with mass squared $-2\mu^2$. What about the π_i fields?

To find their masses we need only the terms quadratic in them. These come from

$$-\frac{\mu^2}{2}(\pi_i\pi_i) - \frac{\lambda}{2}(\langle\sigma\rangle^2 + \pi_i\pi_i)^2 = -\frac{\mu^2}{2}(\pi_i\pi_i) - \frac{\lambda}{2}(-\frac{\mu^2}{2\lambda} + \pi_i\pi_i)^2 \tag{2.11}$$

But we see then that the quadratic terms all vanish! All three of the π_i are massless scalar fields. What looked like four scalar fields with mass squared μ^2 (though that couldn't be right since $\mu^2 < 0$), turned out actually to be one massive scalar (with mass squared $-2\mu^2$) and three massless scalars. What is going on here?

This is an explicit example of Goldstone's Theorem, which states that if a symmetry is broken spontaneously, every "broken generator" gives rise to a massless scalar particle. Here we began with four real fields, all equivalent. The symmetry was $O(4)$. The rotation group $O(N)$ has $N(N-1)/2$ generators, so $O(4)$ has six generators. After the σ got singled out, there were just the three π_i that were equivalent. This remaining $O(3)$ has three generators (its just the rotation group), so we lost (or broke) three generators. There therefore had to be three massless scalars. And so there were.

We can look at this same model in another way. Instead of thinking of four equivalent scalars or of a single σ and a triplet π, we can write

$$\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 - i\phi_2 \\ \phi_3 - i\phi_4 \end{pmatrix}; \qquad \begin{pmatrix} \overline{\phi}^0 \\ -\phi^- \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_3 + i\phi_4 \\ -(\phi_1 + i\phi_2) \end{pmatrix} \tag{2.12}$$

Here we see four particles arranged in a complex doublet, analogous to the K system.

Finally return to the picture $(\sigma, \vec{\pi})$. It is a fact of group theory that the algebra for $O(4)$ is the same as two $SU(2)$s i.e. two rotation groups. For one of these rotation groups, it is clear that σ is a scalar and $\vec{\pi}$ is a vector. If we consider a small rotation

$$e^{-i\delta\vec{\alpha}\cdot\vec{T}} \approx 1 - i\delta\vec{\alpha}\cdot\vec{T} \tag{2.13}$$

where T are the 3×3 rotation matrices, the changes in σ and $\vec{i}$ are

$$\delta\sigma = 0; \qquad \delta\vec{\pi} = -i\delta\vec{\alpha}\cdot\vec{T}\vec{\pi} = -i\delta\vec{\alpha}\times\vec{\pi} \tag{2.14}$$

What is the other rotation group?

It turns out that if we take a small "new" rotation

$$e^{-i\delta\vec{\beta}\cdot\vec{T}'} \approx 1 - i\delta\vec{\beta}\cdot\vec{T}' \tag{2.15}$$

we can define its action by

$$\delta\pi = -\delta\beta\sigma; \qquad \delta\sigma = \delta\vec{\beta}\cdot\vec{\pi} \tag{2.16}$$

How can we see that this works? Just calculate the change in $\sigma^2 + \pi^2$:

$$\begin{aligned} \delta(\sigma^2 + \pi^2) &= 2(\sigma\delta\sigma + \vec{\pi}\cdot\vec{\delta\pi}) \\ &= 2(\sigma(\delta\vec{\beta}\cdot\vec{\pi}) + \vec{\pi}\cdot(-\delta\beta\sigma) = 0 \end{aligned} \tag{2.17}$$

This model also has a discrete symmetry, which we can call parity:

$$\vec{\pi} \to -\vec{\pi}; \qquad \sigma \to \sigma \tag{2.18}$$

so in this sense we can call π pseudoscalar and σ scalar. The two rotation symmetries can be termed "vector" and "axial," since the latter mixes the scalars and pseudoscalars. We see that after σ gets a vacuum expectation value, only the vector symmetry survives.

2.2 Abelian Higgs Model

We've that what you see isn't always what you get. Now we extend this to gauge theories. The simplest gauge theory is electromagnetism. To make it interesting there needs to be a charged particle, so let's use a charged scalar, whose Lagrangian is

$$\mathcal{L} = \partial_\alpha\phi^\dagger\partial^\alpha\phi - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \tag{2.19}$$

The Lagrangian for electromagnetism is

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{2.20}$$

How can we combine these? The answer, of course, is to change

$$\partial_\alpha \to \partial_\alpha + ieA_\alpha \tag{2.21}$$

so that we have

$$\mathcal{L} = (\partial_\alpha - ieA_\alpha)\phi^\dagger(\partial^\alpha + ieA^\alpha)\phi - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \\ -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{2.22}$$

Why is this right? Because it gives correctly the laws of quantum mechanics and agrees with atomic physics. However, we notice that this choice isn't random. It has special properties. Consider the transformation

$$\phi \to e^{-i\theta(x)}\phi; \qquad A_\alpha \to A_\alpha + \frac{1}{e}\partial_\alpha\theta \tag{2.23}$$

The Lagrangian is invariant under this *gauge* transformation. For the history of gauge transformations in electromagnetic theory, see [7].

So this appears to be a theory of a charged scalar with mass μ interacting with the usual electromagnetic field. However, let's ask what happens when $\mu^2 < 0$. First we know that $|\phi|^2$ will have a non-zero vacuum expectation value and in fact

$$\langle|\phi|^2\rangle = \frac{-\mu^2}{2\lambda} \equiv \frac{v^2}{2} \tag{2.24}$$

We know even more. Before this happens, the theory is invariant under

$$\phi \to e^{-i\zeta}\phi \tag{2.25}$$

where ζ is a constant. This symmetry is called $U(1)$. It is simply the group of rotations in the complex plane. It has one generator. But this symmetry is broken once we pick a value for $\langle\phi\rangle$. So we anticipate that we will find one massive scalar and one massless scalar. But we are wrong!

Let's write the old field ϕ in a way that incorporates what we know about the vacuum expectation value:

$$\phi = \frac{v+\rho}{\sqrt{2}}e^{i\chi(x)/v} \tag{2.26}$$

But remember that we can change ϕ by a phase that depends on x if we wish. That just changes A_μ into some new A_μ. Use this freedom to get rid of χ! Next, rewrite the Lagrangian:

$$\mathcal{L} = \frac{1}{2}|(\partial_\alpha + ieA_\alpha)(v+\rho)|^2 - \frac{1}{2}\mu^2(v+\rho)^2$$

$$
\begin{aligned}
& -\frac{1}{4}\lambda(v+\rho)^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
= \;& \frac{1}{2}\partial_\alpha\rho\partial^\alpha\rho - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\frac{\mu^2}{2}\rho^2 + \dots \qquad (2.27)
\end{aligned}
$$

The ... indicates cubic and quartic terms, which represent interactions. Remarkably, what emerges is a massive vector field A_μ, with mass ev, and a scalar field, ρ with mass squared $-2\mu^2 > 0$.

What happened to the massless scalar particle we knew had to be there? It was eaten by the photon, which thereby became massive and thus needed a third degree of freedom. This is our paradigm, due to Peter Higgs [8].

2.3 Non-Abelian Gauge Symmetry

Long before Higgs, Glashow, Weinberg, and Salam, C. N. Yang and R. N. Mills provided [9] the next key ingredient for the Standard Model. The essential feature of the covariant derivative above is the property that if the field ϕ is transformed by a space-time dependent function

$$\phi \to e^{-i\chi Q}\phi \qquad (2.28)$$

where the charge Q is +1 for ϕ then the covariant derivative of ϕ, $D_\mu\phi = (\partial_\mu + ieQA_\mu)\phi$, transforms in the same way:

$$
\begin{aligned}
D_\mu\phi &= (\partial_\mu + ieQA_\mu)\phi \to= (\partial_\mu + ieQA_\mu + ie\frac{1}{e}Q\partial_\mu\chi)e^{-i\chi Q}\phi \\
&= e^{-i\chi Q}D_\mu\phi \qquad (2.29)
\end{aligned}
$$

This means that the whole Lagrangian is invariant under this space-time dependent (local) gauge transformation. The problem solved by Yang and Mills was how to generalize this to a non-Abelian symmetry.

A general Lie algebra of the sort we care about can be written

$$[T^a, T^b] = if_{abc}T^c \qquad (2.30)$$

where, with suitable normalization, f_{abc} is completely antisymmetric. For the rotation group $O(3)$, $f_{abc} = \epsilon_{abc}$. The abstract algebra will have many representations that satisfy the commutation relations. For the rotation group we know these representations are characterized by j and that the dimensionality of the representation is $2j+1$, so that the matrices T are then

$(2j+1) \times (2j+1)$. Some particle multiplet transforming under a representation of this algebra will behave as

$$\phi \to e^{-i\chi^a T^a} \phi \tag{2.31}$$

We are interested in χ^a that depend on space-time, x. Now the ordinary gradient will behave as

$$\partial_\mu \phi \to \partial_\mu e^{-i\chi^a T^a} \phi = e^{-i\chi^a T^a} \partial_\mu \phi + \partial_\mu e^{-i\chi^a T^a} \tag{2.32}$$

We need to add something to ∂_μ to fix up this last piece. By analogy with electromagnetism we guess that the addition must involve the χ^a. To take care of the a index, we need to have gauge fields labeled with the same index: A^a_μ. Thus there must be as many gauge fields as there are generators of the group. Now once we have A^a_μ, we can't combine it with ∂_μ until we get rid of the a index, by writing $T^a A^a_\mu$. Now if χ^a didn't depend on x we could use just the properties of the group to deduce how A^a_μ transforms:

$$T \cdot A_\mu \to e^{-i\chi \cdot T} T \cdot A_\mu e^{i\chi \cdot T} \tag{2.33}$$

where $\chi \cdot T = \chi^a T^a$, $T \cdot A_\mu = T^a A^a_\mu$. So lets try the following: define δA_μ by

$$T \cdot A_\mu \to e^{-i\chi \cdot T} T \cdot A_\mu e^{i\chi \cdot T} + T \cdot \delta A_\mu \tag{2.34}$$

and suppose that the covariant derivative we want is

$$D_\mu \phi = (\partial_\mu + igT \cdot A_\mu)\phi \tag{2.35}$$

Then

$$\begin{aligned} D_\mu \phi &\to (\partial_\mu + ige^{-i\chi \cdot T} T \cdot A_\mu e^{i\chi \cdot T} + igT \cdot \delta A_\mu) e^{-i\chi \cdot T} \phi \\ &= e^{-i\chi \cdot T} \partial_\mu \phi + (\partial_\mu e^{-i\chi \cdot T})\phi + e^{-i\chi \cdot T} igT \cdot A_\mu \phi + igT \cdot \delta A_\mu e^{-i\chi \cdot T} \phi \end{aligned} \tag{2.36}$$

so we take

$$T \cdot \delta A_\mu = \frac{i}{g} (\partial_\mu e^{-i\chi \cdot T}) e^{i\chi \cdot T} \tag{2.37}$$

(It isn't clear that this definition of δA_μ really works, since the right hand side seems to depend on which representation of the T's we are using. That is, it looks like the change in A_μ depends on the kind of scalar particles we have. What would we do if we had two different kinds of scalars? In fact, despite its appearance, the expression gives the same δA_μ no matter what representation we have. You may find it entertaining to prove this. What is required is to show that the express on the right hand side can be written entirely in terms of $Y = \partial_\mu T \cdot \chi$ and its commutators with $X = T \cdot \chi$)

With this rule for the transformation of A_μ we thus find

$$D_\mu \phi \rightarrow e^{-i\chi \cdot T} D_\mu \phi \tag{2.38}$$

The next step is to find the field strength $F_{\mu\nu}$ to associate with A_μ so that it will be well behaved:

$$T \cdot F_{\mu\nu} \rightarrow e^{-i\chi \cdot T} T \cdot F_{\mu\nu} e^{i\chi \cdot T} \tag{2.39}$$

where T are the representation matrices for any representation. Now we see from Eq.(2.38) that

$$D_\mu \rightarrow e^{-i\chi \cdot T} T \cdot D_\mu \; e^{i\chi \cdot T} \tag{2.40}$$

so consider simply

$$[D_\mu, D_\nu] = ig[\partial_\mu T \cdot A_\nu - \partial_\nu T \cdot A_\mu + ig[T \cdot A_\mu, T \cdot A_\nu] \equiv igT \cdot F_{\mu\nu} \tag{2.41}$$

This must transform also as

$$[D_\mu, D_\nu] \rightarrow e^{-i\chi \cdot T} [D_\mu, D_\nu] e^{i\chi \cdot T} \tag{2.42}$$

so this choice of

$$\begin{aligned} T \cdot F_{\mu\nu} &= \partial_\mu T \cdot A_\nu - \partial_\nu T \cdot A_\mu + ig[T \cdot A_\mu, T \cdot A_\nu] \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu \end{aligned} \tag{2.43}$$

works just right.

2.4 Non-Abelian Higgs Model

Now we combine these ingredients, writing a Yang-Mills theory coupled to a set of scalars ϕ that transform according to T, a representation of some non-Abelian group.

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} (D_\mu \phi)^\dagger D^\mu \phi - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \tag{2.44}$$

As usual, we take $\mu^2 < 0$, and find that at the minimum

$$\langle\phi\rangle^\dagger\langle\phi\rangle = \frac{-\mu^2}{2\lambda} \equiv \frac{v^2}{2} \tag{2.45}$$

Now consider the convariant derivative D_μ applied to this vacuum expectation value (which we take to be constant, independent of space or time):

$$D_\mu\langle\phi\rangle = igA^a_\mu T^a\langle\phi\rangle \tag{2.46}$$

For each a, which labels a generator of the group, $T^a\langle\phi\rangle$ is a column vector. In the Lagrangian we find a term quadratic in the A's

$$\frac{1}{2}g^2 A^a_\mu A^{b\mu}(T^a\langle\phi\rangle)^\dagger(T^b\langle\phi\rangle) = \frac{1}{2}m^2_{ab}A^a_\mu A^{b\mu} \tag{2.47}$$

that is, there is a mass-squared matrix for the vector particles A. To find the physical vector particles and their masses we need to diagonalize this matrix. However, there is one observation we can make right away. Suppose we can write

$$T\cdot A_\mu = T^0 A^0_\mu + \sum_{a'} T^{a'}A^{a'}_\mu \tag{2.48}$$

where the linear combination of generators T^0 has the property

$$T^0\langle\phi\rangle = 0 \tag{2.49}$$

Then the m^2_{00} entry is zero and so is $m^2_{a0} = m^2_{0a}$. That means that A^0_μ represents a massless field, despite the Higgs mechanism.

3 Standard Model

To make a model of electrodynamics and weak interactions we need both a neutral gauge boson, the photon, and charged gauge bosons, the Ws. The most economical approach would be to start with a group that has three generators, $SU(2)$. This actually doesn't work and instead the answer turns out to be that the group is $SU(2)\times U(1)$. The three generators of the $SU(2)$ are called $\vec{T}$ and the generator of the $U(1)$ is called $Y/2$ for historical reasons. The Gell-Mann Nishijima relation connects the third component of isospin, the hypercharge (baryon number +strangeness), and the electric charge. In modern language, the Gell-Mann Nishijima relation follows simply from the quantum numbers of the first three quarks, which satisfy

$$Q = T_3 + Y/2 \tag{3.1}$$

We call $\vec{T}$ weak isospin and Y weak hypercharge. We shall arrange that the analog of the Gell-Mann Nishijima relation is true. One thing that is new here is that the weak interactions, which have a $V - A$ character, interact with the left-handed fermions, not the right-handed fermions. Thus the left-handed and right-handed versions of a fermion will have different weak quantum numbers. According to our previous discussion we need to have a scalar multiplet with

$$Q\langle\phi\rangle = 0 \tag{3.2}$$

so that the photon will remain massless.

Because we really have two independent symmetries, we can have two independent coupling constants, g and g'. We call the gauge fields associated with the $SU(2)$ W and the that associated with the $U(1)$, B. Then the covariant derivative is

$$D_\mu = \partial_\mu + ig\vec{T}\cdot\vec{W}_\mu + ig'\frac{Y}{2}B_\mu \tag{3.3}$$

We rewrite this as

$$D_\mu = \partial_\mu + i\frac{g}{\sqrt{2}}[T^+W_\mu^+ + T^-W_\mu^-] + igT_3W_{3\mu} + ig'\frac{Y}{2}B_\mu \tag{3.4}$$

We make the physical interpretation

$$Q = T_3 + Y/2 \tag{3.5}$$

For the scalar field we use the complex doublet described above. The generators $\vec{T}$ are just $\frac{1}{2}\vec{\sigma}$.

$$\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 - i\phi_2\\ \phi_3 - i\phi_4\end{pmatrix}; \qquad \begin{pmatrix}\overline{\phi}^0\\ -\phi^-\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_3 + i\phi_4\\ -(\phi_1 + i\phi_2)\end{pmatrix} \tag{3.6}$$

We define the value of Y on the doublet to be 1. This is consistent with the charges implied by the symbols ϕ^+ and ϕ^0.

The full Lagrangian is

$$\mathcal{L} = -\frac{1}{4}W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (D_\mu\phi)^\dagger D^\mu\phi - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \tag{3.7}$$

Here $W^{\mu\nu}$ and $B^{\mu\nu}$ are the gauge covariant field strengths defined previously.

We still have the freedom to perform gauge transformations. First we use the rotation symmetry to make ϕ have only a "down" component. Next we use the $U(1)$ to change the phase so the field is real. From our previous examples we know that if $\mu^2 < 0$, we will have a minimum of the potential energy if $\langle\phi\rangle = v/\sqrt{2}$. Assemblying all this, we write

$$\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + \rho(x) \end{pmatrix} \tag{3.8}$$

with

$$v^2 = \frac{-\mu^2}{\lambda} \tag{3.9}$$

The field $\rho(x)$ represents a scalar particle, the Higgs boson.

3.1 Gauge Bosons

To find the mass squared matrix of the vector bosons we compute

$$\begin{aligned} D_\mu\langle\phi\rangle &= \left\{i\frac{g}{\sqrt{2}}[T^+W_\mu^+ + T^-W_\mu^-] + igT_3W_{3\mu} + ig'\frac{Y}{2}B_\mu\right\}\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} igW^+v/2 \\ -i\frac{gv}{2\sqrt{2}}W_3 + i\frac{g'v}{2\sqrt{2}}B \end{pmatrix} \end{aligned} \tag{3.10}$$

It is easy now to find

$$(D_\mu\langle\phi\rangle)^\dagger(D^\mu\langle\phi\rangle) = \frac{g^2v^2}{4}W^+W^- + \frac{(g^2+g'^2)v^2}{8}(\frac{gW_3 - g'B}{\sqrt{g^2+g'^2}})^2 \tag{3.11}$$

The first term provides the masses for $W^\pm$:

$$M_W^2 = \frac{g^2v^2}{4} \tag{3.12}$$

while the second provides the mass

$$M_Z^2 = \frac{(g^2+g'^2)v^2}{4} \tag{3.13}$$

for

$$Z = \cos\theta_W W_3 - \sin\theta_W B \tag{3.14}$$

while the orthogonal combination

$$A = \cos\theta_W B + \sin\theta_w W_3 \tag{3.15}$$

is the massless photon.

We have defined the traditon weak mixing angle, $\sin\theta_W$ by

$$\sin\theta_W = \frac{g}{\sqrt{g^2 + g'^2}} \tag{3.16}$$

Now let's rewrite the covariant derivative using

$$\begin{aligned} W_3 &= \sin\theta_W A + \cos\theta_W Z \\ B &= \cos\theta_W A - \sin\theta_W Z \end{aligned} \tag{3.17}$$

This gives us

$$\begin{aligned} D_\mu &= \partial_\mu + i\frac{g}{\sqrt{2}}[T^+W_\mu^+ + T^-W_\mu^-] + igT_3[\sin\theta_W A + \cos\theta_W Z] + ig'\frac{Y}{2}[\cos\theta_W A - \sin\theta_W Z] \\ &= \partial_\mu + i\frac{g}{\sqrt{2}}[T^+W_\mu^+ + T^-W_\mu^-] + \frac{ig}{\cos\theta_W}(T_3 - Q\sin^2\theta_W)Z_\mu + ig\sin\theta_W QA_\mu \end{aligned} \tag{3.18}$$

from which we deduce

$$e = g\sin\theta_W; \qquad \frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2} \tag{3.19}$$

3.2 Fermions in the Standard Model

Finally, we need to make contact with fermions. The Dirac equation

$$(i\partial\!\!\!/ - m)\psi = 0 \tag{3.20}$$

becomes

$$(iD\!\!\!\!/ - m)\psi = 0 \tag{3.21}$$

and there is a term in the Lagrangian

$$\mathcal{L} = \overline{\psi}(iD\!\!\!\!/ - m)\psi \tag{3.22}$$

The ψ multiplets for the Standard Model look like this: For left-handed fermions we have doublets

$$\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \tag{3.23}$$

while for right-handed fermions we have $SU(2)$ singlets. When we say left-handed and right-handed we really mean

$$e_L = \frac{1}{2}(1-\gamma_5)e; \qquad e_r = \frac{1}{2}(1+\gamma_5)e \tag{3.24}$$

These represent left-handed and right-handed particles only when the particles are ultrarelativistic.

In order to get the electric charge to come out right we need to make the following assignments:

	e_L	e_R	ν_L	u_L	u_R	d_L	d_R
T_3	$-1/2$	0	$1/2$	$1/2$	0	$-1/2$	0
$Y/2$	$-1/2$	-1	$-1/2$	$1/6$	$2/3$	$1/6$	$-1/3$
Q	-1	-1	0	$2/3$	$2/3$	$-1/3$	$-1/3$

(3.25)

The mass of a fermion ordinarily arises from an interaction $m\overline{\psi}\psi$, but we can't use this. The reason is that actually

$$\overline{e}e = \overline{e}_L e_R + \overline{e}_R e_L \tag{3.26}$$

but each term on the right combines a weak $SU(2)$ doublet with a weak $SU(2)$ singlet. We can't make a weak isosinglet this way. We would ruin the $SU(2)$ invariance, which is needed to make the model work. The solution is to use the scalar field and write a "Yukawa" coupling: We write the doublets

$$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}; \qquad Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}; \qquad H = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}; \qquad H^c = \begin{pmatrix} \overline{\phi}^0 \\ -\phi^- \end{pmatrix} \tag{3.27}$$

and compose

$$\mathcal{L}_{Yukawa} = g_e[\overline{L}He_R + \overline{e}_R H^\dagger L] + g_d[\overline{Q}Hd_R + \overline{d}_R H^\dagger Q] + g_u[\overline{Q}H^c u_R + \overline{u}_R H^{c\dagger} Q] \tag{3.28}$$

When the scalar field takes on its vacuum expectation value, these terms generate masses:

$$\mathcal{L}_{fermion\ masses} = \frac{v}{\sqrt{2}}[g_e\overline{e}e + g_d\overline{d}d + g_u\overline{u}u] \tag{3.29}$$

so that the Yukawa couplings are related to the fermion masses by

$$m_f = -\frac{g_f v}{\sqrt{2}} \tag{3.30}$$

What about v? It is actually already determined by the weak interactions. The coupling of the charged W to the fermions comes from the gauge interaction

$$\overline{L}[\frac{-g}{\sqrt{2}}T^-\not{W}^-]L \tag{3.31}$$

which gives terms like

$$\frac{-g}{\sqrt{2}}\overline{\mu}_L\not{W}^-\nu_L \tag{3.32}$$

The weak interaction is the result of W exchange and there is a factor of $1/m_W^2$ for its propagator. The interaction responsible for μ decay is thus

$$\left(\frac{g}{\sqrt{2}}\right)^2\overline{\mu}\frac{\gamma_\lambda}{2}(1-\gamma_5)\nu_\mu\frac{1}{M_W^2}\overline{\nu}_e\frac{\gamma^\lambda}{2}(1-\gamma_5)e \tag{3.33}$$

Comparing with the traditional weak interaction Lagrangian

$$\mathcal{L} = \frac{G_F}{\sqrt{2}}\overline{\mu}\gamma_\lambda(1-\gamma_5)\nu_\mu\frac{1}{M_W^2}\overline{\nu}_e\gamma^\lambda(1-\gamma_5)e \tag{3.34}$$

we conclude that

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} = \frac{1}{2v^2} \tag{3.35}$$

so

$$v = 246\ \mathrm{GeV} \tag{3.36}$$

4 Decays of the Higgs Boson

4.1 Decays into Gauge Bosons

The interaction between the Higgs boson and the gauge fields is prescribed by the covariant derivative:

$$\begin{aligned} D_\mu\phi &= [\partial_\mu + \frac{ig}{\sqrt{2}}(T^+W_\mu^+ + T^-W_\mu^-) + \frac{ig}{\cos\theta_W}(T_3 - Q\sin^2\theta_W)Z_\mu + ieQA_\mu] \begin{pmatrix} 0 \\ v+\rho \end{pmatrix} \frac{1}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} igW_\mu^+(v+\rho)/\sqrt{2} \\ \partial_\mu\rho - \frac{ig}{2\cos\theta_W}Z_\mu(v+\rho) \end{pmatrix} \end{aligned} \quad (4.1)$$

Thus

$$(D_\mu\phi)^\dagger(D^\mu\phi) = \frac{1}{2}\left[\frac{g^2}{2}W_\mu^+W^{-\mu}(v+\rho)^2 + \partial_\mu\rho\partial^\mu\rho + \frac{g^2}{4\cos^2\theta_W}Z_\mu Z^\mu(v+\rho)^2\right] \quad (4.2)$$

In addition to the masses of the W and Z, and the kinetic energy term for the Higgs boson, this determines the interaction between the Higgs and the Ws and Zs. Note that there are no couplings to the photon. The trilinear and quadrilinear couplings are given by

$$\frac{g^2}{2}(v\rho + \rho^2/2)[W_\mu^+W^{-\mu} + \frac{1}{4\cos^2\theta_W}Z_\mu Z^\mu] \quad (4.3)$$

Now the decay $H \to W^+W^-$ has the matrix element

$$-i\mathcal{M} = i(g^2v/2)\epsilon_+^*\epsilon_-^* = igm_W\epsilon_+^*\epsilon_-^* \quad (4.4)$$

according to the rules in Appendix B. From Appendix A, we see that the rule for the sum over polarizations is

$$\sum_{\epsilon_+}\epsilon_{+\mu}^*\epsilon_{+\nu} = -g_{\mu\nu} + \frac{p_\mu^+p_\nu^+}{m_W^2} \quad (4.5)$$

so if we sum over both polarizations,

$$\begin{aligned} |\mathcal{M}|^2 &= g^2m_W^2\left(g_{\mu\nu} - \frac{p_\mu^+p_\nu^+}{m_W^2}\right)\left(g^{\mu\nu} - \frac{p^{-\mu}p^{-\nu}}{m_W^2}\right) \\ &= \left(4 - 2 + \frac{(p^+\cdot p^-)^2}{m_W^4}\right) \\ &= \frac{g^2m_H^4}{4m_W^2}\left(1 - 4\frac{m_W^2}{m_H^2} + 12\frac{m_W^4}{m_H^4}\right) \end{aligned} \quad (4.6)$$

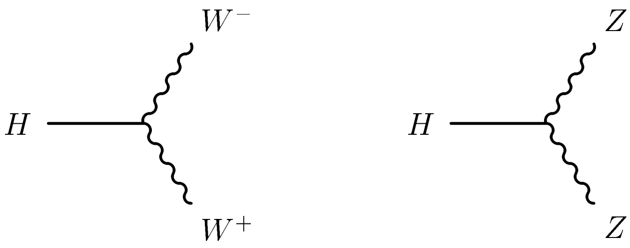


Figure 1: The decays of the Higgs into W and Z pairs are the dominant ones when they are kinematically allowed.

Using Eq.(A.4) and the connection between m_W and G_F

$$\begin{aligned}\Gamma(H \to WW) &= \frac{1}{8\pi}\frac{p_{cm}}{M^2}|\mathcal{M}|^2 \\ &= \frac{1}{8\pi}\frac{(m_H^2 - 4m_W^2)^{1/2}}{2m_H^2}\frac{2G_F}{\sqrt{2}}m_H^4\left(1 - 4\frac{m_W^2}{m_H^2} + 12\frac{m_W^4}{m_H^4}\right) \\ &= \frac{G_F m_H^3}{8\pi\sqrt{2}}\sqrt{1 - \frac{4m_W^2}{m_H^2}}\left(1 - 4\frac{m_W^2}{m_H^2} + 12\frac{m_W^4}{m_H^4}\right) \qquad (4.7)\end{aligned}$$

Similarly

$$\Gamma(H \to ZZ) = \frac{G_F m_H^3}{16\pi\sqrt{2}}\sqrt{1 - \frac{4m_Z^2}{m_H^2}}\left(1 - 4\frac{m_Z^2}{m_H^2} + 12\frac{m_Z^4}{m_H^4}\right) \qquad (4.8)$$

Even if the mass of the Higgs is insufficient for it to decay to two Z bosons, this channel remains important with the modification that one Z becomes virtual. In this case the partial width is [11]

$$\Gamma(H \to ZZ^*) = \frac{G_F m_H m_Z^2}{8\pi^2\sqrt{2}}\frac{\Gamma_Z}{m_z}F(\frac{m_Z^2}{m_H^2}) \qquad (4.9)$$

where

$$\begin{aligned}F(u) &= \frac{3(1 - 8u + 20u^2)}{\sqrt{4u-1}}\cos^{-1}\left(\frac{3u-1}{2u^{3/2}}\right) \\ &\quad -(1-u)(\left(\frac{47}{2}u - \frac{13}{2} + \frac{1}{u}\right) \\ &\quad -\frac{3}{2}(1 - 6u + 4u^2)\ln x \qquad (4.10)\end{aligned}$$

which holds as long as we are not too close to the threshold for the decay to a real Z pair. A similar expression holds for $H \to WW^*$. Note that the cost of not having one of the Z's on shell is roughly a factor Γ_Z/m_Z.

4.2 Decays into Fermion Antifermion Pairs

The coupling of the Higgs boson to fermion antifermion pairs is

$$\frac{g_f}{\sqrt{2}} H\overline{f}f \tag{4.11}$$

while

$$m_f = -\frac{g_f v}{\sqrt{2}} = -\frac{2g_f m_W}{\sqrt{2}g} \tag{4.12}$$

so the coupling is equivalently

$$-\frac{gm_f}{2m_W} H\overline{f}f \tag{4.13}$$

According to Appendix B

$$-i\mathcal{M} = -i\frac{gm_f}{2m_W}\overline{u}(p')v(p) \tag{4.14}$$

The square, summed over final state spins is

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{g^2m_f^2}{4m_W^2}\text{Tr } (\not p' + m_f)(\not p - m_f) \\ &= \frac{g^2m_f^2}{4m_W^2}4(p'\cdot p - m_f^2) =_f \frac{g^2m_f^2}{2m_W^2}(m_H^2 - 4m_f^2). \end{aligned} \tag{4.15}$$

Using the formulae from Appendix A,

$$\begin{aligned} d\Gamma &= \frac{1}{32\pi^2}\frac{p_{cm}}{M^2}|\mathcal{M}|^2 d\Omega \\ \Gamma &= \frac{1}{8\pi m_H^2}\cdot\frac{g^2m_F^2}{2m_W^2}\cdot\frac{1}{2}(m_H^2 - 4m_f^2)^{3/2}; \\ \Gamma(H\to f\overline{f}) &= \frac{G_F}{\sqrt{2}}\frac{m_f^2 m_H}{4\pi}\left(1 - \frac{4m_f^2}{m_H^2}\right)^{3/2} C_f \end{aligned} \tag{4.16}$$

where the color factor C_f is 1 for leptons and 3 for quarks.

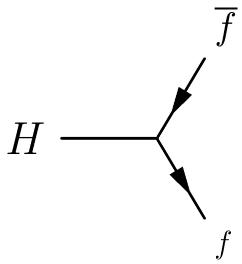


Figure 2: The decay to fermions favors high fermion mass.

For decays to b and t quarks we have

$$\begin{aligned}
\Gamma(H \to b\overline{b}) &= 4.0 \text{ MeV} \left(\frac{m_H}{100 \text{ GeV}}\right)\left(1 - \frac{4m_b^2}{m_H^2}\right)^{3/2} \\
\Gamma(H \to t\overline{t}) &= 24 \text{ GeV} \left(\frac{m_H}{400 \text{ GeV}}\right)\left(1 - \frac{4m_t^2}{m_H^2}\right)^{3/2}
\end{aligned} \tag{4.17}$$

The dominant decay mode of the Higgs boson is to $b\overline{b}$ from the $b\overline{b}$ threshold until WW^* surpasses it around $m_H = 150$ GeV. Once the ZZ channel enters, it has about half the strength of WW. Above its threshold, $t\overline{t}$ is competitive only for $400 < m_H < 600$ GeV, and even there its branching ratio is never more than about half that to ZZ.

4.3 Decays at One Loop to gg, $\gamma\gamma$, and γZ

Other Higgs boson decay occur only through one-loop processes. Consider for example $H \to \gamma\gamma$. There is certainly no direct coupling of this sort. The required form for a (electromagnetically) gauge-invariant coupling is

$$\mathcal{L} \propto HF_{\mu\nu}F^{\mu\nu} \tag{4.18}$$

where here F is the electromagnetic field strength. However, this has dimension $[M]^5$, since each bosonic field has dimension $[M]$ and each derivative also has dimension $[M]$. A renormalizable interaction must have dimension $[M]^4$ or less. Thus, there cannot be such a term in the basic Lagrangian of a renormalizable theory. It follows that this term, if induced at one loop, must be finite.

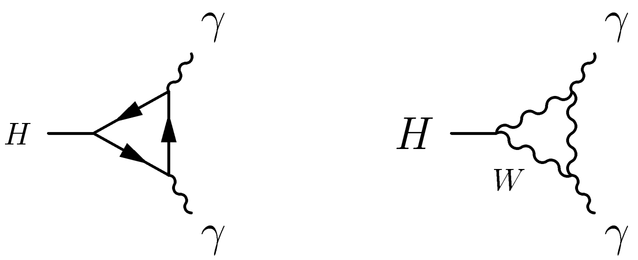


Figure 3: The decay $H \to \gamma\gamma$ receives contributions from fermion loops and W loops.

The Lorentz structure of the amplitude must be

$$\mathcal{M} = A(\epsilon_{1\mu}k_{1\nu} - \epsilon_{1\nu}k_{1\mu})(\epsilon^{2\mu}k^{2\nu} - \epsilon^{2\nu}k^{2\mu}) \tag{4.19}$$

If we take the polarization vectors to be purely spacelike, so $\vec{\epsilon}_1 \cdot \vec{k}_1 = 0$, $\vec{\epsilon}_2 \cdot \vec{k}_2 = 0$ then

$$\begin{aligned} \mathcal{M} &= -2A\vec{\epsilon}_1 \cdot \vec{\epsilon}_2 k_1 \cdot k_2 \\ \sum_{pol} |\mathcal{M}|^2 &= 2|A|^2 m_H^4 \end{aligned} \tag{4.20}$$

In calculating the partial decay width we must remember to divide by two because of the identical particles in the final state.

$$\Gamma = \frac{1}{16\pi}|A|^2 m_H^3 \tag{4.21}$$

The amplitude A receives contributions from W loops and from fermion loops. Both contributions can be written terms of a complex function

$$I(z) = 3\int_0^1 dy \int_0^{1-y} dt \frac{1-4yt}{1-yt/z} \tag{4.22}$$

First note that if $z \to \infty$, then $I(z) \to 1$. Now more generally, we find

$$\begin{aligned} I(z) &= 3[2z + (1-4z)f(z)] \\ f(z) &= 2z\left(\sin^{-1}\frac{1}{2\sqrt{z}}\right)^2; \quad z > 1/4 \\ &= -2z\left(-\cosh^{-1}\frac{1}{2\sqrt{z}} + \frac{i\pi}{2}\right); \quad z < 1/4 \end{aligned} \tag{4.23}$$

The details can be found in [1].

The contribution of the W loop is

$$A_W = \frac{ge^2}{16\pi^2 m_W}\left[\frac{2(2\lambda-1)I(\lambda)+10\lambda-1}{4\lambda-1}\right] \tag{4.24}$$

where $\lambda = m_W^2/m_H^2$.

The contribution from each fermion loop is

$$A_F = -\frac{2}{3}\frac{ge^2}{16\pi^2 m_W}Q_f^2 C_f I(\frac{m_f^2}{m_H^2}) \tag{4.25}$$

where Q_f is the charge of fermion f and C_f is one for charged leptons and 3 for quarks.

We can write the full result then as

$$\Gamma(H\to\gamma\gamma) = \frac{\alpha^2 G_F m_H^3}{128\pi^3\sqrt{2}}\left|``7'' - \frac{4}{3}\sum_f Q_f^2 C_f I(\frac{m_f^2}{m_H^2})\right|^2 \tag{4.26}$$

where

$$``7'' = \frac{4(2\lambda-1)I)(\lambda)+20\lambda-2}{4\lambda-1} \tag{4.27}$$

In the limit of $m_H << m_W$, $``7'' \to 7$.

While the branching ratio to $\gamma\gamma$ is always small, this decay has a distinctive signature. For a Higgs boson too light to decay to ZZ^* or WW^*, this channel may be the detection channel of choice.

The decay to two gluons is analogous, except that of course there is no W loop. The translation from the amplitude for the decay to two photons to the decay to gluons with labels i and j, $i,j = 1,\ldots 8$ is

$$\begin{aligned} e^2 &\to g_s^2 Tr\frac{\lambda^i}{2}\frac{\lambda^j}{2} \\ e^4 &\to (\frac{1}{2}g_s^2\delta^{ij})(\frac{1}{2}g_s^2\delta^{ij}) = 2g_s^4 \end{aligned} \tag{4.28}$$

so

$$\begin{aligned} \Gamma(H\to gg) &= \frac{\alpha_s^2 G_F m_H^3}{36\pi^3\sqrt{2}}|\sum_q I\left(\frac{m_q^2}{m_H^2}\right)|^2 \\ &= 74 \text{ keV } \left(\frac{\alpha_s}{0.1}\right)^2\left(\frac{m_H}{100 \text{ GeV}}\right)^3|\sum_q I\left(\frac{m_q^2}{m_H^2}\right)|^2 \end{aligned} \tag{4.29}$$

This result is enhanced by QCD radiative corrections[12, 13]

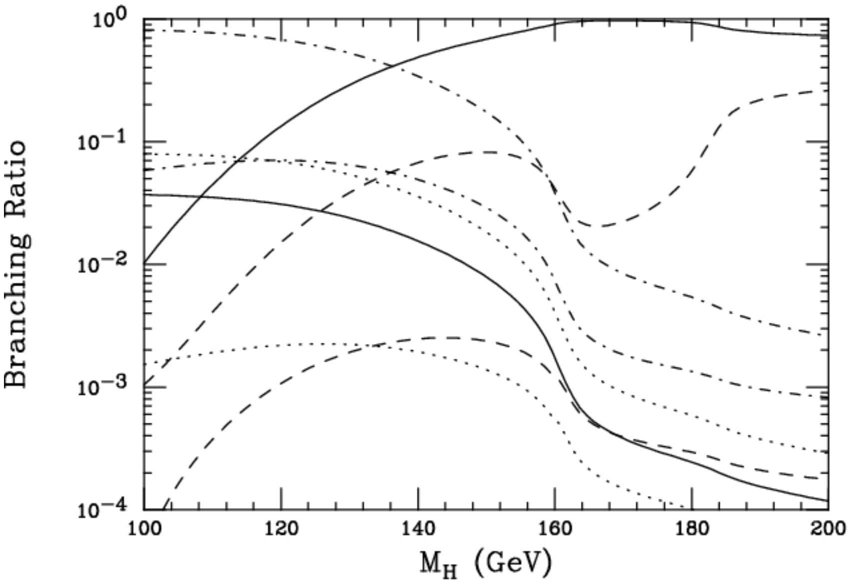


Figure 4: Branching ratios for the various decay modes of the Higgs boson, calculated using HDECAY [14].

4.4 Higgs Width and Branching Ratios

The total width of the Higgs boson as a function of its mass and the branching ratios for the various channels are shown in Fig. 4. Here higher order corrections, not described in the preceding sections have been included. The salient points are these: The $b\bar{b}$ channel dominates at low masses and it supplanted by WW^* near 140 GeV. The WW and ZZ channels dominate above this, challenged only by $t\bar{t}$ between 400 and 500 GeV. The peculiar shape of the curves is the consequence of thresholds, especially that for WW. The partial widths have a smoother behaviour as seen in Fig. 5

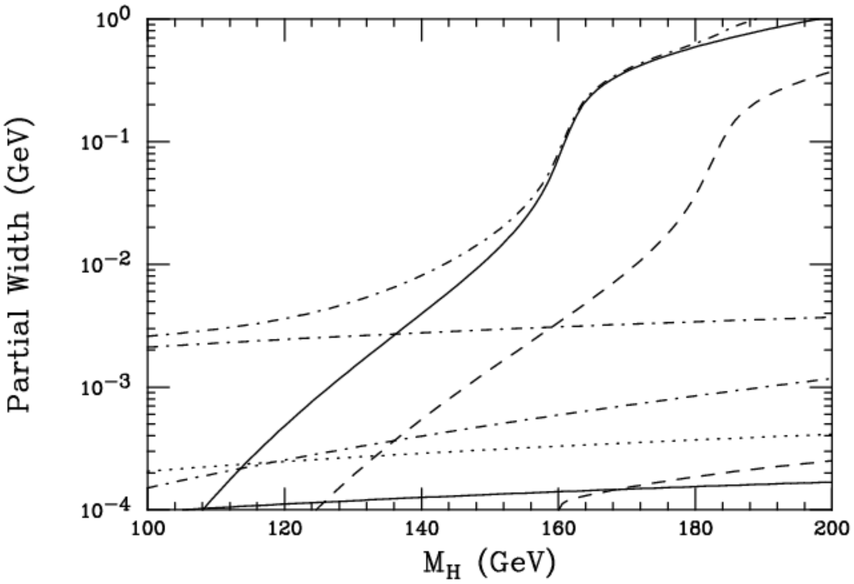


Figure 5: Branching ratios for the various decay modes of the Higgs boson, calculated using HDECAY [14].

5 Higgs Boson Production

5.1 In Lepton Colliders

5.1.1 Resonant Production in e^+e^-, $\mu^+\mu^-$ Collisions

The cross section for resonant production $ab \to R$ is always given by the Breit-Wigner formula:

$$\sigma = \frac{2J+1}{(2S_a+1)(2S_b+1)}\frac{4\pi}{k^2}\frac{\Gamma^2/4}{(E-M)^2+\Gamma^2/4}BR(R\to ab) \tag{5.1}$$

where k is the incident c.m. momentum. At the peak, the factor containing the energy dependence is unity. Consider resonant production of the Higgs boson in e^+e^- collisions. The partial width is, from Sec.4.2

$$\Gamma(H\to e^+e^-) = 1.7\times 10^{-11}\left(\frac{m_H}{100\text{GeV}}\right)\ \text{GeV} \tag{5.2}$$

Altogether, then, at the peak the cross section is

$$\sigma(e^+e^-\to H) \ = \ 4.9\times 10^{7}\ \text{fb}\left(\frac{100\ \text{GeV}}{m_H}\right)^2$$

$$
\begin{aligned}
& \times 1.7 \times 10^{-11} \left(\frac{m_H}{100\text{GeV}}\right) \text{ GeV} \times \frac{1}{\Gamma_H} \\
= & \;\; 8.3 \times 10^{-4} \text{ fb} \left(\frac{100 \text{ GeV}}{m_H}\right) \times \frac{1\text{GeV}}{\Gamma_H} \qquad (5.3)
\end{aligned}
$$

Even below 150 GeV, where the width of the Higgs boson is just a few MeV, this is quite hopeless since the e^+e^- cross section is several units of R where

$$
\text{one unit of } R = \frac{4\pi\alpha^2}{3s} = \frac{87 \text{ nb}}{s(\text{GeV}^2)} = 8.7\text{pb} \left(\frac{(100\text{GeV})^2}{s}\right) \qquad (5.4)
$$

However, the situation is much improved if the colliding particles are muons rather than electrons. Naturally this increases the cross secton by $(m_\mu/m_e)^2 = 4.3 \times 10^4$. Taking the Higgs width to be just that from the decay to $b\bar{b}$, we have

$$
\sigma(\mu^+\mu^- \to Higgs) = 9 \text{ pb} \left(\frac{(100\text{GeV})^2}{m_H}\right)^2 \qquad (5.5)
$$

We see that this makes the signal comparable to the background. A more serious study must include background production from Z^* decays and from Zs resonantly produced after initial state radiation. It also must include consideration of the beam energy spread, which will reduce the peak cross section.

5.1.2 Associated Production of H with Z in e^+e^-

There has been a continuing effort to find the Higgs boson through the processes $e^+e^- \to HZ^*$ and $e^+e^- \to HZ$. The former was proposed by Ioffe and Khoze [15] and calculated in detail by Bjorken [16] The latter was evaluated by Ellis, Gaillard, and Nanopoulos [17].

Let's calculate the cross section through the virtual Z since it is this that seems relevant now that lower mass Higgs bosons have been excluded. We use the Breit-Wigner formula described in Appendix C

$$
\begin{aligned}
\sigma(e^+e^- \to ZH) &= \frac{3}{4}\frac{16\pi}{s}\frac{s\Gamma^2}{(s-m_Z^2)^2}\frac{\Gamma(Z^* \to e^+e^-)}{\Gamma}\frac{\Gamma(Z^* \to ZH)}{\Gamma} \\
&= 12\pi\frac{\Gamma(Z^* \to e^+e^-)\Gamma(Z^* \to ZH)}{(s-m_Z^2)^2} \qquad (5.6)
\end{aligned}
$$

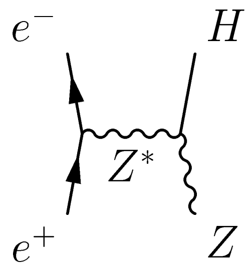


Figure 6: The process $e^+e^- \to HZ$ was the basis for the search at LEP-II.

Here Γ stands for the width of the Z^*, i.e. the width the Z would have if its mass were $\sqrt{s}$. To evaluate $\Gamma(Z^* \to e^+e^-)$ we consider the generic coupling of a Z to a fermion-antifermion pair:

$$\begin{aligned}
\mathcal{M}(Z^* \to e^+e^-) &= \epsilon_\mu \overline{u}\gamma^\mu(g_V + g_A\gamma_5)v \\
|\mathcal{M}|^2_{ave} &= \frac{1}{3}[-g_{\mu\nu}]\,(\mathrm{Tr}\ \not{p}\gamma^\mu(g_V + g_A\gamma_5)\not{q}\gamma^\nu(g_V + g_A\gamma_5)) \\
&= -\frac{1}{3}\mathrm{Tr}\ \not{p}\gamma^\mu\not{q}\gamma_\mu(g_V^2 + g_A^2) \\
&= \frac{8}{3}p\cdot q(g_V^2 + g_A^2) \\
&= \frac{4m_Z^2}{3}(g_V^2 + g_A^2)
\end{aligned} \tag{5.7}$$

Note that we dropped the second part of the polarization sum

$$\sum_{pol} \epsilon_\mu \epsilon_\nu^* = -(g_{\mu\nu} - k_\mu k_\nu / M^2) \tag{5.8}$$

because we assumed the outgoing fermions were light. For light fermions, both the vector and axial vector currents are conserved.

The magic formula for the neutral current in the Standard Model, the expression that gives the couplings is

$$\frac{g}{\cos\theta_W}(T_3 - Q\sin^2\theta_W) \to \frac{g}{\cos\theta_W}(T_3\gamma_\mu\frac{1}{2}(1-\gamma_5) - Q\gamma_\mu\sin^2\theta_W) \tag{5.9}$$

so that

$$\begin{aligned}
g_V &= \frac{g}{\cos\theta_W}(\frac{1}{2}T_{3L} - Q\sin^2\theta_W) \\
g_A &= -\frac{g}{\cos\theta_W}\frac{1}{2}T_{3L}
\end{aligned} \tag{5.10}$$

The formula of Appendix A give, treating the final fermions as massless

$$\begin{aligned}\Gamma(Z \to f\overline{f}) &= \frac{m_Z}{12\pi}(g_V^2 + g_A^2) \\ &= \frac{m_Z}{12\pi}\left(\frac{g}{\cos\theta_W}\right)^2 [(\frac{1}{2}T_{3L} - Q\sin^2\theta_W)^2 + \frac{1}{4}T_{3L}^2] \\ &= \frac{G_F m_Z^3}{24\pi\sqrt{2}}[(2T_{3L} - 4Q\sin^2\theta_W)^2 + 4T_{3L}^2] \\ &= \frac{\alpha m_Z}{48}\frac{1}{\sin^2\theta_W\cos^2\theta_W}[(2T_{3L} - 4Q\sin^2\theta_W)^2 + 4T_{3L}^2] \end{aligned} \quad (5.11)$$

We describe the decay $Z^* \to ZH$ with momenta $p*, p$, and q for the Z^*, Z, and H respectively. The momenta of the decay products in the Z^* rest frame is p_{cm}. From Eq. 4.3 the ZZH coupling is

$$\frac{g^2 vHZ_\mu Z^\mu}{4\cos^2\theta_W} = \frac{gm_Z HZ_\mu Z^\mu}{2\cos\theta_W} \quad (5.12)$$

giving a vertex factor

$$i\frac{gm_Z g_{\mu\nu}}{\cos\theta_W} \quad (5.13)$$

The amplitude squared, averaged over initial polarizations (treating the Z^* as on-shell)

$$\begin{aligned}|\mathcal{M}|^2_{ave} &= \frac{1}{3}\left(\frac{gm_Z}{\cos\theta_W}\right)^2 \left(g_{\mu\nu} - p^*_\mu p^*_\nu/m^{*2}\right)\left(g^{\mu\nu} - p^\mu p^\nu/m^2\right) \\ &= \frac{1}{3}\left(\frac{gm_Z}{\cos\theta_W}\right)^2 \left(4 - 1 - 1 + \frac{(p * \cdot p)^2}{m^2 m^{*2}}\right) \\ &= \frac{1}{3}\left(\frac{gm_Z}{\cos\theta_W}\right)^2 [\frac{p_{cm}^2}{m^2} + 3] \end{aligned} \quad (5.14)$$

The partial width then for $Z^* \to ZH$ is

$$\begin{aligned}\Gamma(Z^* \to ZH) &= \frac{1}{24\pi}\left(\frac{gm_Z}{\cos\theta_W}\right)^2 [\frac{p_{cm}^2}{m^2} + 3]\frac{p_{cm}}{m^{*2}} \\ &= \frac{\alpha p_{cm}}{6\sin^2\theta_W\cos^2\theta_W}\left[\frac{p_{cm}^2}{m^{*2}} + 3\frac{m^2}{m^{*2}}\right] \end{aligned} \quad (5.15)$$

Combining the partial widths, we compute the cross section for $e^+e^- \to ZH$ at center of mass energy squared $m^{*2} = s$:

$$\sigma(e^+e^- \to ZH) = \frac{12\pi}{(s - m_Z^2)^2} \cdot \frac{\alpha m_{Z^*}^2}{48}\frac{1}{\sin^2\theta_W\cos^2\theta_W}[(2T_{3L} - 4Q\sin^2\theta_W)^2 + 4T_{3L}^2]$$

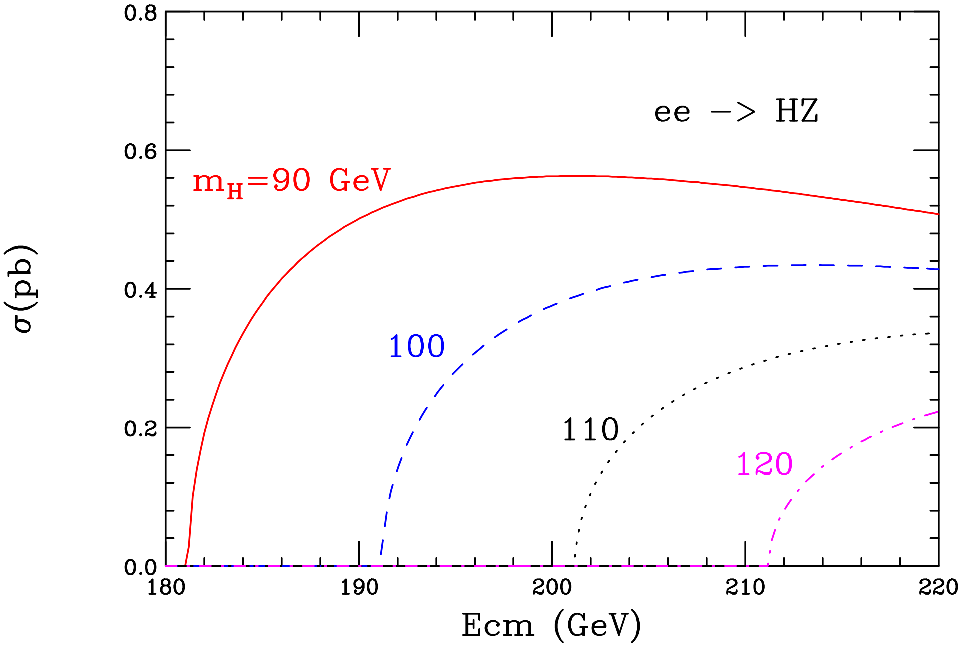


Figure 7: Cross section for $e^+e^- \to HZ$ for various values of the Higgs mass.

$$
\begin{aligned}
&\cdot \frac{\alpha p_{cm}}{6\sin^2\theta_W\cos^2\theta_W}\left[\frac{p_{cm}^2}{m^{*2}}+3\frac{m^2}{m^{*2}}\right] \\
=\; &\frac{\pi\alpha^2}{192\sin^4\theta_W\cos^4\theta_W}\frac{2p_{cm}\sqrt{s}}{(s-m_Z^2)^2}[(1-4\sin^2\theta_W)^2+1] \\
&\times[4\frac{p_{cm}^2}{s}+12\frac{m^2}{s}]
\end{aligned}
\qquad (5.16)
$$

5.1.3 WW and ZZ fusion in e^+e^-

Although originally proposed as a mechanism for Higgs boson production at high energy hadron colliders [18], the WW and ZZ fusion process is relevant to high energy linear electron-positron colliders, as well.

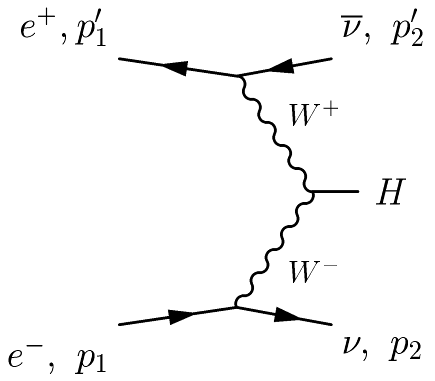


Figure 8: The W fusion process (which can also proceed through a pair of Zs) becomes most effective at very high energy.

These processes are analogous to the two-photon mechanism. We can think of the gauge bosons here as being bremsstrahlung products. There are, however, important differences. While the virtual photons from electrons are produced near the forward direction, the Ws and Zs have transverse momenta of order m_W. Moreover, it is the longitudinal Ws and Zs that turn out to be the most important.

Consider, in particular, $e^+e^- \to \overline{\nu}\nu H$ and let the initial electron momentum be p_1, the initial positron momentum be p_2, the ν momentum p_1', the $\overline{\nu}$ momentum p_2'. The matrix element is

$$-i\mathcal{M} = (igm_W)g_{\mu\nu}\frac{ig}{\sqrt{2}}\frac{\overline{u}(p_1')\gamma^\mu\frac{1}{2}(1-\gamma_5)u(p_1)}{(q_1^2-m_W^2)}\frac{ig}{\sqrt{2}}\frac{\overline{u}(p_2')\gamma^\nu\frac{1}{2}(1-\gamma_5)u(p_2)}{(q_2^2-m_W^2)} \tag{5.17}$$

where $q_1 = p_1 - p_1'$, $Q_2 = p_2 - p_2'$. Squaring and averaging over initial spins

$$|\mathcal{M}|^2 = g^6 m_W^2 \frac{1}{4}\frac{1}{4}\frac{\mathrm{Tr}\ [\not{p}_1'\gamma^\mu \not{p}_1\gamma^\nu\frac{1}{2}(1-\gamma_5)]}{(q_1^2-m_W^2)^2}\frac{\mathrm{Tr}\ [\not{p}_2'\gamma^\mu \not{p}_2\gamma^\nu\frac{1}{2}(1-\gamma_5)]}{(q_2^2-m_W^2)^2} \tag{5.18}$$

Using Eqs. (B.26, B.27),

$$|\mathcal{M}|^2 = g^6 m_W^2 \frac{1}{4}\frac{1}{4}\frac{16 p_1'\cdot p_2'\ p_1\cdot p_2}{(q_1^2-m_W^2)^2(q_2^2-m_W^2)^2} \tag{5.19}$$

If we write the four-momenta explicitly,

$$p_1 = E(1,0,0,1); \qquad p_2 = E(1,0,0,-1) \tag{5.20}$$

$$p_1' = (\sqrt{x_1^2E^2 + p_{\perp 1}^2}, \vec{p}_{\perp 1}, x_1 E); \quad p_2' = (\sqrt{x_2^2E^2 + p_{\perp 2}^2}, \vec{p}_{\perp 2}, -x_2 E) \tag{5.21}$$

we find

$$\begin{array}{rclrcl} q_1^2 &=& -2p_1\cdot p_1' \approx -\frac{p_{\perp 1}^2}{x_1} & q_2^2 &=& -2p_2\cdot p_2' \approx -\frac{p_{\perp 2}^2}{x_2} \\ 2p_1'\cdot p_2' &=& x_1x_2 s; & 2p_1\cdot p_2 &=& s \end{array} \tag{5.22}$$

The four-momentum of the Higgs boson, k, comes from combining the momenta of the virtual Ws: $q_1 + q_2 = k$, so

$$\begin{aligned} m_H^2 &= (q_1+q_2)^2 = \left(2E - \sqrt{x_1^2E^2 + p_{\perp 1}^2} - -\sqrt{x_2^2E^2 + p_{\perp 2}^2}\right)^2 - (x_1E - x_2E)^2 - (\vec{p}_{\perp 1} + \vec{p}_{\perp 2})^2 \\ &\approx 4E^2(1-x_1)(1-x_2) - (2 - x_1 - x_2)\left(\frac{p_{\perp 1}^2}{x_1} + \frac{p_{\perp 2}^2}{x_2}\right) - (\vec{p}_{\perp 1} + \vec{p}_{\perp 2})^2 \end{aligned} \tag{5.23}$$

We do the phase space calculation explicitly, noting that the flux factor, Eq.(A.11), here is just $s/2$:

$$\begin{aligned} d\sigma &= \frac{(2\pi)^4}{2s}\frac{d^3p_1'}{(2\pi)^3 2p_1'}\frac{d^3p_2'}{(2\pi)^3 2p_2'}\frac{d^3k}{(2\pi)^3 2E_H} \\ &\quad \times \delta^4(p_1' + p_2' + k - p_1 - p_2)\frac{s^2x_1x_2}{4} \\ &\quad \times \frac{g^6 m_W^2}{(q_1^2 - m_W^2)^2(q_2^2 - m_W^2)^2} \end{aligned} \tag{5.24}$$

We get rid of the Higgs momentum in the usual way by writing

$$\frac{d^3k}{(2\pi)^3 2E_H} = \frac{d^4k}{(2\pi)^3}\delta(k^2 - m_H^2) \tag{5.25}$$

and then integrating d^4k to remove the δ^4. This leaves us

$$\begin{aligned} d\sigma &= \frac{g^6 m_W^2}{(2\pi)^5 2s}\frac{d^3p_1'}{(2p_1'}\frac{d^3p_2'}{2p_2'}\frac{s^2x_1x_2}{4} \\ &\quad \times \frac{\delta((q_1+q_2)^2 - m_H^2)}{\left(\frac{p_{\perp 1}^2}{x_1} + m_W^2\right)^2\left(\frac{p_{\perp 2}^2}{x_2} + m_W^2\right)^2} \end{aligned} \tag{5.26}$$

This is still not manageable, but if we ignore the pieces containing $p_\perp$ in the expression for $(q_1+q_2)^2$, we have simply

$$\begin{aligned} d\sigma &= \frac{1}{16\pi^2}\left(\frac{\alpha}{\sin^2\theta_W}\right)^3 s m_W^2 \\ &\quad \times \frac{dx_1 dx_2 d^2p_{\perp 1} d^2p_{\perp 2}}{\left(\frac{p_{\perp 1}^2}{x_1}+m_W^2\right)^2\left(\frac{p_{\perp 2}^2}{x_2}+m_W^2\right)^2} \\ &\qquad \times\delta(s(1-x_1)(1-x_2)-M_H^2) \end{aligned} \tag{5.27}$$

We can do the transverse momentum integrals directly and then the integrals over the fractions of the incident energy x_1, x_2, given to the Ws:

$$\begin{aligned} d\sigma &= \frac{1}{16}\left(\frac{\alpha}{\sin^2\theta_W}\right)^3 s m_W^2 \frac{x_1}{m_W^2}\frac{x_2}{m_W^2} dx_1 dx_2 \delta(s(1-x_1)(1-x_2)-M_H^2) \\ \sigma &= \frac{1}{16}\left(\frac{\alpha}{\sin^2\theta_W}\right)^3 \frac{s}{m_W^2}\int_0^{1-m_H^2/s} dx \frac{x}{s(1-x)}\left(1-\frac{m_H^2}{s(1-x)}\right) \\ &= \frac{1}{16 m_W^2}\left(\frac{\alpha}{\sin^2\theta_W}\right)^3 \left[\left(1+\frac{m_H^2}{s}\right)\ln\frac{s}{m_H^2}-2\left(1-\frac{m_H^2}{s}\right)\right] \end{aligned} \tag{5.28}$$

Note that although a heavy particle with mass m_H is being produced, the cross section is not suppressed by a factor $1/m_H^2$. The cross section is "anomalously large."

The cross section can be reduced, without approximation to a double integral[19]. The result is

$$d\sigma = \frac{\alpha^3}{15 x_W^3}\left(\frac{4m_W^2}{s^2}\right)\frac{2(1+\cos\theta)}{(1-\eta)(1-\zeta)} J(1+\frac{2m_W^2}{(1-\eta)s}, 1+\frac{2m_W^2}{(1-\zeta)s}, \cos\theta) d\eta d\zeta \tag{5.29}$$

where η is the fraction of the electron's energy given to the W^-, ζ is the fraction of the positron's energy given to the W^+, and

$$\cos\theta = 1-2\frac{\zeta\eta - m_H^2/s}{(1-\eta)(1-\zeta)} \tag{5.30}$$

is the negative of the cosine of the angle between the outgoing neutrinos. The kinematic boundary is given by

$$\zeta\eta \geq m_H^2/s; \qquad \zeta+\eta \leq 1+m_H^2/s \tag{5.31}$$

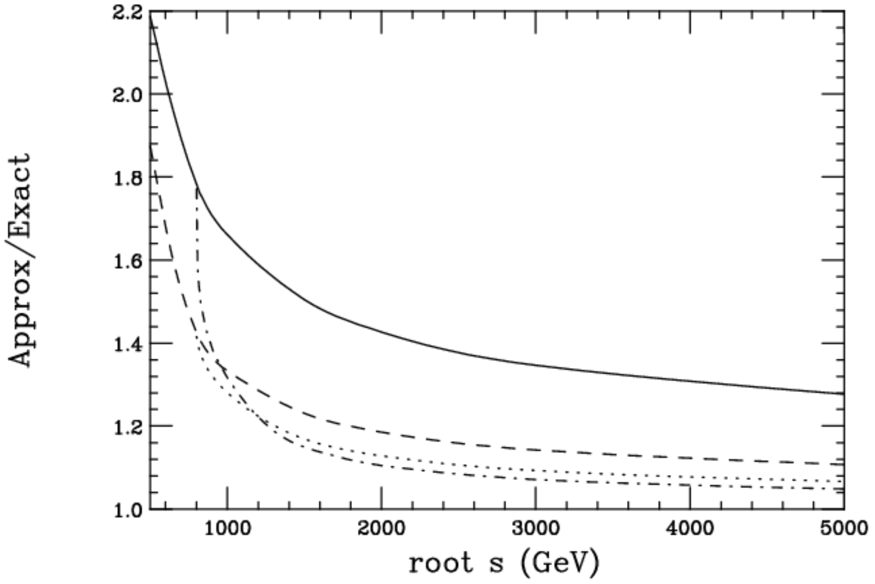


Figure 9: Ratio of the approximation, Eq. (5.28), to the exact cross section, Eq. (5.29), for $e^+e^- \to \nu\overline{\nu}H$. The solid curve is for $m_H = 100$ GeV, the dashed for $m_H = 300$ GeV, the dotted for $m_H = 500$ GeV, and the dashdot for $m_H = 700$ GeV.

The function J is somewhat messy:

$$\begin{aligned} J(x,y,\cos\theta) &= 4\pi\left\{\frac{3}{\Delta^2}\left(\frac{1}{\sqrt{\Delta}}\tanh^{-1}\frac{\sqrt{\Delta}}{xy-\cos\theta}-\frac{xy-\cos\theta}{(x^2-1)(y^2-1)}\right)(x-y\cos\theta)(y-x\cos\theta)\right. \\ &\qquad \left. +\frac{\cos\theta}{\Delta^{3/2}}\tanh^{-1}\frac{\sqrt{\Delta}}{xy-\cos\theta}+\frac{x^2+y^2-3xy\cos\theta+1}{\Delta(x^2-1)(y^2-1)}\right\} \end{aligned} \tag{5.32}$$

The ratio of the standard approximation, Eq. (5.28), to the exact answer is shown in Fig. 9). Not surprisingly, the approximation gives too large an answer, especially at low energies. There is the suggestion that there are corrections of order m_W^2/m_H^2.

5.2 Production at Hadron Colliders

At the Tevatron Collider and LHC each single particle provides a spectrum of quarks, antiquarks, and gluons. The effective cross section is thus convolution of parton distributions with parton-level cross sections:

$$d\sigma = \int dx_1 dx_2 f_1(x_1) f_2(x_2) d\hat{\sigma} \tag{5.33}$$

The canonical exposition of this paradigm was given for the ill-starred SSC by Eichten, et al. (EHLQ) [20].

Suppose that at the parton level the process is reonant production, e.g. $q\overline{q} \to W$. In the narrow width approximation, the Breit Wigner formula simplifies:

$$\begin{aligned} \hat{\sigma} &= \frac{2J+1}{(2S_a+1)(2S_b+1)} \frac{4\pi}{k^2} \frac{\Gamma^2/4}{(E-M)^2+\Gamma^2/4} BR(R \to ab) \\ &\to \frac{2J+1}{(2S_a+1)(2S_b+1)} \frac{4\pi^2}{k^2} \Gamma m \delta(\hat{s} - m^{)} BR(R \to ab) \\ &\to (2J+1) \frac{4\pi^2}{m} \Gamma(R \to ab) \delta(\hat{s} - m^2) \end{aligned} \tag{5.34}$$

In the last line we relied on there being two polarization states for each incident parton, whether fermionic or gluonic.

The cross section for producing the resonance is thus

$$\sigma = \frac{(2J+1)4\pi^2}{m^3} \Gamma(R \to ab) \tau \int \frac{dx}{x} f_a(x) f_b(\tau/x) \equiv \frac{(2J+1)4\pi^2}{m^3} \Gamma(R \to ab) \tau \frac{d\mathcal{L}}{d\tau} \tag{5.35}$$

where $\tau = m^2/s$.

5.2.1 Gluon Fusion

If the indications from the masses of the W, Z, and t are correct, the Higgs boson is likely to be found at a relatively small mass, not far above the limit near 115 GeV set at LEP II. The dominant mechanism for its production would then be gluon fusion. The cross section, at lowest order is simply

$$\sigma = \left(\frac{1}{8} \cdot \frac{1}{8} \cdot 2\right) \frac{4\pi^2}{m_H} \Gamma(H \to gg) \delta(\hat{s} - m_H^2) \tag{5.36}$$

where $\hat{s}$ is the gg invariant mass squared. The factors $(1/8)^2 2$ arise because only 1/8 of the time do colliding gluons have the right colors to annihilate and because the width summed over all 8 gluons. On the other hand, the

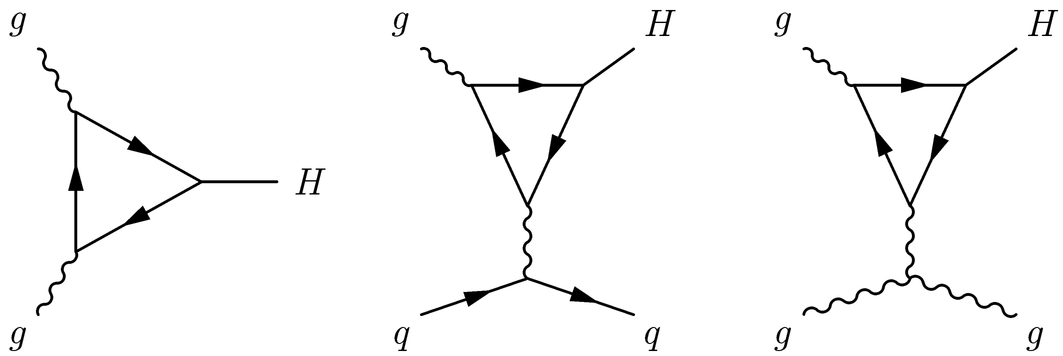


Figure 10: Three contributions to Higgs producction: Lowest order $gg \to H$, $qg \to qH$, and $gg \to Hg/$

decay rate included a factor of $1/2$ for identical outgoing particles, which needs to be removed. This gives us

$$\sigma = \frac{\alpha_s^2}{576\pi v^2}\tau\frac{d\mathcal{L}}{d\tau} \tag{5.37}$$

where we approximated

$$|\sum_q I\left(\frac{m_q^2}{m_H^2}\right)|^2 \to 1 \tag{5.38}$$

assuming that $m_t/m_H >> 1$ and dropping the small contributions from the lighter quarks.

The QCD radiative corrections to this have been the subject of extensive work [26, 29, 13]. At the next level in QCD we need to consider new physical processes, $qg \to Hq$, $gg \to Hg$, as well as virtual corrections to $gg \to H$. See Fig. 10.

In addition to the new processes, there are radiative corrections to the underlying $gg \to H$. See Fig.11

Since the dominant contribution is from the t quark loop, one approach is to ignore the other quarks and then treat the t as heavy compared to the Higgs. Then the t quark disappears leaving behind an effective interaction between the Higgs and the gluons:

$$\mathcal{L}_{Hgg} = \frac{1}{4}\frac{\beta(\alpha_s)}{1+\gamma_m(\alpha_s)}G^a_{\mu\nu}G^{a\mu\nu}\frac{H}{v} \tag{5.39}$$

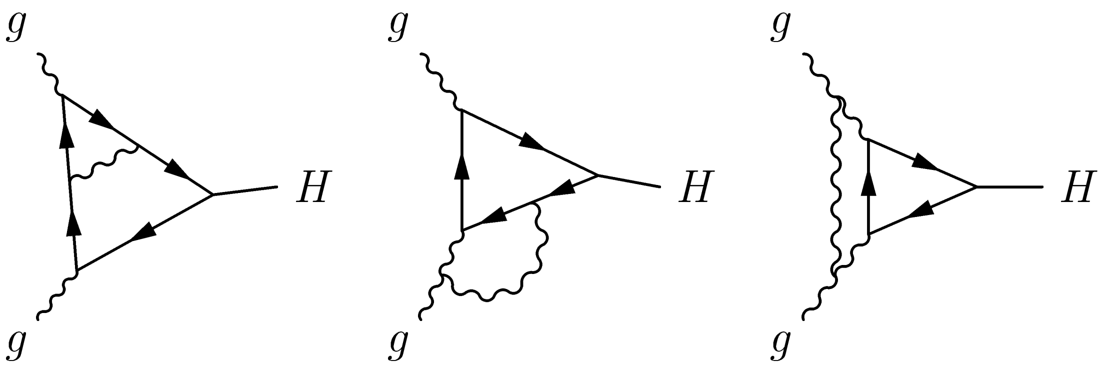


Figure 11: Some virtual radiative corrections to $gg \to H$.

where β is the β function describing the evolution of the strong coupling constant, but only that part due to the t quark, and where γ_m is the anomalous dimension of top quark from renormalizing the Yukawa coupling. Explicitly

$$\begin{aligned}\beta &= \frac{\alpha_s}{3\pi}\left(1+\frac{19\alpha_s}{4\pi}\right) \\ \gamma_m &= \frac{2\alpha_s}{\pi}\end{aligned} \tag{5.40}$$

This approach provides a clever and effective way to a reliable answer[28, 29]. However, the full calculation has been done.

In either the full calculation or in the shortcut, it is necessary to handle carefully infinities that arise both in the virtual and real contributions. The infinities in $qg \to qH$ arise from emission of the virtual gluon by the incoming quark. This is a universal phenomenon and is treated by "factorization," that is, absorbing this effect back into the structure functions. Some of the infinities in $gg \to gH$ are of this same form. In addition, there are cancellations between the virtual processes in $gg \to H$ and the real process $gg \to H$.

When all is said and done, the resulting change,

$$K = \frac{\sigma(pp \to HX)}{\sigma(pp \to H)} \tag{5.41}$$

is between two and three at LHC. In large part, this arises simply from a rescaling of the lowest order result by the factor

$$K_{rescaling} = 1 + \frac{\alpha_s(\mu)}{\pi}\left(\pi^2 + \frac{11}{2}\right)$$

$$= \quad 1 + 4.89\alpha_s(\mu) \tag{5.42}$$

The dependence on the scale μ at which α_s is evaluated is an artifact. A truely complete calculation will turn out not to depend on μ.

5.2.2 W and Z content of fermions

We can turn the tables on the calculation of WW fusion by thinking of the Ws as partons in an electron. Return to Eq. (5.27)

$$\begin{aligned} &\frac{1}{16\pi^2}\left(\frac{\alpha}{\sin^2\theta_W}\right)^3 s m_W^2 \frac{dx_1 dx_2 d^2p_{\perp 1} d^2p_{\perp 2}}{\left(\frac{p_{\perp 1}^2}{x_1} + m_W^2\right)^2 \left(\frac{p_{\perp 2}^2}{x_2} + m_W^2\right)^2} \\ &= \frac{4\pi^2}{m}\Gamma(R \to ab) f_a(1-x_1) f_b(1-x_2) \end{aligned} \tag{5.43}$$

The appearance of $1-x$ is due to our having used x to represent the fraction of the electron's energy retained by the final-state electron. Now use

$$\Gamma(H \to W_L W_L) = \frac{G_F}{\sqrt{2}}\frac{m_H^3}{8\pi} = \frac{g^2}{8m_W^2}\frac{m_H^3}{8\pi} = \frac{4\pi\alpha}{8\sin^2\theta_W m_W^2}\frac{m_H^3}{8\pi} \tag{5.44}$$

to find the distribution of the W partons inside the electron:

$$\frac{g^2}{16\pi^3}\frac{dy}{y}\frac{d^2p_\perp}{m_W^2 + p_\perp^2/(1-y)} \to \frac{g^2}{16\pi^2}(1-y)\frac{dy}{y} \tag{5.45}$$

where $y = 1 - x$ represents the fraction of the electron's momentum given to the W.

Of course we might just as well think of this as giving the W content of a u or d quark in a proton. We can generalize the production of Higgs bosonsby WW fusion to imagine simply scattering longitudinal Ws off each other [21, 22]. If the mass of the Higgs boson is low enough, we simply recover the results above. However, as the mass of the Higgs boson increases, it is less and less appropriate to considerate it as a real particle. Since its width grows at the cube of its mass, it becomes so wide it is hard to exact where it is. Equivalently, an increasing mass means an increasing λ, which in turn means increasing interactions.

5.2.3 Strongly Interacting W and Z

Indeed, the sector of the Higgs boson and the longitudinal Ws and Z becomes strongly interacting if the Higgs boson is very heavy. Moreover, these interactions are quite analogous to the interactions of pions. In fact, they may be more "pion-like" than pions themselves.

To see this, return to the Lagrangian

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \tag{5.46}$$

where

$$\phi^\dagger\phi = \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) \tag{5.47}$$

Then if $\mu^2 < 0$, some expectation value is non-zero. We write

$$\begin{aligned} \phi_3 &= \sigma = \langle\phi_3\rangle + H = \langle\sigma\rangle + H \\ v^2 &= -\mu^2/\lambda \end{aligned} \tag{5.48}$$

We can rewrite the Lagrangian as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu\vec{\pi}\partial^\mu\vec{\pi} + \frac{1}{2}\partial_\mu H\partial^\mu H + \mu^2 H^2 \\ &\quad + \frac{\mu^2}{v}H(H^2+\pi^2) \\ &= \frac{1}{2}\partial_\mu\vec{\pi}\partial^\mu\vec{\pi} + \frac{1}{2}\partial_\mu H\partial^\mu H - m_H^2 H^2 \\ &\quad - \frac{m_H^2}{2v}H(H^2+\pi^2) - \frac{m_H^2}{8v^2}(H^2+\pi^2)^2 \end{aligned} \tag{5.49}$$

It is natural to expect that we can use the scalars $\vec{\pi}$ as surrogates for the longitudinal W and Z since these putative massless scalars were eaten in the symmetry breaking. Moreover, it is correct: the equivalence theorem [25] establishes this. As an example, let us calculate the decay rate for $h \to W^+w^-$, just using the identification of $\vec{\pi}$ with W^+, w^-, Z. We write $\pi^2 = 2w^+w^- + zz$ (adhering to the convention that these scalar representations of the longitudinal gauge bosons are indicated by lower case letters) and determine the matrix element from the Lagrangian Eq.(5.49):

$$-i\mathcal{M} = -i\frac{m_H^2}{2v}\cdot 2 \tag{5.50}$$

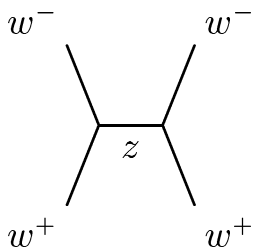
Now from the equations in Appendix A,

$$\begin{aligned}
\Gamma &= \frac{1}{8\pi}\frac{p}{M^2}|\mathcal{M}|^2 \\
&\approx \frac{1}{8\pi}\frac{m_h/2}{m_H^2}\frac{m_H^4}{v^2} \\
&= \frac{1}{8\pi}\frac{G_F}{\sqrt{2}}m_H^3
\end{aligned} \tag{5.51}$$

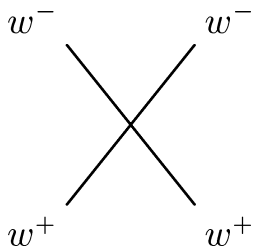
in agreement with Eq.(4.7) in the limit $m_H \to \infty$ which we are considering.

Buoyed by this success, let us calculate all the channels of WW scattering using this same Lagrangian.

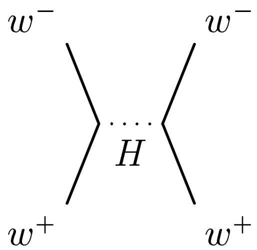
First we calculate $w^+w^- \to w^+w^-$



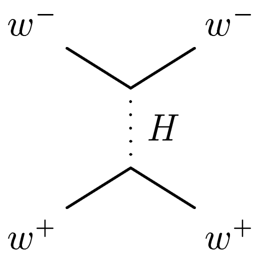
$$-i\mathcal{M} = 0$$



$$-i\mathcal{M} = -\frac{im_H^2}{8v^2} \cdot 4 \cdot 2 \cdot 2 = -\frac{2im_h^2}{v^2}$$



$$-i\mathcal{M} = \left(-\frac{im_H^2}{2v}\right)^2 \frac{i}{s-m_H^2} \cdot 2 \cdot 2$$



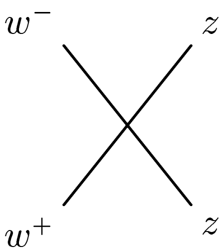
$$-i\mathcal{M} = \left(-\frac{im_H^2}{2v}\right)^2 \frac{i}{t-m_H^2} \cdot 2 \cdot 2$$

For the sum, we find

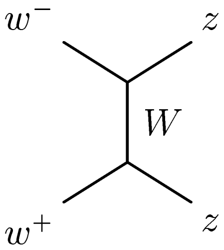
$$-i\mathcal{M} \quad = \quad -\frac{im_H^2}{v^2}\left(2 + \frac{m_H^2}{s - m_H^2} + \frac{m_H^2}{t - m_H^2}\right)$$

$$
\begin{aligned}
&= -\frac{im_H^2}{v^2}\left(\frac{s}{s-m_h^2}+\frac{t}{t-m_H^2}\right) \\
&\approx -\frac{iu}{v^2} \qquad (5.52)
\end{aligned}
$$

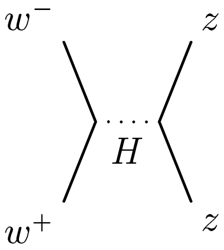
For $w^+w^- \to zz$ we have



$$-i\mathcal{M} = -\tfrac{im_H^2}{8v^2} \cdot 4 \cdot 2 = -\tfrac{im_H^2}{v^2}$$



$$-i\mathcal{M} = 0$$

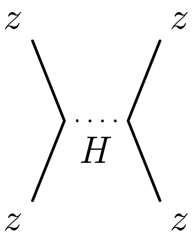


$$-i\mathcal{M} = \tfrac{i}{s-m_H^2}\left(-\tfrac{im_H^2}{2v}\right)^2 \cdot 2 \cdot 2 = -\tfrac{im_H^4}{v^2(s-m_H^2)}$$

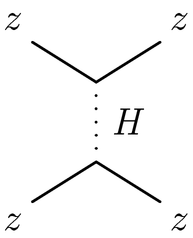
Summing these we find

$$-i\mathcal{M} = -\frac{im_H^2}{v^2}\left(1 + \frac{m_H^2}{s-m_H^2}\right) = -\frac{im_H^2}{v^2}\frac{s}{s-m_H^2} \approx \frac{is}{v^2} \tag{5.53}$$

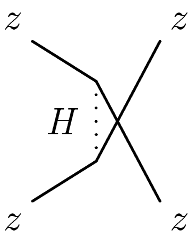
For $zz \to zz$ we have



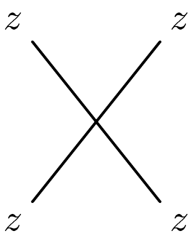
$$-i\mathcal{M} = \frac{i}{s-m_H^2}\left(-\frac{im_H^2}{2v}\right)^2 \cdot 2 \cdot 2$$



$$-i\mathcal{M} = \frac{i}{t-m_H^2}\left(-\frac{im_H^2}{2v}\right)^2 \cdot 2 \cdot 2$$



$$-i\mathcal{M} = \frac{i}{u-m_H^2}\left(-\frac{im_H^2}{2v}\right)^2 \cdot 2 \cdot 2$$



$$-i\mathcal{M} = -\frac{im_H^2}{8v^2} \cdot 4!$$

Here the sum is

$$-i\mathcal{M} = -\frac{m_H^4}{v^2}\left(\frac{i}{s-m_H^2} + \frac{i}{t-m_H^2} + \frac{i}{u-m_H^2}\right) - 3i\frac{m_H^2}{v^2} = 0 \qquad (5.54)$$

In summary

$$
\begin{aligned}
\mathcal{M}(w^+w^- \to w^+w^-) &= \frac{u}{v^2} \\
\mathcal{M}(w^+w^+ \to w^+w^+) &= \frac{s}{v^2} \\
\mathcal{M}(w^+w^- \to zz) &= -\frac{s}{v^2} \\
\mathcal{M}(zz \to zz) &= 0
\end{aligned}
\tag{5.55}
$$

where the second amplitude is obtained from the first by crossing ($s \to u$).

Unitarity is expressed [see Appendix B]

$$2\Im\mathcal{M}(p_1, p_2, p_1', p_2') = -(2\pi)^4 \sum_f d\Phi_n \mathcal{M}(p_1', p_2', f)^* \mathcal{M}(p_1, p_2, f) \tag{5.56}$$

and

$$d\Phi_2 = \frac{(2\pi)^{-6}}{4\sqrt{s}} p\, d\Omega \tag{5.57}$$

If we consider just s waves and elastic scattering,

$$\Im\mathcal{M} = -\frac{1}{16\pi}\frac{2p}{\sqrt{s}}|\mathcal{M}|^2 \tag{5.58}$$

We implicitly assumed a two-body final state with non-identical particles. If we choose a new normalization,

$$\mathcal{A} = -\frac{p}{8\pi\sqrt{s}}\mathcal{M} \tag{5.59}$$

then we have conveniently

$$\Im\mathcal{A} = |\mathcal{A}|^2 \tag{5.60}$$

If the two-body state is made of identical particles, the phase space integration extends only over half of 4π and we have

$$\Im\mathcal{A} = \frac{1}{2}|\mathcal{A}|^2 \tag{5.61}$$

In the standard case we can satisfy unitarity with

$$\mathcal{A} = e^{i\delta}\sin\delta \tag{5.62}$$

whereas for identical particles we need

$$\mathcal{A} = 2e^{i\delta}\sin\delta \tag{5.63}$$

Consider a situation in which we have only two-body states, some channels with indentical particles and others with non-identical particles. Then unitarity reads

$$\Im a_{fi} = \sum_{x\neq\overline{x}} a^*_{xf}a_{xi} + \frac{1}{2}\sum_{x=\overline{x}} a^*_{xf}a_{xi} \tag{5.64}$$

We can reduce this to a known problem with the substitution

$$\begin{array}{llll} a_{fi} & = b_{fi} & \overline{f}\neq f, & \overline{i}\neq i \\ a_{fi} & = \sqrt{2}b_{fi} & \overline{f}\neq f, & \overline{i}= i \\ a_{fi} & = \sqrt{2}b_{fi} & \overline{f}= f, & \overline{i}\neq i \\ a_{fi} & = 2b_{fi} & \overline{f}= f, & \overline{i}= i \end{array} \tag{5.65}$$

It is possible to select phases so that all the s-wave amplitudes are symmetric: $a_{fi} = a_{if}$. The b amplitudes satisfy

$$\Im b_{fi} = \sum_x b^*_{xf}b_{xi} \tag{5.66}$$

the solution to which is found by writing the matrix equations

$$\begin{aligned} S &= I + 2ib \\ S^\dagger S &= I \end{aligned} \tag{5.67}$$

In other words, take S unitary. Then

$$b = \frac{S-I}{2i} \tag{5.68}$$

is a solution. If we diagonalize the S matrix, then its non-zero elements are of the form $e^{2i\delta_a}$, where $a = 1, ...$ runs over the eigenchannels.

In the case of s-wave scattering of ws and zs we have only the $I = 0$ and $I = 2$ channels and we anticipate that all we need are $\delta_{I=0}$ and $\delta_{I=2}$. In our peculiar limit of s-wave relativistic scattering

$$\mathcal{A} = -\frac{1}{16\pi}\frac{2k}{\sqrt{s}}\mathcal{M} = -\frac{1}{16\pi}\mathcal{M} \tag{5.69}$$

we write our s-wave amplitudes as (with $s_0 = 16\pi v^2$=(1.7 TeV)^2)

$$\begin{aligned} \mathcal{A}(w^+w^+ \to w^+w^+) &= -s/s_0 \\ \mathcal{A}(w^+w^- \to w^+w^-) &= \frac{1}{2}s/s_0 \\ \mathcal{A}(w^+w^- \to zz) &= s/s_0 \\ \mathcal{A}(z \to zz) &= 0 \end{aligned} \tag{5.70}$$

Reading the Clebsch-Gordan coefficients out of the Particle Data Book (and changing the sign of zz since we wrote $\pi^2 = 2w^+w^- + zz$ in violation of the Condon and Shortley convention) we expect

$$\begin{aligned} \mathcal{A}(w^+w^+ \to w^+w^+) &= a_2 \\ \mathcal{A}(w^+w^- \to w^+w^-) &= \frac{1}{6}a_2 + \frac{1}{3}a_0 \\ \mathcal{A}(w^+w^- \to zz) &= -(\frac{1}{3}a_2 - \frac{1}{3}a_0) \\ \mathcal{A}(zz \to zz) &= \frac{2}{3}a_2 + \frac{1}{3}a_0 \end{aligned} \tag{5.71}$$

all of which are satisfied by

$$a_2 = -s/s_0; \qquad a_0 = 2s/s_0 \tag{5.72}$$

The remaining question is what are the unitarity restrictions on a_2 and a_0. Of course our Born-level calculations cannot satisfy unitarity because the amplitudes are purely real.

Explicitly, we have

$$\begin{aligned} \Im\mathcal{A}(w^+w^- \to w^+w^-) &= [|\mathcal{A}(w^+w^- \to w^+w^-)|^2 + \frac{1}{2}|\mathcal{A}(w^+w^- \to zz)|^2 \\ \Im[\frac{1}{6}a_2 + \frac{1}{3}a_0] &= [|\frac{1}{6}a_2 + \frac{1}{3}a_0|^2 + \frac{1}{2}|\frac{1}{3}a_2 - \frac{1}{3}a_0|^2] \end{aligned}$$

$$
\begin{aligned}
&= [\frac{1}{12}|a_2|^2 + \frac{1}{6}|a_0|^2] \\
\Im\mathcal{A}(w^+w^- \to zz) &= [\frac{1}{2}\mathcal{M}(zz \to zz)^*\mathcal{A}(w^+w^- \to zz) + \mathcal{A}(zz \to w^+w^-)^*\mathcal{A}(w^+w^- \to w^+w^-) \\
\Im[-(\frac{1}{3}a_2 - \frac{1}{3}a_0)] &= [\frac{1}{2}(\frac{2}{3}a_2^* + \frac{1}{3}a_0^*)(-\frac{1}{3}a_2 + \frac{1}{3}a_0) + (\frac{1}{3}a_2^* - \frac{1}{3}a_0^*)(\frac{1}{6}a_2 + \frac{1}{3}a_0)] \\
&= -\frac{1}{6}|a_2|^2 + \frac{1}{6}|a_0|^2
\end{aligned}
\tag{5.73}
$$

and so on. Evidently the solution is simply

$$
\Im a_I = \frac{1}{2}|a_I|^2; \qquad \Im(1/a_I) = -\frac{1}{2} \tag{5.74}
$$

so

$$
a_I = 2e^{i\delta_I}\sin\delta_I \tag{5.75}
$$

This suggests an ad hoc procedure, the K-matrix technique, for unitarizing a Born amplitude. The prescription is

$$
\begin{aligned}
\frac{1}{(a/2)} &= \frac{1}{(a_{Born}/2)} - i \\
a &= 2\frac{2/a_{Born} + i}{4/a_{Born}^2 + 1}
\end{aligned}
\tag{5.76}
$$

We can apply this to each of our non-zero s-wave amplitudes to obtain $a_I^K, I = 0, 2$. We then have the unitarized amplitudes

$$
\begin{aligned}
a^K(w^+w^- \to w^+w^-) &= \frac{1}{6}a_{I=2}^K + \frac{1}{3}a_{I=0}^K \\
a^K(w^+w^- \to zz) &= -\frac{1}{3}a_{I=2}^K + \frac{1}{3}a_{I=0}^K \\
a^K(w^+w^+ \to w^+w^+) &= a_{I=2}^K
\end{aligned}
\tag{5.77}
$$

with

$$
a_{Born\ I=2} = -s/s_0; \qquad a_{Born\ I=0} = 2s/s_0 \tag{5.78}
$$

In terms of these, the cross sections are

$$
\sigma(w^+w^- \to w^+w^-) = \frac{16\pi}{s}|a^K(w^+w^- \to w^+w^-)|^2
$$

$$
\begin{aligned}
\sigma(w^+w^- \to zz) &= \frac{8\pi}{s}|a^K(w^+w^- \to zz)|^2 \\
\sigma(w^+w^+ \to w^+w^+) &= \frac{8\pi}{s}|a^K(w^+w^+ \to w^+w^+)|^2
\end{aligned}
\tag{5.79}
$$

Note that unitarity is satured when $a_I = 2i$ for the elastic channels. For the inelastic $w^+w^- \to zz$, the maximum occurs when the amplitudes are at opposite sides of the circle of elastic scattering, e.g. $\delta_0 = \pi, \delta_2 = 0$. The limiting amplitudes are thus

$$
\begin{aligned}
a^K(w^+w^- \to w^+w^-) &= i \\
a^K(w^+w^- \to zz) &= \frac{2i}{3} \\
a^K(w^+w^+ \to w^+w^+) &= 2i
\end{aligned}
\tag{5.80}
$$

and the limiting cross sections are

$$
\begin{aligned}
\sigma(w^+w^- \to w^+w^-) &= \frac{16\pi}{s} \\
\sigma(w^+w^- \to zz) &= \frac{16\pi}{s}\frac{4}{9}\frac{1}{2} \\
\sigma(w^+w^+ \to w^+w^+) &= \frac{16\pi}{s} \cdot 4 \cdot \frac{1}{2}
\end{aligned}
\tag{5.81}
$$

The results are shown in Figs. 12-14, where we use as the unit of cross section

$$
\sigma_0 = \frac{16\pi}{s_0} = \frac{16\pi}{(1.7\ \mathrm{TeV})^2} = 6.8\ \mathrm{nb} \tag{5.82}
$$

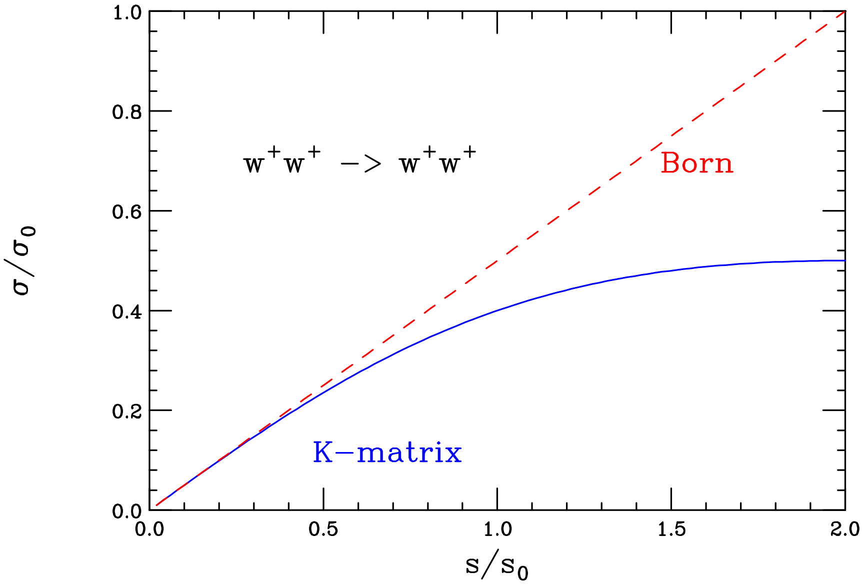


Figure 12: $W^+W^+ \to W^+W^+$ scattering in Born approximation and with K-matrix unitarization.

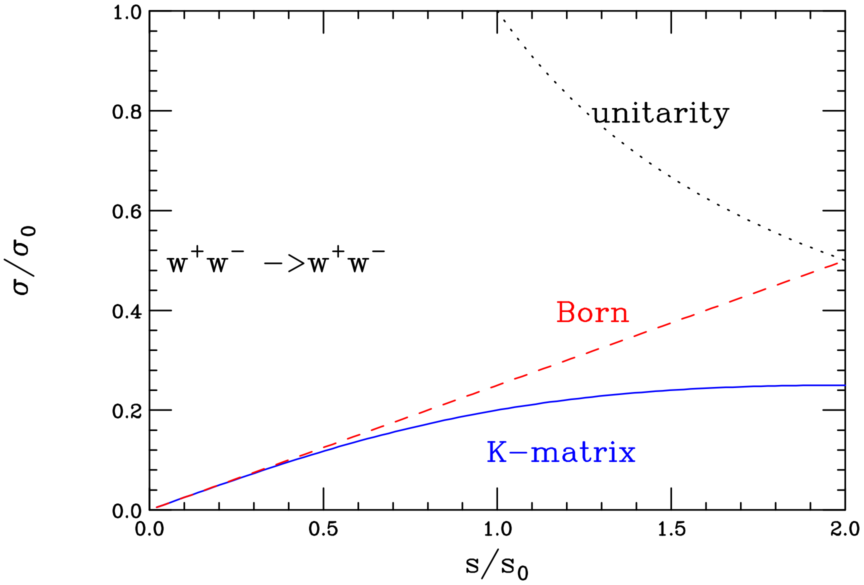


Figure 13: $W^+W^- \to W^+W^-$ scattering in Born approximation and with K-matrix unitarization.

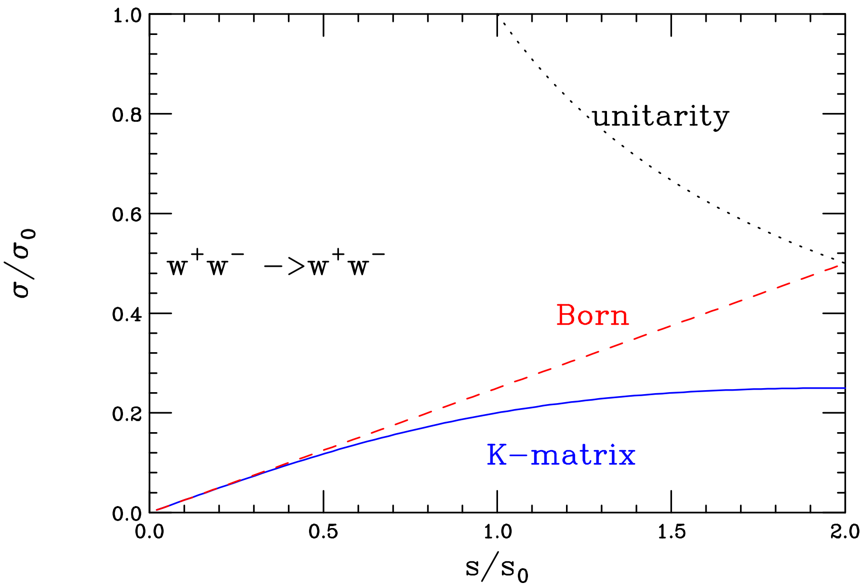


Figure 14: $W^+W^- \to ZZ$ scattering in Born approximation and with K-matrix unitarization.

6 Bounds on the Higgs Mass from Theoretical Considerations

In the Standard Model it seems that the Higgs mass could be anything. We know only that

$$m_H^2 = 2\lambda\ v^2 \tag{6.1}$$

and since we don't know λ, m_H is unspecified. Of course if λ is too large, our theory will not be perturbative.

In fact, we can say something more than this. The reason is that λ specifies a fixed coupling. We know, however, that it is often more insightful to think of couplings as being dependent on the mass scale or distance scale at which they are measured. This is even the case in electrodynamics. We know that the long-distance value of the electromagnetic coupling, the fine structure constant, is about $\alpha = 1/137$. However, if we get close to a heavy nucleus we find that strength of the electric field is not really $z\alpha/r^2$, be something larger, as a result of the vacuum polarization. As we penetrate the cloud of virtual positrons surrounding the nucleus, we see more and more charge, so the field increases faster than $1/r^2$. The opposite effect is seen in asymptotic freedom, where at short distances interactions become weaker, rather than stronger.

In scalar field theories, like those describing just the Higgs sector, the behavior is like that in electrodynamics: at short distances interactions become stronger.

Let us see how this happens. Begin with our very first Lagrangian.

$$\mathcal{L} = \partial_\alpha\phi\partial^\alpha\phi - \frac{1}{2}\mu^2\phi^2 - \frac{1}{2}\lambda\phi^4 \tag{6.2}$$

The amplitude for two-to-two scattering is just (remember: i times the Lagrangian)

$$-i\mathcal{M}_0 = -12i\lambda \tag{6.3}$$

The factor twelve arises from $1/2$ times the 4! ways we have of picking the ϕ fields to attach to the four lines in the diagram.

Now lets go to the next order. To do this, we need to specify in the incoming and outgoing momenta. Actually, it must be enough to specify the Lorentz invariant quantities, s, t, u, where

$$s \;=\; (p_1 + p_2)^2 = (p_2 - p_2')^2;$$

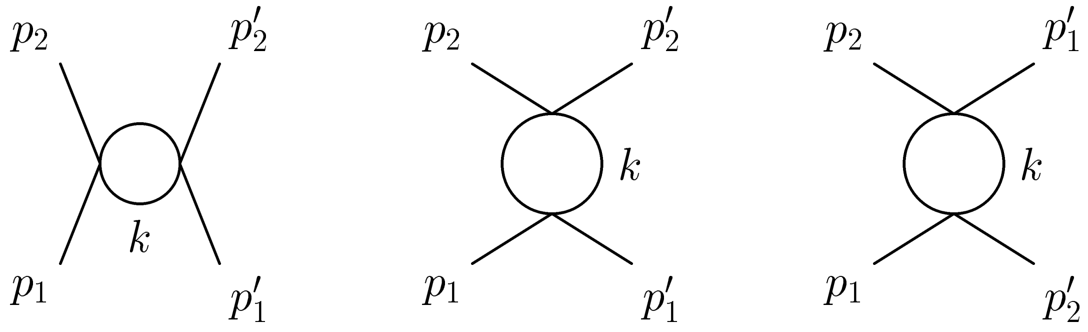


Figure 15: Second order scattering in ϕ^4.

$$
\begin{aligned}
t &= (p_1 - p_1')^2 = (p_2 - p_2')^2; \\
u &= (p_1 - p_2')^2 = (p_2 - p_1')^2
\end{aligned} \tag{6.4}
$$

For a real scattering event, these always statisfy

$$
s + t + u = 4m^2 \tag{6.5}
$$

but we can consider off-shell scattering if we wish. Since we are ultimately interested in high-energy scattering, we take $s = t = u = Q^2 < 0$. By working at spacelike momenta we'll stay away from confusing things like real particle production. Now consider a diagram in which p_1 and p_2 first interact to produce a new pair, which subsequently interacts to produce p_2 and p_2'. The amplitude is

$$
-i\mathcal{M}_{s,2} = (-i\lambda/2)^2 \cdot (24)^2 \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2} \frac{i}{(k-Q)^2 - \mu^2} \tag{6.6}
$$

Here we forget about spontaneous break down and use the apparent scalar mass μ. The four-momentum Q is just $Q = p_1 + p_1'$. The factor of 1/2 comes from the loop and makes sure we don't count it twice. We use the Feynman trick

$$
\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2} \tag{6.7}
$$

to write

$$
\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - \mu^2} \frac{1}{(k-Q)^2 - \mu^2} \quad = \quad \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(1-x)(k^2 - \mu^2) + x((k-Q)^2 - \mu^2)]^2}
$$

$$
\begin{aligned}
&= \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - 2xk\cdot Q + xQ^2 - \mu^2)]^2} \\
&= \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k - xQ)^2 - x^2Q^2 + xQ^2 - \mu^2)]^2}
\end{aligned} \tag{6.8}
$$

Since at large momenta we have d^4k/k^4, the integral diverges at large values of k. If we had fewer than four dimensions, say n dimensions, we would have d^nk/k^4 and the integral would converge. This is the trick of dimensional regularization. The rules of thumb are given in Appendix D

Using the equations there,

$$
-i\mathcal{M}_{s,2} = (-i\lambda/2)^2 \cdot (24)^2 \cdot \frac{-i}{16\pi^2} \int_0^1 dx\Gamma(\epsilon)(\mu^2 + x^2Q^2 - xQ^2)^{-\epsilon} \tag{6.9}
$$

where

$$
\epsilon = 2 - \frac{n}{2} \tag{6.10}
$$

The ϵ exponent is handled by the expansion $A^\epsilon = e^{\epsilon A} = 1 + \epsilon A + \ldots$.

Now we have calculated only one of three diagrams possible. The vertex with the incoming momentum p_1 might have p_1' or p_2' incoming, too, in place of p_2. These amplitudes would be indicated $\mathcal{M}_{t,2}$ and $\mathcal{M}_{u,2}$. Altogether we get a factor of three:

$$
\mathcal{M}_0 + \mathcal{M}_2 = 12\lambda - \frac{27\lambda^2}{2\pi^2} \int_0^1 dx[\Gamma(\epsilon) - \ln(\mu^2 + x^2Q^2 - xQ^2)] \tag{6.11}
$$

The way that renormalization is carried out here is that we now say "Oh, I meant that the amplitude should be $-12i\lambda$, when $Q^2 = -\mu^2$ not this divergent and complicated mess. I'll add a piece $\delta\lambda$ to fix this up." The amplitude thus becomes

$$
\mathcal{M}_0 + \mathcal{M}_2 = 12\lambda + 12\delta\lambda - \frac{27\lambda^2}{2\pi^2} \int_0^1 dx[\Gamma(\epsilon) - \ln(\mu^2 + x^2Q^2 - xQ^2)] \tag{6.12}
$$

where

$$
12\delta\lambda - \frac{27\lambda^2}{2\pi^2} \int_0^1 dx[\Gamma(\epsilon) - \ln(\mu^2 - x^2\mu^2 + x\mu^2)] = 0 \tag{6.13}
$$

which is to say

$$
\begin{aligned}
\mathcal{M}_0 + \mathcal{M}_2 &= 12\lambda + \frac{27\lambda^2}{\pi^2} \int_0^1 dx \ln \frac{\mu^2 + x^2Q^2 - xQ^2}{\mu^2 - x^2\mu^2 + x\mu^2} \\
&\approx 12\lambda + \frac{27\lambda^2}{2\pi^2} \ln \frac{Q^2}{\mu^2}
\end{aligned} \tag{6.14}
$$

This is, in fact, the beginning of a geometric series:

$$\begin{aligned}
\mathcal{M} &= 12\lambda[1+\frac{9\lambda}{8\pi^2}\ln\frac{Q^2}{\mu^2}+\ldots \\
&= \frac{12\lambda}{1-\frac{9\lambda}{4\pi^2}\ln\frac{Q}{\mu}} \\
&\equiv 12\lambda_Q
\end{aligned} \tag{6.15}$$

We see that the effective coupling, λ_Q grows with Q and eventually blows up. This is the so-called Landau pole. Taken literally, this would tell us that we can't really work at scales beyond

$$\Lambda_{\text{Landau}} = \mu e^{4\pi^2/(9\lambda)} \tag{6.16}$$

Of course, we can't take this completely literally because we have used perturbation theory and this would fail before the coupling became infinite. Still the result is suggestive. To make this semi-quantitative, let's take $\mu^2 = m_H^2 = 2\lambda v^2$. Let's also assume that $m_H < \frac{1}{2}\Lambda_{\text{Landau}}$ since we wouldn't really know what we were talking about if $m_H > \Lambda_{\text{Landau}}$. Then

$$m_H^2 = 2\lambda v^2 < \frac{2\times 4\pi^2 v^2}{9\ln 2} \tag{6.17}$$

which gives

$$m_H < 875 \text{ GeV} \tag{6.18}$$

A better treatment uses not a single scalar, but four scalars, as we noted in the presentation of the Standard Model. We then find

$$m_H^2 = 2\lambda v^2 < \frac{4\pi^2 v^2}{3\ln 2} \tag{6.19}$$

which gives

$$m_H < 1.07 \text{ TeV}. \tag{6.20}$$

An alternative approach is to insist that the simple model remain adequate up to some scale Λ. In the four-scalar model, this tells us that

$$m_H^2 = 2\lambda v^2 < \frac{4\pi^2 v^2}{3\ln\Lambda/v} \tag{6.21}$$

where we make the assumption that v is the appropriate scale. In the extreme we might take $\Lambda \approx 10^{16}$ GeV so $m_H < 160$ GeV.

Rather than sum the geometric series, we can analyze the behavior of λ_Q with a differential equation - the renormalization group equation. Directly we see that

$$\frac{1}{\lambda(Q)} \equiv \frac{1}{\lambda_Q} = \frac{1}{\lambda} - \frac{9}{4\pi^2} \ln \frac{Q}{\mu} \tag{6.22}$$

so that

$$\frac{1}{\lambda(Q)^2} \frac{\partial \lambda(Q)}{\partial \ln Q} = \frac{9}{4\pi^2} \tag{6.23}$$

The renormalization group equations for the couplings can be determined systematically [24]. With the conventional normalization

$$\begin{aligned} \lambda &= \frac{m_H^2}{2v^2} \\ g_t &= -\frac{m_t}{v} \end{aligned} \tag{6.24}$$

the equation for λ reads, with $t = \ln Q^2$, [26]

$$\frac{d\lambda}{dt} = \frac{1}{16\pi^2}\left[12\lambda^2 + 12\lambda g_t^2 - 12g_t^4 - \frac{3}{2}\lambda(3g^2 + g'^2) + \frac{3}{16}(2g^4 + (g^2 + g'^2)^2)\right] \tag{6.25}$$

We see that the purely self coupling drives λ to grow with increasing t if λ is large. However, if λ is small, it is the Yukawa coupling to the t quark that dominates and this drives λ down. The theory won't make sense if λ goes negative because then the potential energy is not bounded below. So we can go beyond the scale Λ where $\lambda(\Lambda) = 0$. Now dropping all but the Yukawa driving term,

$$\lambda(\Lambda) = \lambda(v) - \frac{3}{4\pi^2} g_t^4 \ln(\Lambda^2/v^2) \tag{6.26}$$

so this sets a limit

$$m_H^2 > \frac{3v^2}{2\pi^2} g_t^4 \ln(\Lambda^2/v^2) \approx 68 \text{ GeV} \sqrt{\ln(\Lambda/v)} \tag{6.27}$$

7 Radiative Corrections from the Higgs Boson

It is a fundamental consequence of quantum mechanics that there are physical effects from particles that are not physical produced, but exist only virtually. It is in this way that we can hope to see the Higgs boson before any accelerator actually produces it. An especially sensitive test is provided by the relation between the masses of the Z, W, t, and Higgs.

We have already seen that in lowest order the masses of the Z and W and the Fermi constant are given by

$$\begin{aligned} m_W^2 &= \frac{g^2 v^2}{4} \\ m_Z^2 &= \frac{(g^2 + g'^2) v^2}{4} \\ G_F &= \frac{1}{\sqrt{2} v^2} \end{aligned} \tag{7.1}$$

and

$$\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2} \tag{7.2}$$

To fix the parameters of the Standard Model we need three quantities. While in the Lagrangian it is g, g', and v that seem fundamental, we can use in their place any three physical quantities determined by them. The traditional choice is α_{em}, G_F, and m_Z. These are extremely well measured. Unfortunately, it is not $\alpha_{em}(Q = 0)$ that really shows up in the calculations, but $\alpha_{em}(Q = m_Z)$.

The W mass is an especially interesting case. In the Standard Model there is a special relation connecting the W and Z masses and the mixing angle that described the gauge interactions:

$$m_W = \cos\theta_W m_Z \tag{7.3}$$

At lowest order, the W and Z masses are set by the vacuum expectation value of the Higgs boson. We might have had other scalar multiplets besides the one we chose. In general, the gauge boson masses come from

$$(D_\mu \langle\phi\rangle)^\dagger D_\mu \langle\phi\rangle \tag{7.4}$$

If we have several multiplets we can write

$$D_\mu\langle\phi\rangle \to \sum_\phi \left[\frac{g}{\sqrt{2}}(T^+W_\mu^+ + T^-W_\mu^-) + gT_3W_3 - g'T_3B\right]\langle\phi\rangle \tag{7.5}$$

where we used $\langle\phi\rangle = (T_3 + Y/2)\langle\phi\rangle = 0$ Now we can rewrite masses as

$$(D_\mu\langle\phi\rangle)^\dagger D_\mu\langle\phi\rangle = \sum_\phi \left\{\frac{g^2}{2}[T(T+1) - T_3^2]2W^+W^- + T_3^2(g^2+g'^2)Z^2\right\}\langle\phi\rangle^2 \tag{7.6}$$

The ratio of the squares of the masses is thus

$$\frac{m_W^2}{m_Z^2} = \frac{g^2}{g^2+g'^2}\frac{\sum_\phi[T(T+1) - T_3^2]\frac{1}{2}\langle\phi\rangle^2}{T_3^2\langle\phi\rangle^2} \tag{7.7}$$

If all the Higgs multiplets with vacuum expectation values have $T = 1/2$, the second factor is unity and the relation $m_W = \cos\theta_W m_Z$ holds.

Provided there is just the one Higgs doublet breaking electroweak symmetry, this relation has small, calculable corrections. At Born (tree) level we have

$$m_W^2 = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2}G_F m_Z^2}}\right)m_Z^2 \tag{7.8}$$

With radiative corrections this becomes

$$m_W^2 = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4\pi\alpha(1+\Delta r)}{\sqrt{2}G_F m_Z^2}}\right)m_Z^2 \tag{7.9}$$

The sensitivity of m_W^2 to Δr is most easily seen by writing

$$\begin{aligned}
m_W^2 &= \frac{1}{2}\left(1 + \sqrt{1 - \sin^2 2\theta_W(1+\Delta r)}\right)m_Z^2 \\
\frac{1}{m_W^2}\frac{\partial m_W^2}{\partial\Delta r} &= -\frac{\sin^2\theta_W}{1 - 2\sin^2\theta_w} \approx -0.36 \\
\frac{1}{m_W}\frac{\partial m_W}{\partial\Delta r} &\approx -0.18
\end{aligned} \tag{7.10}$$

so that a shift of Δr by 0.01 changes m_W by 145 MeV.

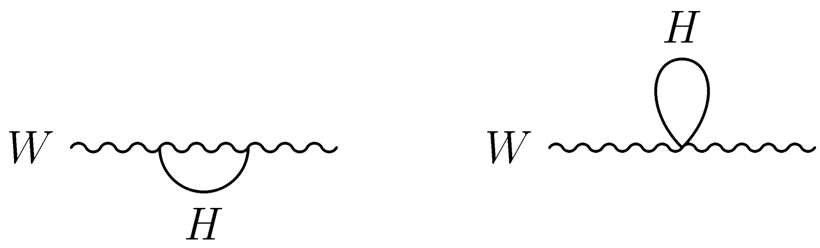


Figure 16: Higgs boson contributions to the W mass.

Here Δr contains all kinds of loops inserted into the propagators of the photon, A, and W. Explicitly [27, 1]

$$\Delta r = \Delta\alpha - \Delta\rho\frac{c^2}{s^2} + \frac{g^2}{s^2}[\Pi'_{33}(m_Z^2) - 2s^2\Pi'_{3Q}(m_Z^2)] + \frac{g^2}{s^2}(s^2 - c^2)\Pi'_{11}(m_W^2) \quad (7.11)$$

where

$$\begin{aligned} s &= \sin\theta_W \\ c &= \cos\theta_W \end{aligned} \quad (7.12)$$

and where

$$\begin{aligned} \Delta\alpha &= e^2[\Pi_{QQ}(m_Z^2) - \Pi_{QQ}(0)] \\ \Delta\rho &= \frac{g^2}{m_W^2}[\Pi_{11}(0) - \Pi_{33}(0)] \\ \Pi'(q^2) &= [\Pi(q^2) - \Pi(0)]/q^2 \approx \frac{\partial\Pi(q^2)}{\partial q^2} \end{aligned} \quad (7.13)$$

The polarization tensor $i\Pi_{Q_1Q_2}$ is obtained by calculating the diagram with the charges Q_1 and Q_2 at the vertices (and without the coupling constant). Thus Π_{11} is the polarization tensor for a charged W, while Π_{33} is the polarization tensor for a Z, but using only the coupling that goes with the W_3. We can ignore questions of gauge choice and look only at $g_{\mu\nu}$ type terms.

Just to show how these things work, let us isolate the leading contribution from the Higgs boson. Now this won't contribute of course to $\Delta\alpha$ since the neutral Higgs boson doesn't contribute to the running of the electromagnetic coupling. Similarly there is no contribution to Π_{3Q} since Q here stands for the electric charge coupling. For our purposes, we can approximate $\Pi'_{33}(m_Z^2) = \Pi'_{11}(m_W^2) = \Pi'_{33}(0)$. Altogether then, the contribution of the Higgs to Δr is

$$\Delta r_H = \frac{c^2}{s^2}\frac{g^2}{m_W^2}[\Pi_{33}(0) - \Pi_{11}(0)] + 2g^2\Pi'_{33}(0) \tag{7.14}$$

There are two kinds of self energy diagrams for the W and Z with Higgs boson loops. The diagrams where the emission and absorption of the Higgsis at a single point have the properties

$$\Pi_{33}(0) = \Pi_{11}(0); \qquad \Pi'_{33}(0) = 0 \tag{7.15}$$

and thus don't contribution to Δr_H. The other diagrams involve the ZZH and WWH couplings. From Eq. (4.2) we see that the vertex factors are

$$\begin{aligned} WWH : (ig^2v/2)g_{\alpha\beta} &= igm_W g_{\alpha\beta} \\ ZZH : (ig^2v/2\cos^2\theta_W)g_{\alpha\beta} &= (igm_W/\cos^2\theta_W)g_{\alpha\beta} \end{aligned} \tag{7.16}$$

where α and β get tied to the polarization indices of the gauge bosons. Now when we calculate Π_{33} we couple only to the W_3 part of the Z, the piece that has coefficient $\cos\theta_W$. This just cancels the $\cos\theta_W$ factor in the vertex. Thus the integrals for Π_{33} and Π_{11} are identical except that Π_{33} has m_Z^2 in it, while Π_{11} has m_W^2.

For either Π_{11} or Π_{33} we have

$$\begin{aligned} i\Pi_{\alpha\beta}(q^2) &= (im)^2\int\frac{d^nk}{(2\pi)^4}\frac{i}{(k+q)^2 - m_H^2}\frac{-i(g_{\alpha\beta} - k_\alpha k_\beta/m^2)}{k^2 - m^2} \\ &= (im)^2\int_0^1 dx\int\frac{d^nk}{(2\pi)^4}\frac{g_{\alpha\beta} - (k' - xq)_\alpha(k' - xq)_\beta/m^2}{(k'^2 - x^2q^2 + xq^2 - xm_H^2 - (1-x)m^2)^2} \end{aligned} \tag{7.17}$$

Since we are concerned solely with the $g_{\alpha\beta}$ piece of the answer, we retain only $g_{\alpha\beta}$ and $k'_\alpha k'_\beta$ in the numerator. We make the replacement

$$k'_\alpha k'_\beta \to k'^2 g_{\alpha\beta}/n \tag{7.18}$$

to obtain

$$i\Pi(q^2) = -\int_0^1 dx\int\frac{d^nk'}{(2\pi)^4}\frac{m^2 - k'^2/n}{(k'^2 - \mu^2)^2} \tag{7.19}$$

where m is m_Z or m_W and

$$\mu^2 = xm_H^2 + (1-x)m^2 - x(1-x)q^2 \tag{7.20}$$

Now using the formulae of Appendix D,

$$\begin{aligned} i\Pi(q^2) &= \frac{-i\Gamma(2-n/2)}{16\pi^2}\int_0^1 dx \frac{m^2+\mu^2/(2-n)}{(\mu^2)^{2-n/2}} \\ i\frac{\partial\Pi(q^2)}{\partial q^2} &= \frac{-i\Gamma(2-n/2)}{16\pi^2}\int_0^1 dx \left[\frac{(2-n/2)m^2/\mu^2 - 1/2}{(\mu^2)^{2-n/2}}\right][-x(1-x)] \end{aligned} \tag{7.21}$$

We expand

$$(\mu^2)^{2-n/2} = 1 + (2-n/2)\ln\mu^2 = 1 + (2-n/2)[\ln m_H^2 + \mathcal{O}(m^2/m_H^2) \tag{7.22}$$

so

$$\begin{aligned} i\Pi(q^2) &= \frac{-i\Gamma(2-n/2)}{16\pi^2}\int_0^1 dx[m^2+\mu^2/(2-n)][1-(2-n/2)\ln m_H^2] \\ i\Pi(0) &= \frac{i\Gamma(2-n/2)}{16\pi^2}[m^2 + \frac{m_H^2+m^2}{4-2n}][1-(2-n/2)\ln m_H^2] \end{aligned} \tag{7.23}$$

and

$$\begin{aligned} i(\Pi_{33}(0)-\Pi_{11}(0)) &= \frac{-i\Gamma(2-n/2)}{16\pi^2}(m_Z^2-m_W^2) \\ &\quad \times[1-\frac{1}{4-2n}][1+(2-n/2)\ln m_H^2] \end{aligned} \tag{7.24}$$

We are only interested in the m_H dependence. Indeed the terms we have computed don't converge by themselves. Only when we include all the diagrams with internal Ws and Zs will the sum converge, since the Higgs is an integral part of the gauge sector. The surviving m_H dependent piece is

$$\Pi_{33}(0) - \Pi_{11}(0) = \frac{3}{4}(m_Z^2 - m_W^2)\frac{1}{16\pi^2}\ln m_H^2 \tag{7.25}$$

In the expression for Π' we can drop the term suppressed by m^2/m_H^2 and expand as before to find

$$\begin{aligned} i\frac{\partial\Pi(q^2)}{\partial q^2} &= \frac{-i\Gamma(2-n/2)}{16\pi^2}\int_0^1 dx[x(1-x)/2][1-(2-n/2)\ln m_H^2] \\ &\to \frac{i}{16\pi^2}\frac{1}{12}\ln m_H^2 \end{aligned} \tag{7.26}$$

Combining all this,

$$\begin{aligned}
\Delta r_H &= \frac{c^2}{s^2}\frac{g^2}{m_W^2}[\Pi_{33}(0)-\Pi_{11}(0)]+2g^2\Pi'_{33}(0)\\
&= \frac{g^2}{16\pi^2}[\frac{3}{4}+\frac{2}{12}]\ln m_H^2\\
&= \frac{\sqrt{2}G_F}{4\pi^2}\frac{11}{12}\ln m_H^2\\
&\approx 2.5\times 10^{-3}\ln m_H^2 \qquad (7.27)
\end{aligned}$$

This is really just an indication since we've taken m_H large compared to m_W and m_Z.

We anticipate that quark loops will shift the relative masses of the W and Z. However, this effect is suppressed if the quark doublets are degenerate. To see this, consider Eq.(7.11). For such a doublet, there is an isospin symmetry so $\Pi_{11}=\Pi_{33}$, killing $\Delta\rho$. If we ignore the difference between evaluating Pi at m_Z and m_W, then we can write

$$\frac{g^2}{s^2}[\Pi'_{33}(m_Z^2)-2s^2\Pi'_{3Q}(m_Z^2)]+\frac{g^2}{s^2}(s^2-c^2)\Pi'_{11}(m_W^2)=-2s^2\Pi'_{3Y/2}(m_Z^2) \qquad (7.28)$$

which vanishes when we sum over the two quarks, which have the same $Y/2$ but opposite T_3.

The top quark contributes importantly to the shift in the W mass because the $t-b$ system breaks this "custodial" isospin so strongly. The contribution to Δr is

$$\begin{aligned}
\Delta r_t &= -\frac{3G_F c^2}{8\pi^2\sqrt{2}s^2}\left\{m_t^2+m_b^2-\frac{2m_t^2m_b^2}{m_t^2-m_b^2}\ln\frac{m_t^2}{m_b^2}\right\}\\
&= -0.036\left(\frac{m_t}{175\text{ GeV}}\right)^2 \qquad (7.29)
\end{aligned}$$

We can now determine how much the predicted W mass shifts for a given shift in m_t or m_H:

$$\begin{aligned}
\frac{1}{m_W}\delta m_W &= -0.18\delta\Delta r=-0.18[5.0\times 10^{-3}\delta\ln m_H-0.072\frac{\delta m_t}{m_t}]\\
&= -9.0\times 10^{-4}\delta\ln m_H+1.3\times 10^{-2}\frac{\delta m_t}{m_t} \qquad (7.30)
\end{aligned}$$

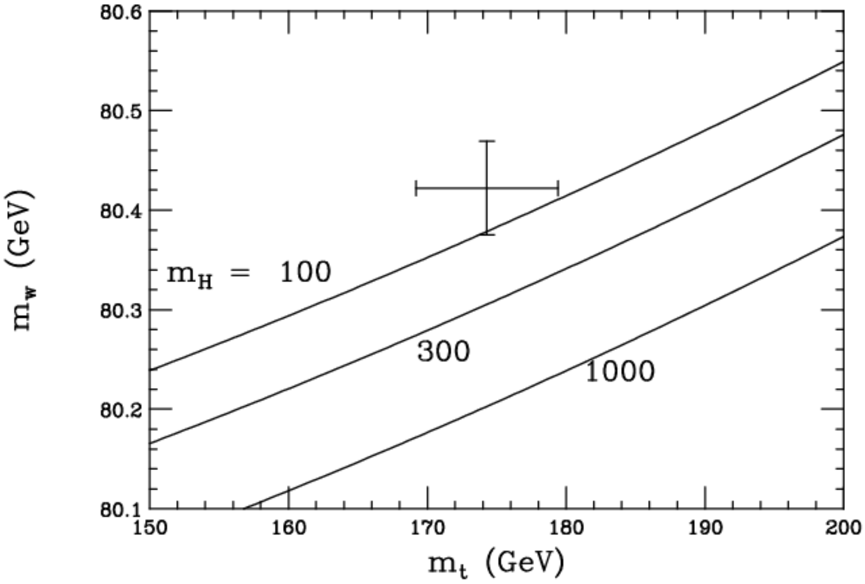


Figure 17: The current values for m_t and m_W are shown, with their uncertainties, along with curves determined by radiative corrections 7.31.

A thorough treatment**??**, including higher order corrections gives the interpolating formula

$$\begin{aligned} m_W(\mathrm{GeV}) &= 80.3829 - 0.0579 \ln(m_H/100) - 0.008[\ln(m_H/100)]^2 \\ &\quad -0.517 \left(\frac{\delta\alpha_H^{(5)}}{0.0280} - 1 \right) + 0.543 \left[\left(\frac{m_t}{175} \right)^2 - 1 \right] \\ &\quad -0.085 \left(\frac{\alpha_s(M_Z)}{0.118} - 1 \right) \end{aligned} \tag{7.31}$$

where the $\overline{\mathrm{MS}}$ scheme has been used and where all masses are in GeV. Curves for $m_H = 100, 300, 1000$ GeV are shown together with the values reported by the Particle Data Group in 2001 [$m_t = 174.3 \pm 5.1$ GeV, $m_W = 80.422 \pm 0.047$ GeV] in Fig.17.

A Phase Space

Decay rates and cross sections are obtained from the invariant matrix element, $\mathcal{M}$, by evaluating the phase space

$$d\Phi = \Pi \frac{d^3 p_i}{(2\pi)^3 2E_i} \delta^4(P - \sum p_i) \tag{A.1}$$

for the process in question. For the decay of a particle of mass . M,

$$d\Gamma = \frac{(2\pi)^4}{2M} |\mathcal{M}|^2 d\Phi \tag{A.2}$$

For a two-body process (decay or scattering),

$$d\Phi_2 = \frac{(2\pi)^{-6}}{4\sqrt{s}} p_{cm} d\Omega_{cm} \tag{A.3}$$

where s is the c.m. energy squared and p_{cm} is the momentum of either final state particle in the c.m. Thus for a two-body decay

$$d\Gamma = \frac{1}{32\pi^2} |\mathcal{M}|^2 \frac{p_{cm} d\Omega_{cm}}{M^2} \tag{A.4}$$

For a three-body decay or for scattering to a three-body final state phase space can be evaluated in several ways. The Dalitz plot form describes the orientation of the final state plane in the c.m. by three Euler angles, α, β, and γ. The angles α and γ are integrated from 0 to 2π, while β varies between 0 and π. The final state particles have energies E_1, E_2, and E_3. The phase space is given by

$$d\Phi_3 = \frac{1}{8(2\pi)^9} d\alpha d\cos\beta d\gamma dE_1 dE_2 \tag{A.5}$$

Alternatively, we can treat the problem as a quasi-two-body process, viewing particles 1 and 2 as forming a resonance of mass m_{12}. We then find

$$d\Phi_3 = \frac{1}{8(2\pi)^9 M} q_{12} p_3 dm_{12} d\Omega_{12} d\Omega_3 \tag{A.6}$$

where

$$\begin{aligned} q_{12} &= \text{momentum of 1 in } 1-2 \text{ rest frame} \\ p_3 &= \text{momentum of 3 in overall rest frame} \\ d\Omega_{12} &= \text{solid angle in } 1-2 \text{ rest frame} \\ d\Omega_3 &= \text{solid angle in overall rest frame} \end{aligned} \tag{A.7}$$

We can always decompose n-body phase space by grouping together the first m particles and the last $n-m$ particles:

$$q_{1,m} = \sum_{i}^{m} p_i; \qquad q_{m+1,n} = \sum_{m+1}^{n} p_i \tag{A.8}$$

with the result

$$\begin{aligned} d\Phi_n(P; p_1, \ldots p_m, p_{m+1} \ldots p_n) &= d\Phi_2(P; q_{1,m}, q_{m+1,n}) \\ &\qquad \times (2\pi)^3 dq_{1,m}^2 d\Phi_m(q_{1,m}, p_1, \ldots p_m)) \\ &\qquad\qquad \times (2\pi)^3 dq_{m+1,n}^2 d\Phi_{n-m}(q_{m+1,n}, p_{m+1}, \ldots p_n) \end{aligned} \tag{A.9}$$

Cross sections are given generally by the expression

$$d\sigma = \frac{(2\pi)^4 |\mathcal{M}|^2}{4\text{flux factor}} d\Phi \tag{A.10}$$

where

$$\begin{aligned} \text{flux factor} &= \sqrt{(k_1 \cdot k_2)^2 - m_1^2 m_2^2} \\ &= k_{cm}\sqrt{s} = k_{lab} m_{target} \end{aligned} \tag{A.11}$$

where the k's present the incoming momenta.

For two-body to two-body scattering

$$\frac{d\sigma}{d\Omega_{cm}} = \frac{1}{64\pi^2 s} \frac{k'_{cm}}{k_{cm}} |\mathcal{M}|^2 \tag{A.12}$$

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s} |\mathcal{M}|^2 \tag{A.13}$$

where k, k' are the initial and final c.m. momenta and $t = (k - k')^2$ is the four-momentum transfer squared.

The connection to the old-fashioned non-relativistic scattering amplitude f_{cm} is

$$f_{cm} = -\frac{1}{8\pi\sqrt{s}} \mathcal{M} \tag{A.14}$$

$$\frac{d\sigma}{d\Omega_{cm}} = |f_{cm}|^2 \tag{A.15}$$

$$\sigma_{tot} = \frac{4\pi}{p_{cm}} \Im f_{cm}(0^\circ) \tag{A.16}$$

B Feynman Rules

The Feynman rules for spontaneously broken gauge theories are non-trivial. The canonical reference is Fujikawa, Lee, and Sanda [10]. Ignoring these important changes, we recall the traditional rules:

- Each vertex is $i \times \mathcal{L}$. This follows from the appearance of $\exp(-i\mathcal{L})$ in the evaluation of the S-matrix.
- The calculated amplitude is $-i\mathcal{M}$. This follows from the relation between the S matrix and the invariant amplitude. For example, for two-body to two-body scattering

$$\begin{aligned} \langle p_1' p_2' | S | p_1 p_2 \rangle &= \langle p_1' p_2' || p_1 p_2 \rangle \\ &\quad -i(2\pi)^4 \delta(p_1' + p_2' - p_1 - p_2) \frac{\mathcal{M}(p_1, p_2, p_1', p_2')}{[(2\pi)2E_1'(2\pi)E_2'(2\pi)E_1(2\pi)E_2]^{1/2}} \end{aligned} \tag{B.1}$$

The state normalization is

$$\begin{aligned} \langle p' | p \rangle &= \delta^3(p - p'), \\ I &= \int d^3p |p\rangle\langle p| \end{aligned} \tag{B.2}$$

so that unitarity, $S^\dagger S = I$ gives

$$2\Im\mathcal{M}(p_1, p_2, p_1', p_2') = -(2\pi)^4 \sum_f d\Phi_n \mathcal{M}(p_1', p_2', f)^* \mathcal{M}(p_1, p_2, f) \tag{B.3}$$

- The factor associated with a gradient term in the vertex depends on whether the momentum flows into or out of the ertex. Since the expansion of a field in momentum space is

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}(2\omega)^{1/2}} [a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}], \tag{B.4}$$

the factors are

$$\begin{aligned} &\partial_\mu \to +ik_\mu \text{ for outgoing;} \\ &\partial_\mu \to -ik_\mu \text{ for incoming;} \end{aligned} \tag{B.5}$$

.

- For each incoming or outgoing particle that has spin is there is a wave fuction. For a massive spin-one particle of momentum q there is a polarization vector ϵ (incoming) or ϵ^* (outgoing) with

$$
\begin{aligned}
\epsilon \cdot q &= 0 && \text{(B.6)}\\
\sum_\epsilon e^*_\mu e_\nu &= -g_{\mu\nu} + q_\mu q_\nu / m^2 && \text{(B.7)}
\end{aligned}
$$

  This is the unitary gauge, which we use for W and Z. For photons, we can drop the second term as long as we confirm that the electromagnetic current is properly conserved ($J \cdot q = 0$).

- For a Dirac particle we have the spinors $u(p,s)$ (incoming) and $\overline{u}(p,s)$ (outgoing) and for its anti-particle $v(p,s)$ (outgoing) and $\overline{v}(p,s)$ (incoming). These obey the relations

$$
\begin{aligned}
(\not{p} - m)u &= \overline{u}(\not{p} - m) = 0; && \text{(B.8)}\\
(\not{p} + m)v &= \overline{v}(\not{p} + m) = 0; && \text{(B.9)}\\
u(p, s_z = 1/2) &= \sqrt{E+m}\begin{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{pmatrix}; && \text{(B.10)}\\
u(p, s_z = -1/2) &= \sqrt{E+m}\begin{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{pmatrix}; && \text{(B.11)}\\
v(p, s_z = 1/2) &= \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{pmatrix}; && \text{(B.12)}\\
v(p, s_z = -1/2) &= \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{pmatrix}; && \text{(B.13)}\\
s \cdot p &= 0 && \text{(B.14)}\\
u(p,s)\overline{u}(p,s) &= (\not{p} + m)\frac{1}{2}(1 + \gamma_5 \not{s}); && \text{(B.15)}\\
v(p,s)\overline{v}(p,s) &= (\not{p} - m)\frac{1}{2}(1 + \gamma_5 \not{s}); && \text{(B.16)}\\
\sum_s u(p,s)\overline{u}(p,s) &= (\not{p} + m); && \text{(B.17)}\\
\sum_s v(p,s)\overline{v}(p,s) &= (\not{p} - m); && \text{(B.18)}\\
\overline{u}(p,s)u(p,s) &= 2m && \text{(B.19)}
\end{aligned}
$$

$$
\begin{aligned}
\overline{v}(p,s)v(p,s) &= -2m && \text{(B.20)}\\
u(p,s) &= \frac{\not p + m}{\sqrt{2m(E+m)}}u(0,s); && \text{(B.21)}\\
v(p,s) &= \frac{-\not p + m}{\sqrt{2m(E+m)}}v(0,s); && \text{(B.22)}\\
& && \text{(B.23)}
\end{aligned}
$$

Some useful trace relations are

$$
\begin{aligned}
\mathrm{Tr}\ \not a \not b &= 4a\cdot b && \text{(B.24)}\\
\mathrm{Tr}\ \not a \not b \not c \not d &= 4(a\cdot b\, c\cdot d - a\cdot c\, b\cdot d + a\cdot d\, b\cdot c) && \text{(B.25)}\\
\mathrm{Tr}\ \not a \gamma_\mu \not b \gamma_\nu \mathrm{Tr}\ \not c \gamma^\mu \not d \gamma^\nu &= 32(a\cdot c b\cdot d + a\cdot d b\cdot c) && \text{(B.26)}\\
\mathrm{Tr}\ \not a \gamma_\mu \not b \gamma_\nu \gamma_5 \mathrm{Tr}\ \not c \gamma^\mu \not d \gamma^\nu \gamma_5 &= 32(a\cdot c b\cdot d - a\cdot d b\cdot c) && \text{(B.27)}\\
& && \text{(B.28)}
\end{aligned}
$$

Other Dirac matrix identities:

$$
\gamma_\mu \not a \gamma^\mu = -2\not a \qquad \text{(B.29)}
$$

- The propagators are, for a scalar

$$
\frac{i}{q^2 - m^2}; \qquad \text{(B.30)}
$$

for a Dirac particle

$$
\frac{i}{\not q - m} \qquad \text{(B.31)}
$$

for a massive vector in unitary gauge

$$
-i\frac{g^{\mu\nu} - q^\mu q^\nu/m^2}{q^2 - m^2} \qquad \text{(B.32)}
$$

For the photon we drop the second term in the numerator.

C Breit Wigner Formula

There may be no formula in particle physics as useful as the Breit Wigner formula for a cross section near a resonance. The traditional non-relativistic version reads

$$\sigma = \frac{2J+1}{(2s_1+1)(2s_2+1)} \cdot \frac{4\pi}{k^2} \cdot \frac{\Gamma^2/4}{(E-E_0)^2+\Gamma^2/4} \cdot \mathrm{BR_{in}BR_{out}} \tag{C.1}$$

Here J is the spin of the resonance and E_0 is the resonant energy. The spins of the incident particles are s_1 and s_2. The c.m. energy of the collision is k. The full-width at half maximum is Γ. This is also the lifetime of the resonant state. The branching ratios are for the resonance decaying into the initial and final states. We see the unitarity bound for scattering in a single partial wave in the first two factors.

In a relativistic setting we usually find a form like

$$\sigma = \frac{2J+1}{(2s_1+1)(2s_2+1)} \cdot \frac{4\pi}{k^2} \cdot \frac{m^2\Gamma^2}{(s-m^2)^2+m^2\Gamma^2} \cdot \mathrm{BR_{in}BR_{out}} \tag{C.2}$$

arising from the Feynman propagator. The difference between the two forms vanishes at the resonance itself and thus is equivalent to a non-resonant contribution.

D Dimensional Regularization

Consider the integral

$$\int \frac{d^n p}{(p^2-m^2)^\alpha} = \int_{-\infty}^{\infty} \frac{dp_0 d^{n-1}q}{(p_0^2-q^2-m^2+i\epsilon)^\alpha} \int_{-\infty}^{\infty} \frac{dp_0 q^{n-2} dq d\Omega_{n-2}}{(p_0^2-q^2-m^2+i\epsilon)^\alpha} \tag{D.3}$$

First let's see how to do the n-dimensional angular integral. We can determine this from a single example:

$$\begin{aligned} \int d^m q e^{-q^2} &= \pi^{m/2} = \int q^{m-1} dq d\Omega_{m-1} e^{-q^2} \\ &= \int_0^\infty v^{(m-1)/2} \frac{dv}{2v^{1/2}} e^{-v} \int d\Omega_{m-1} \\ &= \frac{1}{2}\Gamma(m/2) \int d\Omega_{m-1} \end{aligned} \tag{D.4}$$

So we find

$$d\Omega_{m-1} = \frac{2\pi^{m/2}}{\Gamma(m/2)} \tag{D.5}$$

Now do the p_0 integral in the complex plane, picking up the pole of order α at $p_0 = \sqrt{q^2+m^2}$:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dp_0 d^{n-1}q}{(p_0^2 - q^2 - m^2 + i\epsilon)^\alpha} &= \frac{-2\pi i}{(\alpha-1)!}\int q^{n-2}dq \left(\frac{\partial}{\partial p_0}\right)^{\alpha-1} (p_0 + \sqrt{q^2+m^2})^{-\alpha} \int d\Omega_{n-2} \\
&= \frac{-2\pi i}{(\alpha-1)!}[-\alpha(-\alpha-1)\cdots(-2\alpha+2)]2^{-2\alpha+1} \\
&\quad \times \int q^{n-2}dq\sqrt{q^2+m^2})^{-2\alpha+1} d\Omega_{n-2}
\end{aligned} \tag{D.6}$$

Since

$$\begin{aligned}
[-\alpha(-\alpha-1)\cdots(-2\alpha+2)] &= (-1)^{\alpha-1}\frac{(2\alpha-2)!}{(\alpha-1)!} = (-1)^{\alpha-1}2^{2\alpha-2}(\alpha-3/2)\cdots(1/2)\frac{\Gamma(1/2)}{\Gamma(1/2)} \\
&= (-1)^{\alpha-1}2^{2\alpha-2}\frac{\Gamma(\alpha-1/2)}{\Gamma(1/2)}
\end{aligned} \tag{D.7}$$

we find

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dp_0 d^{n-1}q}{(p_0^2 - q^2 - m^2)^\alpha} &= \frac{\pi i}{(\alpha-1)!}(-1)^\alpha m^{n-2\alpha}\frac{\Gamma(\alpha-1/2)}{\Gamma(1/2)}\frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \\
&\quad \times \int_0^\infty du u^{n-2}(u^2+1)^{-\alpha+1/2}
\end{aligned} \tag{D.8}$$

Now

$$\begin{aligned}
\int_0^\infty du u^{n-2}(u^2+1)^{-\alpha+1/2} &= \frac{1}{2}\int_0^\infty dv v^{(n-3)/2}(1+v)^{-\alpha+1/2} \\
&= \frac{1}{2}\frac{\Gamma((n-1)/2)\Gamma(\alpha-n/2)}{\Gamma(\alpha-1/2)}
\end{aligned} \tag{D.9}$$

Combining the results,

$$\int \frac{d^n p}{(p^2-m^2)^\alpha} = \int_\infty^\infty \frac{dp_0 d^{n-1}q}{(p_0^2-q^2-m^2)^\alpha} = \frac{i\pi^{n/2}\Gamma(\alpha-n/2)}{\Gamma(\alpha)}m^{n-2\alpha}(-1)^\alpha \tag{D.10}$$

It is trivial from this to deduce

$$\int \frac{d^n p p^2}{(p^2 - m^2)^\alpha} = \frac{i\pi^{n/2}\Gamma(\alpha - n/2 - 1)}{\Gamma(\alpha)} m^{n-2\alpha}(-1)^{\alpha+1} m^2 \frac{n}{2} \tag{D.11}$$

Moreover using

$$g_{\mu\nu} g^{\mu\nu} = n \tag{D.12}$$

we can conclude

$$\int \frac{d^n p p^\mu p^\nu}{(p^2 - m^2)^\alpha} = \frac{i\pi^{n/2}\Gamma(\alpha - n/2 - 1)}{\Gamma(\alpha)} m^{n-2\alpha}(-1)^{\alpha+1} m^2 \frac{g^{\mu\nu}}{2} \tag{D.13}$$

These three integrals, plus the Feynman tricks

$$\begin{aligned} \frac{1}{AB} &= \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} \\ \frac{1}{ABC} &= \int_0^1 dx \int_0^{1-x} dy \frac{1}{[xA + yB + (1-x-y)C]^3} \\ \frac{1}{\Pi_{i=1}^n D_i} &= (n-1)! \int dt_1 \ldots dt_n \delta(1 - \sum_k t_k)[\sum_{j=1}^n t_j D_j]^{-n} \end{aligned} \tag{D.14}$$

are enough to get you out of any tough spot.

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