This Is

MEGA-Mathematics!

STORIES AND ACTIVITIES FOR MATHEMATICAL THINKING PROBLEM-SOLVING AND COMMUNICATION

The Los Alamos Workbook

Nancy Casey

Mike Fellows

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Mega-Math is a project of the C-3 Group at the Los Alamos National Laboratories in Los Alamos, New Mexico. The goal of the project is to bring the experience and content of 20th century mathematics to elementary school students and their teachers.

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An electronic copy of this book and related materials are available in postscript format via anonymous ftp from `ftp.cs.uidaho.edu`, directory is `pub/mega-math`. Consult the `readme` file in that directory.

The most current version of MegaMath materials is available on the World Wide Web: `http://www.c3.lanl.gov/~captors/mega-math`

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Nancy Casey and Mike Fellows

Victoria, October, 1993

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Chapter 1

About MegaMathematics

The purpose of this workbook is to provide some opportunities for children and their teachers to experience mathematics in ways it is experienced by mathematicians and scientists. Mathematics is lively and exciting; it is a field more akin to art and poetry than many people think.

When research mathematicians get together, they exchange stories and favorite problems. They marvel at the way some things were solved, and share their amazement that certai problems that sound simple don't have solutions. They talk about what they have been doing, what they are working on, and what they wish they could learn if they only had the time. In short, they inspire each other. We hope that young mathematicians in school classrooms will have the same experience.

Perhaps one of the best-kept secrets about mathematics is that it is easy to get a feel for the kinds of questions that are at the wide-open frontiers of knowledge. We already know that students enjoy speculating about what happened to the dinosaurs, pondering the shape of the universe or imagining the Big Bang. We think that they can get as much intriguing puzzlement from wondering about the size of infinity and finding out that there are some very simple-sounding problems that would take huge computers longer than the estimated age of the universe to solve. We are concerned that so much of the "good stuff" doesn't make it's way into the math classrooms until graduate school.

Good mathematics ultimately comes from and returns to good stories — and questions that bug you! We will have succeeded if you, and the children in your care, enjoy the stories and activities and come away from them with some annoying unanswered questions.

In this book, you will be introduced to the Hotel Infinity and hear of the adventures of the proprietor Sam Cantori and his helper George. We're willing to bet our shoelaces that you'll enjoy exploring the mathematics of knots, even though the bubbles you'll blow with the knot-shaped bubble-blowers will still be the familiar kind.

In the chapters on "Map-Coloring," "Machines that Eat Your Words" and "Ice Cream

Stands" you will be introduced to some important puzzles and ideas of computer science. When you visit "Unusual School" you will need to bring the kind of thinking tools that mathematicians rely on.

All of the activities involve hands-on exploration, and lots of opportunities for mathematical thinking, problem-solving and communication. All of the topics are real live important areas of current mathematical research. When mathematicians consider these subjects, they do just what you will be doing—wrestling with interesting ideas, and having that special kind of fun that comes from tenacious and satisfying thinking.

Chapter 2

The Most Colorful Math of All

2.1 Introduction

Overview

Believe it or not, coloring is a profound mathematical topic with many multi-million-dollar industrial applications. The problem presented here has been of interest to mathematicians for over a hundred years. Some versions of the problem are easy to solve, others have yielded only recently to relentless efforts that would not have been possible without the aid of powerful computers. In all of its aspects, the basic problem is the same: How many colors do you need?

With a few crayons or markers and some hand-drawn maps, children can quickly find themselves grappling with some of the same conundrums that contemporary mathematicians do.

Big Ideas and Key Concepts

• Four color theorem • Complexity • Induction • Proof

NCTM Curriculum Strands

• Mathematics as Problem Solving • Mathematics as Reasoning • Mathematical Connections • Mathematics as Communication

Evaluation

It is important for students who do these activities to experience the problem-solving process, and to understand the types of problem solving that mathematicians do. They can begin to learn that the "right answers " that are so often associated with mathematics are sometimes elusive, and not often found quickly. They should also be learning that they don't always need to go to an authority to evaluate the quality of their own answers.

Students should begin to recognize that they find a variety of strategies for solving problems on their own. They should be trying to articulate those strategies clearly, and finding out that the more they try to do so, the more adept they become. They should also be thinking about where their ideas come from, whether they are flashes of insight, the result of careful plodding, things they discovered from listening to or watching others, or ideas that came to them while they were explaining something to someone else.

With all of this to discover, it is certainly not necessary for students to "learn" the best ways to color maps!

It is also not very important for students to understand the idea of induction as presented here, or even accept the proofs that rely on induction in the activities. It is a good idea to use the word, simply for the purpose of exposing students to it. What is really important, however, is that students be exposed to the idea that *truth* is central to mathematics, and it is important for mathematicians of all ages to think about the *ways* that they use to find out if something is really true or not. The theme of thinking about how one thinks about problems pervades mathematics.

The Mathematics Behind the Maps

How many different colors do you need to color a map so that countries that share a boundary line are not colored the same color? In the mid-1800's this problem became known among a small group of mathematicians as a popular unsolved problems. Even at that time, it was common knowledge among map-makers that 4 colors seemed to be sufficient to color a map. This wasn't actually proved, however, until 1976.

Even though we now know that 4 colors is enough to color any map, some maps can be colored with fewer colors—three colors or even two. Finding a method to determine exactly how many colors is needed for any map continues to daunt mathematicians today. Some mathematicians believe that a fast method to find out the minimum number of colors needed for a map is impossible (that is, simply does not exist and hence can never be found). Given any map, there is, however, a reasonably fast way to tell whether or not you can color it with two colors, and we do know you won't ever need more than four colors; so the hard problem is figuring out if you can do it with three. For some maps, we presently don't know any better way to tell if three colors will be enough than to try every possible combination of three colors.

To get an idea of how impractical this would be, select one of the maps on these pages,

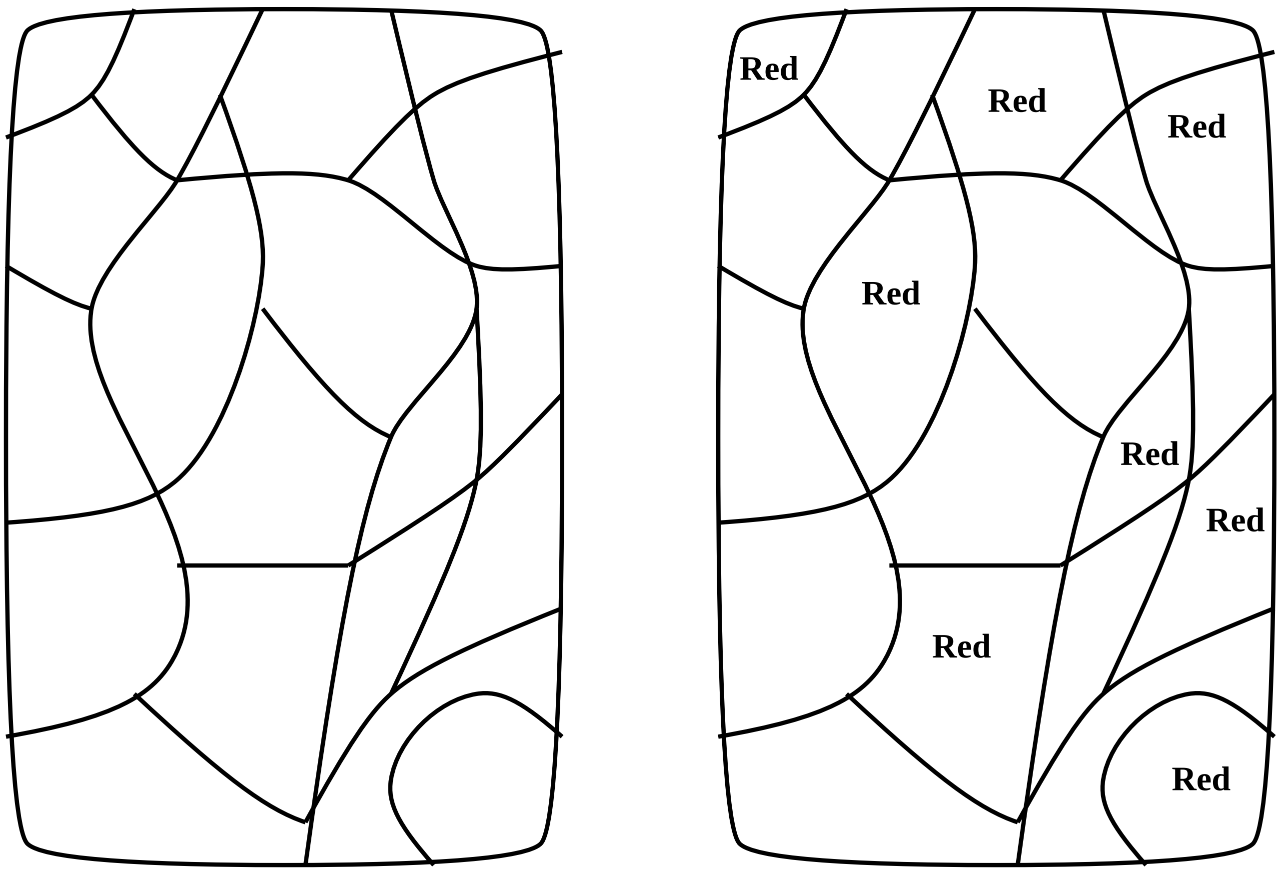


Figure 2.1: The Polka Dot Strategy for coloring a map. One color is used to color as many regions as possible before beginning to use another color.

and try to find a systematic way of counting all the different ways you could color it with just three colors. If you don't lose track completely, the number will be very large before you have gotten a very good start. And as maps go, these maps are rather small. For very large maps, it would take the fastest computers imaginable longer than the projected age of the universe to check out all the possibilities! The question of whether there is a fast method to find out if three colors is enough (and if so, what it is!) is known as the **P? =?NP Question**. This is widely regarded as the most important unsolved problem in computer science, mathematical logic, and the foundations of mathematics.

Children will typically discover a variety of interesting strategies for finding a coloring of a map that uses a small number of colors. Some of these are described below. It is important to give them a chance to discover these exciting ideas on their own. This is certainly not something to "teach" them.

One of the strategies they may discover is the *Polka Dot Strategy* of using one color to exhaustion before taking up another one. (See Figure 2.1.) This has a colorful mathematical name. It is known as the *Greedy Algorithm*. Greedy Algorithms are one of the four or five most basic and general algorithm design strategies used in computer science. (There is another example of a Greedy Algorithm in the chapter **Algorithms and Ice Cream For All**.

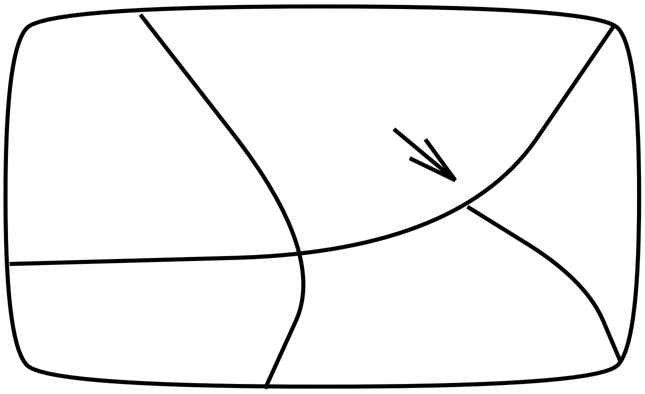


Figure 2.2: A map that cannot be colored with two colors. Why?

Other discoveries that students can make provide a rich field of play for logical reasoning and deduction. For example, some children will be able to explain why the map shown in Figure 2.2 cannot be colored with two colors. At the juncture indicated, three neighborhoods meet, each of which borders the other two, so they must all have different colors. Such an explanation is a **deduction**, part of the idea of **proof** which is at the heart of mathematics.

Surprisingly, the map in Figure 2.5 can be colored with two colors. Some children may discover a method for finding a 2-coloring (if there is one) by following a "chain of logical deductions" as illustrated in Figure 2.3 below:

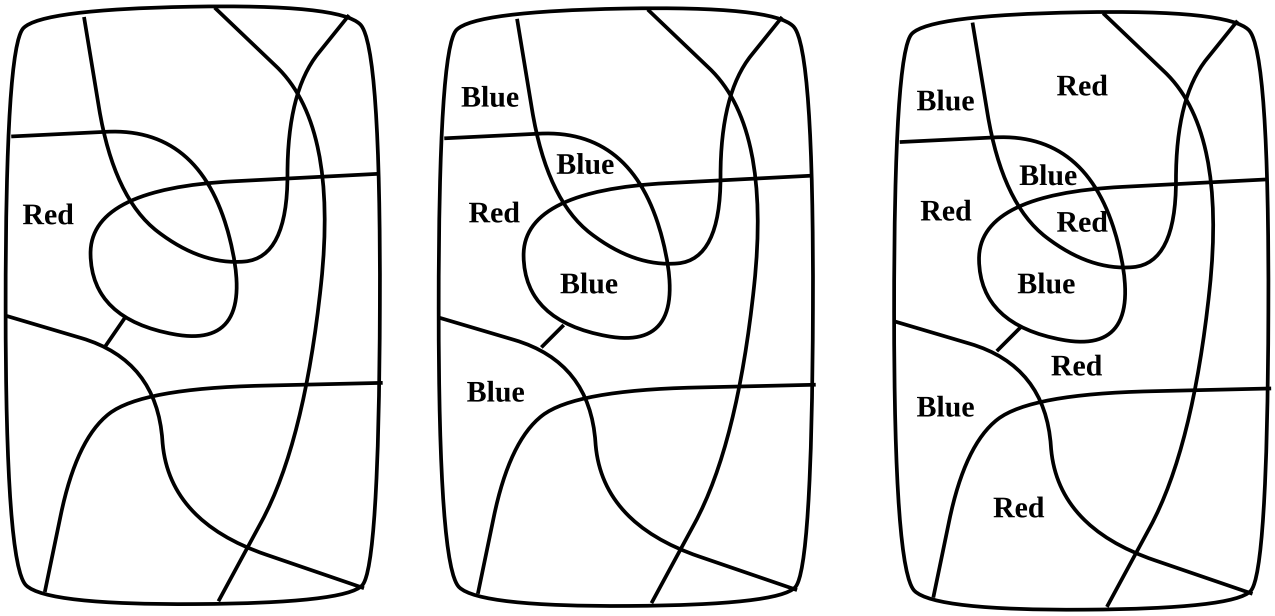


Figure 2.3: Chain of deductions for a 2-coloring.

In the leftmost map, one neighborhood has been colored *red*. In any *red/blue* 2-coloring of the map, the bordering neighborhoods must be colored *blue* as shown in the center map. And, in turn, the neighborhoods bordering those must be colored *red*, as shown in the rightmost map, and so on. We will either succeed in 2-coloring the entire map, or get stuck. In the

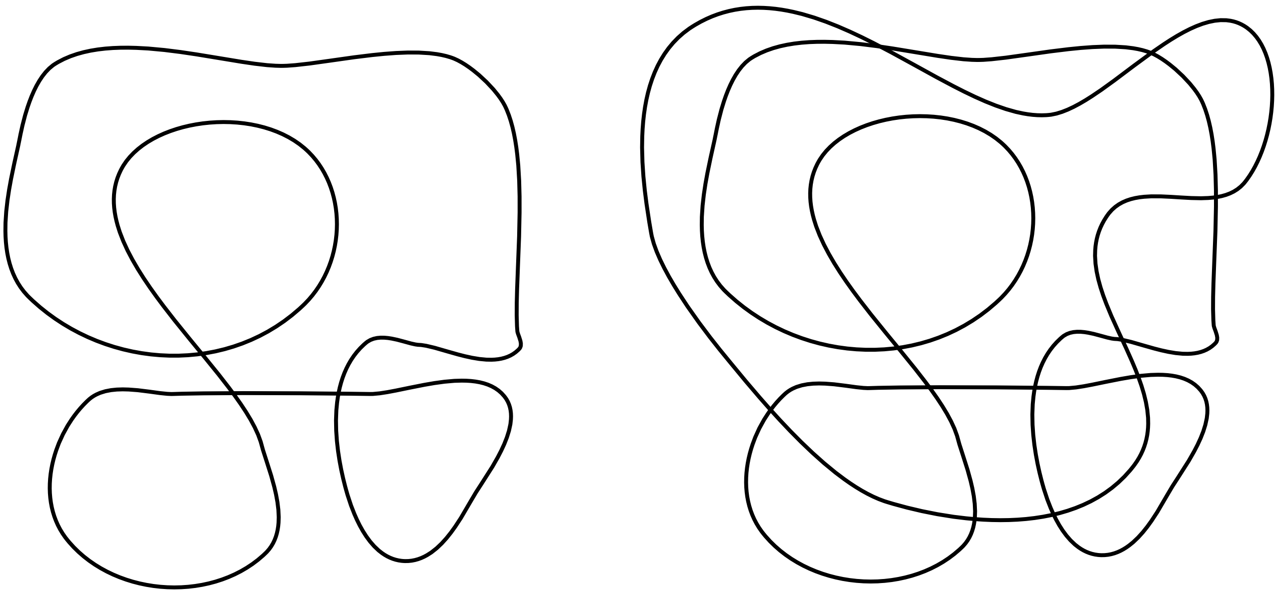


Figure 2.4: Simple closed curves make maps that are 2-colorable

latter case, we know that a 2-coloring is impossible, since we only did what we were logically forced to do up to that point.

Notice the "checkerboard" character of a 2-coloring. Such checkerboard 2-colorings play a role in the mathematical study of knots and surfaces in space.

After a while, some children may prefer to work with unifix cubes (or other colored marking pieces) to represent the colors assigned to the neighborhoods. This clearly (and concretely) represents a higher level of abstraction than the more-or-less permanent assignment of a color to a neighborhood using crayons. In fact, this is the manipulative employment of the concept of a **variable**, one of the most fundamental ideas in mathematics. The neighborhood is the **variable** that is assigned the **value**, a color. (Although students will probably not be very interested in the concept of variable and its usefulness, you will observe that working with "changeable markers" is a powerful problem-solving strategy for this problem.)

There is a simple (and surprising?) way to make "complicated" maps that can be 2-colored. The recipe is this: place pen to paper and draw a curve that returns to the starting point. The curve that you draw can intersect and cross itself any number of times. The map on the left in Figure 2.4 is such a curve. You can draw any number of such curves on top of one another (as in the map on the right in Figure 2.4), and surprisingly enough, the resulting map will be two-colorable.

The reason why this is true is explored in the activity **Make a Two-Colorable Map** using string and colored marking pieces. The activity is a manipulative introduction to the idea of **proof by induction** a fundamental idea that students have been introduced traditionally in high school.

The basic principle of proof by induction is to begin by looking at the simplest possible case, and then add elements to that simplest case in a step-by-step orderly fashion that is systematic enough to eventually cover all the cases. If you can show why, at each step, what you know about the simplest case must still be true, you can argue that it will always be true.

It is important that students work with a variety of maps when they experiment with map coloring. The maps that are included with the activities in this chapter are meant as a starting point only. Additional maps are easy to draw, and you will learn a lot about map coloring by experimenting with maps to give to your students that will challenge them and let them apply what they know. And, of course, students can make their own maps.

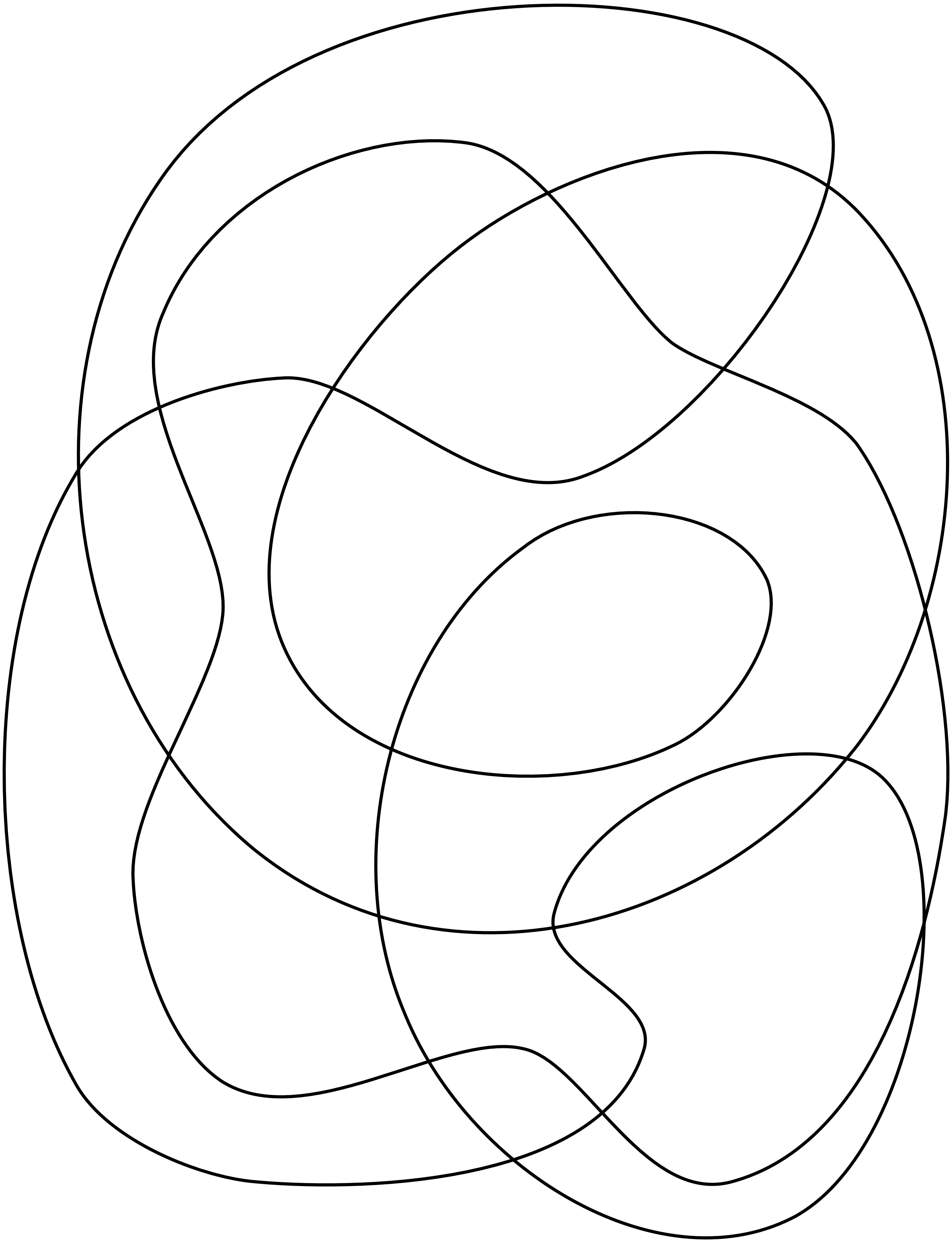


Figure 2.5: A Map to Color

2.2 The Poor Map Colorer

Description

Students experiment with coloring several maps so that they can understand what the map coloring questions are all about and try out some of their own ideas for answering them.

Instructions

1. Read or tell the story of **The Poor Map-Colorer**.
2. Pass out the copies of the black line masters (start with Figure 2.5).
3. Invite the children to work singly or in groups (whichever they prefer) to try to discover the smallest number of colors needed to color the map.
4. After some problem-solving, have the students discuss (and perhaps write about) their results, their strategies and reasoning and the observations that they made about the problem.
5. Replay with more complex maps, such as in Figure 2.6. In order to have a variety of maps to work with, you can draw additional ones that will be appropriate for your students.
6. Show children that they can draw their own maps to share with one another and try to color.
7. Show students how they can experiment with various colorings without using up so many maps, or having to take the time to color them by marking regions with unifix cubes or other colored marking pieces instead of coloring them in.
8. Read or tell **A Television Story**. What does the problem in this story have in common with the map coloring problem?

The Story of the Poor Map Colorer

Once upon a time there lived a poor map-colorer whose job it was to color the maps of the neighborhoods of the kingdom. In order for such a map to appear pleasing to eye, neighborhoods sharing a border were required to have different colors. (This is a sensible requirement that is observed in the craft of map-coloring even to this very day, as it makes the boundaries between the neighborhoods easier to see at a glance.)

In those long-ago days crayons were rare and quite expensive, and the poor map-colorer was often obliged to try to do her job with just a very few crayons. She often spent many

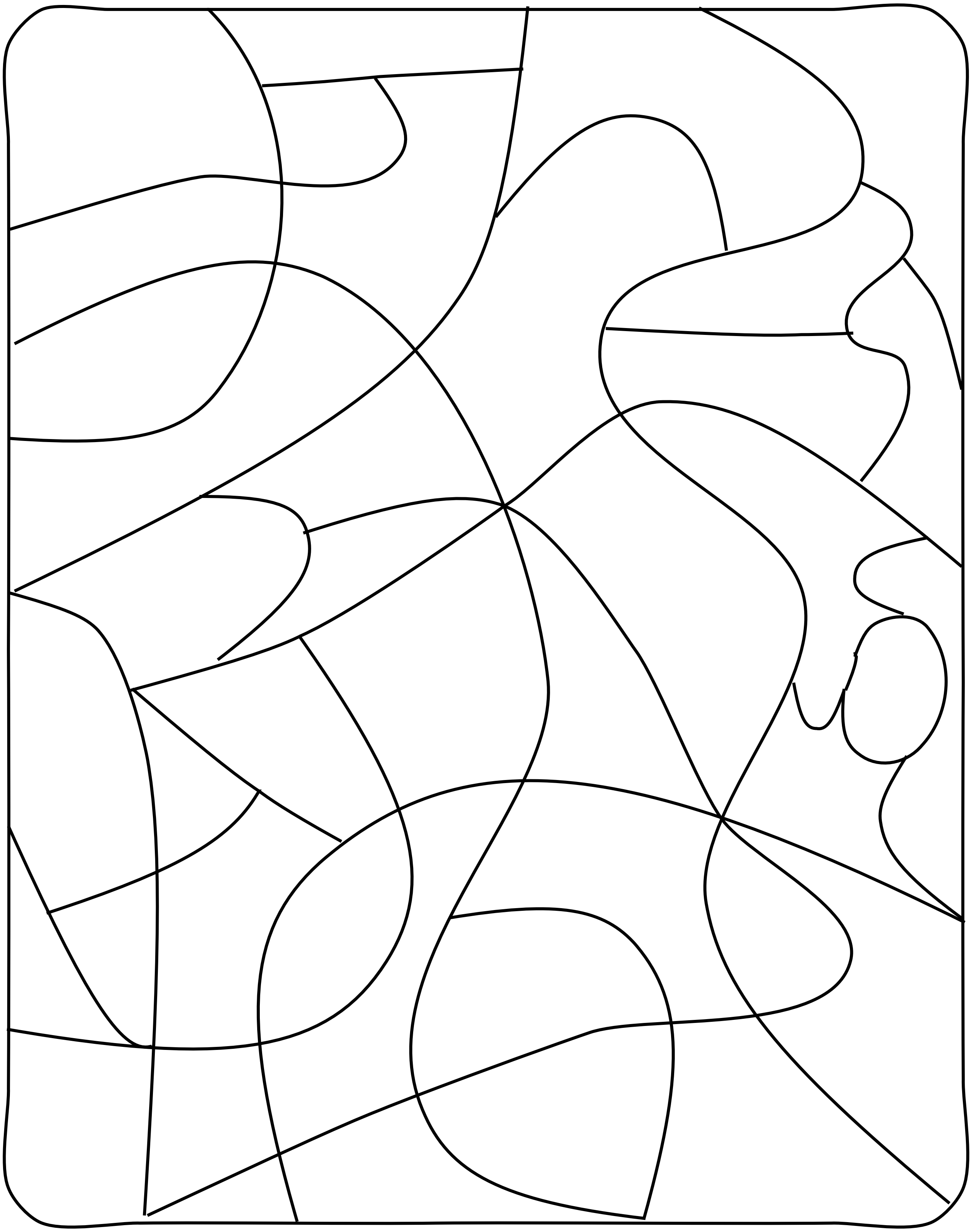


Figure 2.6: A Map to Color

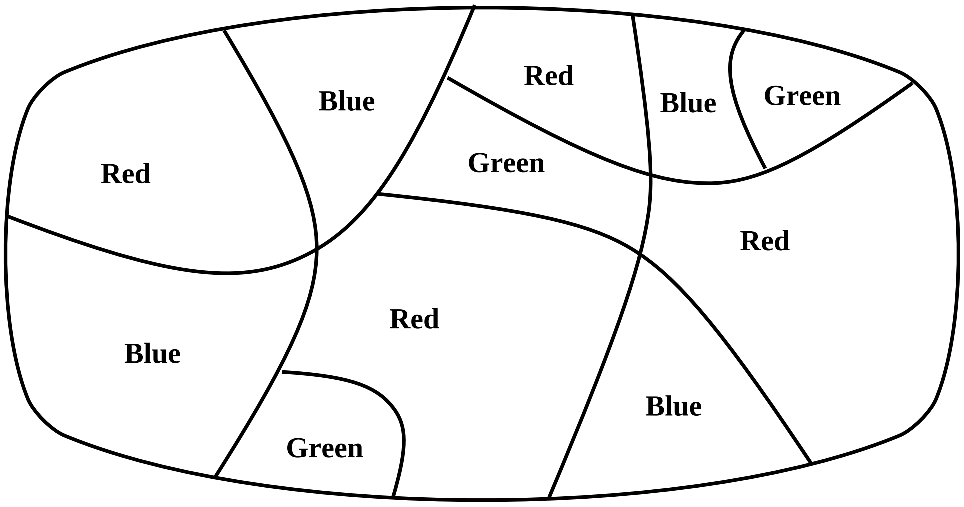


Figure 2.7: The Poor Map Colorer's Map

hours trying to figure out the smallest number of crayons that a map would need, and even to see if might be possible to color a map with only 3 or 4 crayons. This seemed sometimes to be a very hard problem. Can you help?

She colored the map in Figure 2.7 with three colors. Notice that two neighborhoods just touching at a corner can be the same color.

A Television Story

After a while, television came to the kingdom and each neighborhood was given a television transmitter to transmit all of the important news of the neighborhood, and to broadcast the entertainment that the people of the neighborhood liked best. (This was, as you can tell, in the days before Cable.) At first the transmitters had only one frequency or channel — but this made for a problem. The people living near a border sometimes picked up transmissions from their own neighborhood as well as the one over the boundary line. The two signals got mixed up and fuzzy. They couldn't tell *what* they were watching!

It was decided to solve the problem by making enough new channels so that it could be arranged that the channel assigned to each neighborhood could be different from the channel assigned to any bordering neighborhood. To make a new channel required in those days an expensive *tuning crystal*. (They also used special electronic vacuum tubes ... but that's another story.) Any way you looked at it, making new channels was going to be expensive. How many channels do they need?

Discussion

1. Talk about the maps that you colored. How many colors did you use to color each of

them?

2. How did you go about coloring the maps? Write down a description of the strategies that the students used. It might be useful to give

   the strategies names.

3. Are there certain strategies that worked better for one map than they did for another?

4. Is there any strategy that works for all maps? Are there good strategies for determining if a particular number of colors is enough?

5. Can you look at a map and tell for certain how many colors it will need? Can you find a map for which you are sure that it needs 3 colors, or 4 colors? Show students some maps and have them guess how many colors they think it will take. What do you pay attention to when you are making such a guess?

6. What other maps did you make that were interesting to try and color? What made them interesting? Were they easy or hard to do?

7. How is the Television Story like the story of the Poor Map Colorer?

2.3 Making a Two-Colorable Map

Description

It is quite easy to make a map that can be colored with two colors. Students can make these maps and experiment with them to see if they think this is really true. The activity includes a manipulative proof by induction (using colored marking pieces and string) that these maps can indeed be two-colored.

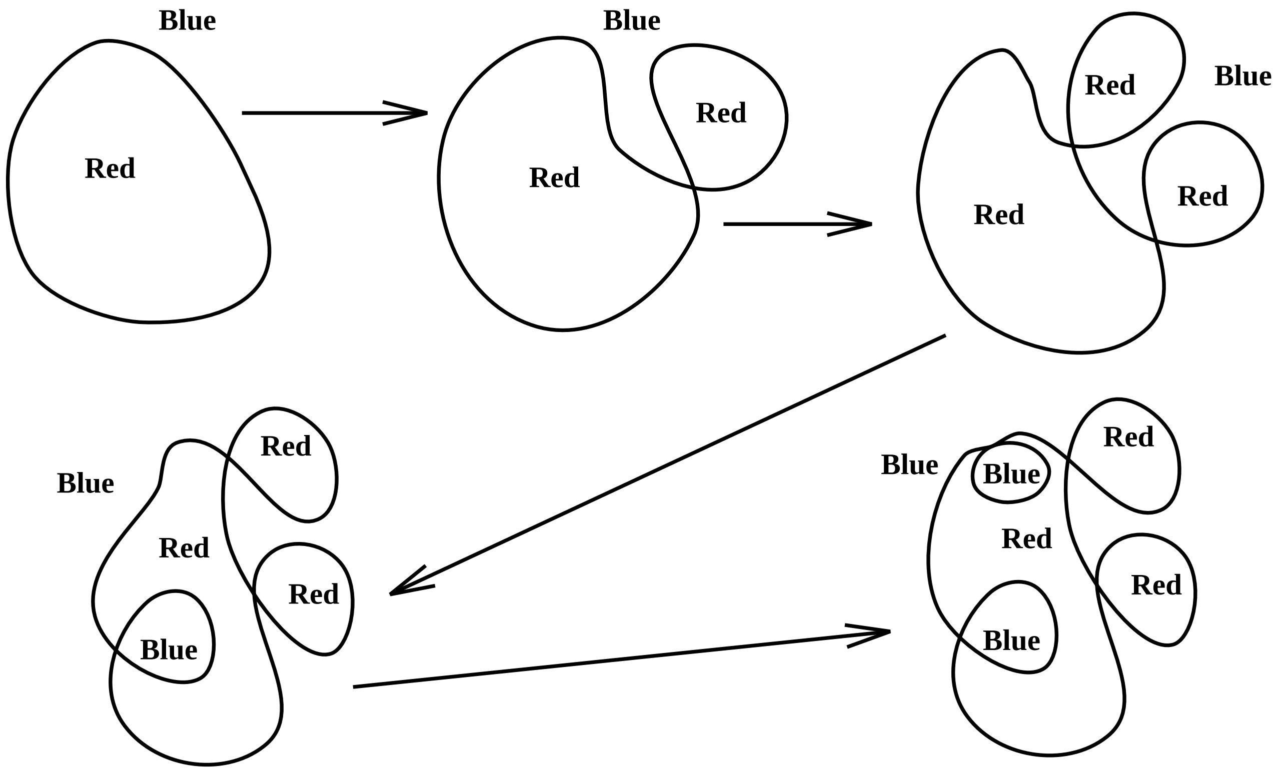


Figure 2.8: Showing that a closed loop can be 2-colored. Begin with a closed loop of string and colored marking pieces to show two different colors for the inside and outside. Then manipulate the string, adding more colored markers as shown in the diagrams. Can you twist the string around in a way that requires you to use a third color?

Materials

- Pencil, paper and coloring tools
- String
- Two colors of marking pieces, such as unifix cubes or poker chips

Instructions

1. Have students draw a map by putting their pencil down on a clean sheet of paper and drawing a long curving line that goes anywhere on the page. The pencil cannot stop or be lifted off the page until it meets the point at which it started. The pencil line can cross itself any number of times and at any point.

2. Tell the students that it has been said that any map that is drawn this way can be colored with two colors. Ask them if they think this is true, and list their reasons why and why not.

3. Have the students work together and experiment with coloring the maps that they have made. They should all be two-colorable. If they find one that is not, check it together very carefully to see that it has been made properly. Be sure that students understand that drawing a map that is so complex it is hard to verify whether it can be two-colored or not is not the same as drawing a map that *cannot* be two-colored.

4. Using a large loop of string and marking pieces of two colors, follow the steps shown in Figure 2.8. Discuss what happens at each step and why the map remains two colorable. It is a good idea to write this down. Is there any possible way to twist and cross the loop of string so that the resulting "map" will not be two-colorable? What is the relationship between the loop of string and the way that the students drew the maps on their paper?

   NOTE: It is important for students to think about these questions, to manipulate the string, and look for a relationship between the string and the maps that they drew. It is *not* important that the students become convinced that the proof is valid (although it is) nor that they understand it in terms of the principle of induction.

5. Figure 2.9 shows the same process step-by-step using two loops of string. Show this to students. Ask them if this shows that if they draw a second closed loop on top of the first one the map will still be two-colorable. Can you always make a two-colorable map by drawing one of these closed loops on top of a two-colorable map?

Discussion

1. Invite the students to talk about what it means to be certain of something. What is the difference between *doubting* that something is true and *knowing* it is false?

2. What makes you certain that you can always make a map that you can color with two colors using this method? What makes you not so sure that this will always work?

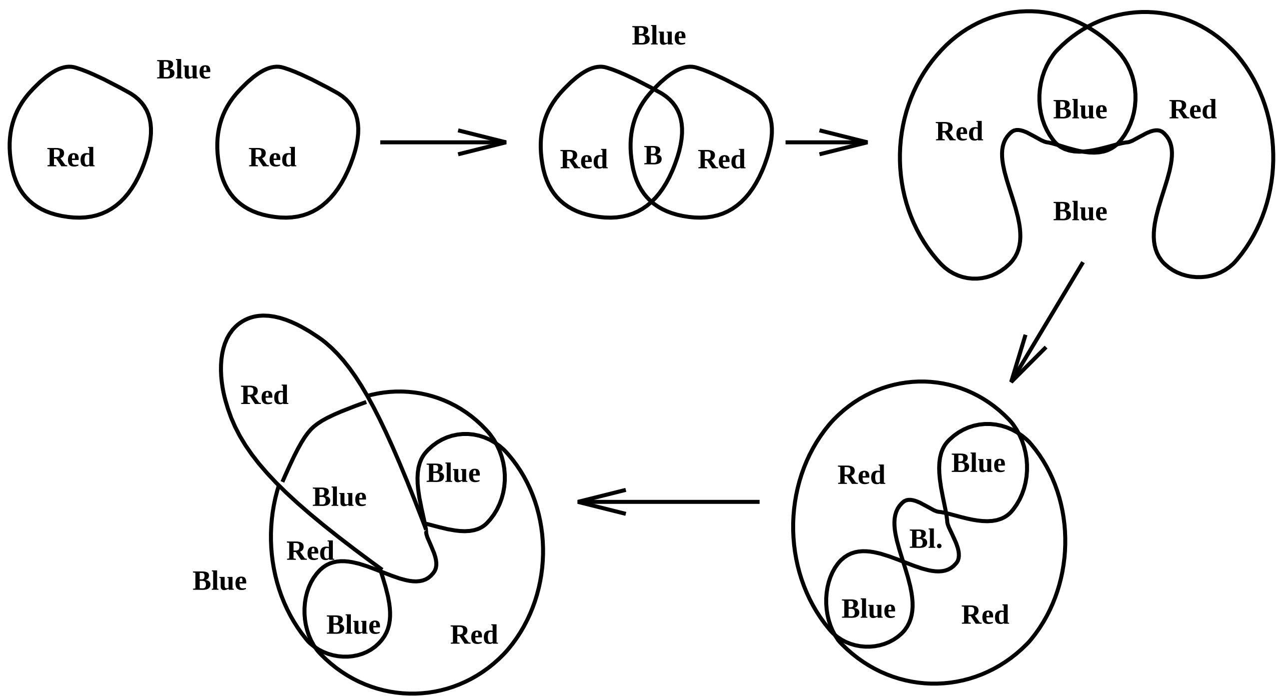


Figure 2.9: Showing that many closed loops will be 2-colorable. Begin with two loops of string and colored marking pieces as shown in the diagram. Overlap the loops and add marking pieces as shown. Try to overlap and twist the strings in such a way as to require 3 colors.

Chapter 3

Untangling the Mathematics of Knots

3.1 Introduction

Overview

Knots have been studied extensively by mathematicians for the last hundred years. Recently the study of knots has proved to be of great interest to theoretical physicists and molecular biologists. One of the most peculiar things about the study of knots is that a category of objects as simple as knots could be so rich in profound mathematical connections.

This section presents a variety of activities which help students explore knots made from pieces of rope. Students will make and verify observations about knots, classify them, combine them, and find ways to determine if two knots are alike. The activities outlined here can be combined to form a single lesson about mathematical knots, or a larger investigative unit extending over a longer period of time. They are roughly sequenced in order of increasing difficulty and challenge, although all of the earlier activities are not strict prerequisites for the later ones.

Finding ways to make precise spoken or written statements about an inherently spatial and manipulative experience is a meaningful and interesting challenge for all students. Teachers can help students learn to do this by helping them develop classroom conventions for naming knots, parts of knots, groups of knots, or for labeling parts of knots to make them easier to talk about.

Although presentations and discussions are appropriate as a whole-class activity when studying knots, most of the other activities work best when students work individually, in pairs and small groups. It is important for each student to be able touch and twist the knots that they are thinking about.

Big Ideas and Key Concepts

• Knot theory • Topology • Operations • Equivalence • Invariance • Identity • Proof

NCTM Curriculum Strands

• Mathematics as Communication • Mathematics as Reasoning • Spatial Sense • Patterns and Relationships • Concept of Number and Number Operations • Probability and Statistics

Vocabulary

• mathematical knot • closed loop • knot addition • prime knot • link • braid • crossing • mirror image • equivalence • open problems • topological equivalence • component

Materials and Preparation

Students will need a variety of knots to manipulate and examine. In the activity **Classroom Menagerie of Mathematical Knots** students who have had some introduction to knots can make the knots that they will use in other investigations. Making knots themselves gives students tactile insight into their structure.

In planning and preparing for activities, the teacher will want to make sample knots beforehand. Large knots for demonstration and display can be made from 2-3 foot lengths of rope that is about 3/4 inch in diameter. Students can work individually or in groups with smaller knots made from 12-inch lengths of rope that is about 1/4 inch in diameter.

The ends of the ropes from which the knots are tied must be joined with tape. Be sure to use tape that is sturdy and will hold the knot together securely, even with much handling. The type of tape you need depends on the material from which the rope is made. Often neither cellophane nor masking tape is adequate and duct tape must be used.

Often the ends of ropes will fray before they can be taped. They should be stabilized by melting or taping ahead of time. It is very frustrating for students to try to reproduce a knot accurately if the rope is decomposing in their hands as they work!

Evaluation

The goal in the study of knots is for students to manipulate various knots, make observations and draw conclusions about them, and understand some of the ideas and questions that come from the branch of mathematics called Knot Theory.

No simple and complete way has yet been found to articulate the complexities of objects as seemingly simple as knots. Students should not be expected to grasp the all intricacies of what mathematicians have discovered about knots or how they describe that knowledge. The goal is for students to understand the variety of questions and problems that can be posed about knots, and to experience some of the problem-solving techniques that mathematicians have tried to unlock their mysteries.

Translating spatial experience into verbal exposition is a challenge for all of us, so it can be difficult and frustrating for students to talk about what they see and understand when studying knots. Do not despair! After all, much of the history of Knot Theory is the story of mathematicians seeking precise ways to describe knots, classify them, and articulate what occurs when they are transformed or combined. Teachers should look for improvement and progress in students' ability to communicate, and use discussion and examples to support it.

If students begin to understand that stating that something is true has special meaning in mathematics, they have made great strides in understanding the subtleties of mathematical truth. It is more important that they understand the *validity* of the question, "How can I be sure this is true?" than that they produce correct mathematical proofs. Just raising this issue is sufficient for students who have not thought about it before. They will not lack experiences for reexamining issues of truth, conjecture, and falsehood in a variety of other contexts. The purpose of raising them here is to create possibilities for analytical thinking and problem solving, not to frustrate or confuse them in their efforts to find things out.

Knots, Links and Other Mathematical Tangles

The mathematical theory of knots originated in the 19th century, but knots have been of interest since ancient times. Knots appear in illuminated manuscripts, sculpture, painting and other art forms from all over the world. As early as human beings used any kind of rope, they probably began inventing knots, and sailors and scouts alike can attest to their variety and usefulness.

The mathematical theory of knots has made major advances in the past decade. One of the most exciting developments has been the discovery of deep connections between knot theory and the branch of physics that studies the fundamental particles and forces that are the building blocks of the universe. It has also been found that DNA is sometimes knotted, and knots may play a role in molecular biology.

Mathematicians envision knots as closed loops, or paths that you can trace round and round with your finger. It is as though the two free ends of tangled rope have been spliced together. When knots are drawn or represented on paper, the places where the rope crosses itself are shown as a broken line and a solid line. The intent is to show that the part of the rope represented by the broken line is passing under the part represented by the solid line.

A knot is a mathematical object, just like number is, and mathematicians ask many of the same questions about knots as they have asked about numbers. One of these questions is,

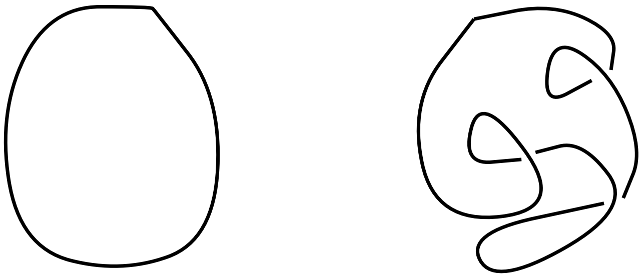


Figure 3.1: The Zero Knot. Notice that the knot on the right can be untangled, without cutting, to look like the knot on the left.

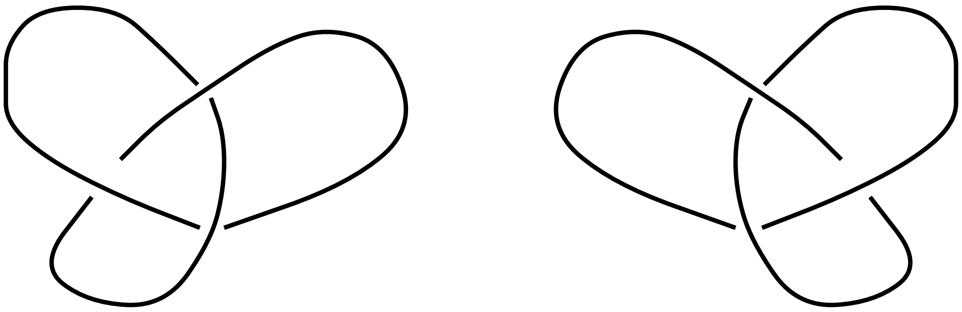


Figure 3.2: Trefoil Knots. Surprisingly, no amount of twisting or deforming (without cutting) of the knot on the left can make it look like the knot on the right.

"Are these two knots equal?"

When two knots seem to be very different, it could very well be that one is just an extra-twisted-up verstion of the other, and that one knot can be transformed into the other by twisting and turning, but without cutting the rope and actually unknotting it. See Figures 3.1 and 3.2.

This is the concept of knot **equivalence**. Knot theorists are still seeking a straightforward and general method for determining whether two knots are equivalent. When students try to determine what it is about two knots that make them the same, even though they may be twisted about so that their forms appear different, they are grappling with the concept of **topological equivalence**. This notion is a very powerful idea that plays many roles throughout mathematics.

What is **knot addition** and how does it work? You can "add" two knots together by making a cut in each one, and, without unknotting, splice the ends together so that each end is joined to an end from the other knot. An example of knot addition is shown in Figure 3.3. Notice that the **Zero Knot** in Figure 3.1 is so named not just because it makes the shape of an "O". The Zero Knot behaves like the number 0. When you add the Zero Knot to another knot, there is a little bit more rope, but the knot itself is unchanged.

What are the basic knot building blocks? Knot addition shows us how two knots can be added together to make a more complex knot. How does this work in reverse? Can you always break a complicated knot into two simpler ones that add together to form it? Of course

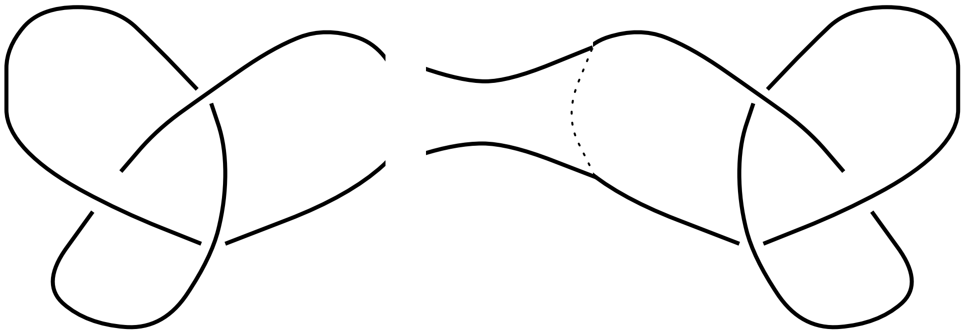


Figure 3.3: Knot addition. With a little bit of surgery, two trefoil knots have been added together.

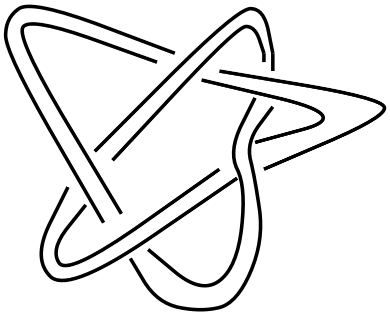


Figure 3.4: Two knots in one! It is called a **link.**

the answer to that question implies that we know what complex and simple knots are! This question is analogous to thinking about prime numbers (2, 3, 5, 7, 11, etc.).

You cannot find two counting numbers that you can multiply together to give you a prime number (except the prime number and 1, of course). All numbers that are not prime can be produced by multiplying together a unique combination of prime numbers. Is the same thing true with knots? Indeed, there are **prime knots**—very many of them—that exist in knot addition. Determining whether a knot is made of smaller building blocks is not always very easy, but it is interesting and challenging to try.

Are there negative knots? When we add the negative number -2 to the number 2 the result is 0. Are there pairs of knots that you can add together and end up with the Zero Knot?

Can knots double up? What do you get, for example, if you begin with a knot, then take a second piece of rope and make a knot out of it that is woven into the first knot? (See Figure 3.4.) The result is what mathematicians call a **link**. Links can be made up of any number of connected knots, and we can ask all the same questions about links as we can about knots–and then we can invent some more. Perhaps this gives you some idea why, although the idea of a knot is very simple, the territory is enormously vast.

Where do **braids** fit in? If you take a braid that is made of three or four, or any number of strands, you can splice the ends in a variety of ways. You can turn it into a knot, or you

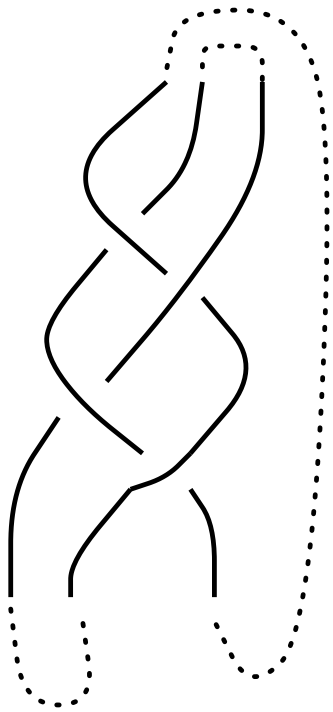


Figure 3.5: A braid. A variety of knots and links can be made from braids by joining their ends.

can turn it into a link made of several intertwined knots. (See Figure 3.5) As usual, questions abound, and the potential for discoveries is great.

Seeking information about knots by experimenting with them and reasoning about them leads to some observations about mathematical proof and the idea of mathematical truth. We know for certain, for example, that the two knots in Figure 3.1 are equivalent. When we defined knot equivalence, we said that two knots were equivalent when one can be transformed into the other without cutting and unknotting. With a little bit of experimentation, we can demonstrate how to do this for these two knots. We could even write up a set of instructions so that someone else could do it, too.

We can try for a long time to rearrange the knots in Figure 3.2 so that one is the same

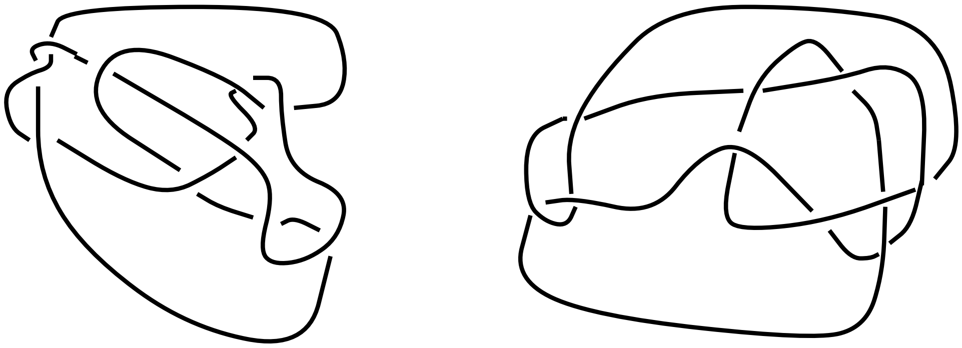


Figure 3.6: Can these two knots be rearranged (without cutting) to look alike?

as the other. How long is long enough? What about the two knots in Figure 3.6? It took 75 years before someone showed that these two knots were equivalent. In mathematics, you can prove that something *can* be done by demonstrating how to do it. You cannot prove that something *cannot* be done by saying, "I tried and tried for a very long time. I tried very hard, but I couldn't do it." Often, it is very tricky to prove mathematically that something can't be done.

Sometimes you can prove that something is impossible by demonstrating how, if it *were* possible, something else that we know is true (because we proved it) would turn out to be false. Other times it is possible to show that you have tried everything, and since there is nothing left to try, what you were trying to do was impossible. Often mathematicians translate problems into other forms so that they can take advantage of what is known in other branches of mathematics to discover a proof. This was the technique that knot theorists used to prove that the two knots in Figure 3.2 are not equivalent. The proof of something which seems so simple, turned out to be very long and advanced.

When children explore knot theory they will be able to demonstrate, after varying amounts of puzzling and experimentation, that various knots are equivalent, that certain knots can be decomposed into simpler forms, and so forth. They will invent questions of their own and perhaps answer them. In certain cases they will undoubtedly be surprised to discover that things which they thought were impossible could be done after all.

The experience of having open questions and unsolved problems mount up faster than closed issues and clear answers might be frustrating or uncomfortable at first. But this has always been the situation in mathematics (and science generally). Although Discoveries, solutions, and answers abound, they are rare in relation to the number of things that are not known. This is why discovering something new or solving a problem is a reason to be pleased and excited.

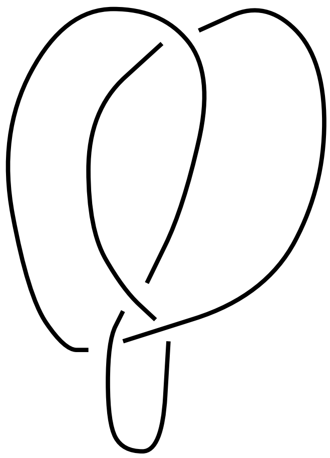


Figure 3.7: The figure-eight knot

3.2 Knot Observations

Description

In this activity, students become familiar with mathematical knots by observing them. By putting these observations into words they gain experience in using language to describe spatial properties. They will undoubtedly discover many basic properties of knots.

Materials

A set of about five large knots that will be the object of the students' study. These can be any knots that you find interesting and that will stimulate a variety of observations on the part of the students. The following knots would be suitable:

- the zero knot (Figure 3.1)
- right- and left-handed trefoil knots (Figure 3.2)
- the figure-eight knot (Figure 3.7)
- either or both of the knots in Figure 3.6

Make the knots from 2-3 foot pieces of rope that is about 3/4 inches in diameter. Be sure they are sturdy and can withstand handling.

Instructions

1. Have students brainstorm about what they already know about knots, What is a knot? Where and how they are used? How they are made?

2. Tell the students that knots are important to mathematicians, too. By studying knots, mathematicians have developed theories that are useful in many areas of science.

3. Show the students the knots that they will initially study. Point out that these are **mathematical knots**. The knots that mathematicians think about have their ends joined to form a closed loop. One reason for doing this is to be able to move the knots all around, pulling and twisting on the loops to see what happens, and be sure that the knots don't come apart.

4. Explain that one of the things that mathematicians do when they study something like knots is to look at a few of them and begin making clear and precise observations about them. At first they notice the simplest things. They try to describe what they notice as clearly as they can so that when they talk to each other about them, each one understands what the other one means and they don't get confused. Often mathematicians who live far away from each other study the same things, and they write each other letters and talk over the phone. So they have to be sure when they describe something that the other person gets the right picture.

5. Explain that another reason mathematicians make and collect observations is that it often turns out that when you list the simple and easy things, other things begin occurring to you, and you notice things that are more complicated and interesting.

6. Have the students begin making observations about knots. This can be done either with the large group, or you can ask the students to work in smaller groups that rotate among the knots. In small groups, students should collect and write down their observations and then report back to the larger group. Make sure the students understand that no observation is too silly or too small. If they have trouble saying exactly what they mean, have them help one another to figure out the best way to put it in words. If they need to, it is okay to invent new words to make something clear, as long as everyone understands what the new word means and they can explain it to anyone else.

7. While the students are making their observations, it is important that they be able to touch and manipulate the knots as well as look at them.

Discussion

1. List student observations. Verify with the group that they perceive what is being described.

2. Combine observations that are similar or redundant. Have students agree on the wording.

3. Students may touch on some or all of the following:

- The closed-loop form of mathematical knots.
- How the Zero Knot isn't very "knotty."
- How the two trefoil knots are very similar.
- How a knot can be tangled to look different, then returned to its former shape.
- Ways to distinguish differences between knots.
- How the "knottiness" of a knot might be measured.

4. The above list is not exhaustive of the kinds of observations that students can make. Save their list, refer to it often when studying knots, and add new items to it as they are discovered.

5. If there is not agreement about some points, put question marks next to them.

6. When you finish up the activity, help the students to see that they learned very many things about knots by using their own minds and senses. They haven't studied any information from the outside yet.

7. Help the students to begin generating and listing questions about knots that they could investigate. Explain that when we take a close look at something that we have never examined carefully before, not only do we see new things, but we also start wondering about things that we never thought of before. After looking at knots, what are some questions that come into your mind?

3.3 Classroom Menagerie of Mathematical Knots

labelknot:menag

Description

By making a variety of knots students will:

- Experience the structure of knots.
- Explore the relationship between knot diagrams and physical knots.
- Become adept at making physical knots from diagrams so that they can quickly and accurately produce knots that they want to experiment with in other activities.

At the end of the activity, each student or small group of students will have many knots to examine and work with while doing other activities with them.

Materials

- For each student or each group:
  - 9-12 inch sections of 1/8" rope, prepared so that the ends will not fray.
  - Knot diagrams for the knots that they will make (Figure 3.8).
  - Tape that will hold the end of the knots securely.
- One large set of the knots that students will make for reference and display.

Instructions

1. If this is the students' first experience with knots, give them some background about the mathematical study of knots. Show how mathematical knots are closed loops with the ends tied together.
2. Explain that each student or group will be making several knots. By making knots yourself, you will be able to understand a lot of things about the structure of knots. Then we will all have quite a few knots to study and experiment with.
3. Have the students look at the knot in the upper left hand corner of the knot diagram (Figure 3.8), a **trefoil** knot. Draw this knot on the board and show students how the broken line in the diagram means that the strand of rope goes under the other strand when the two cross.

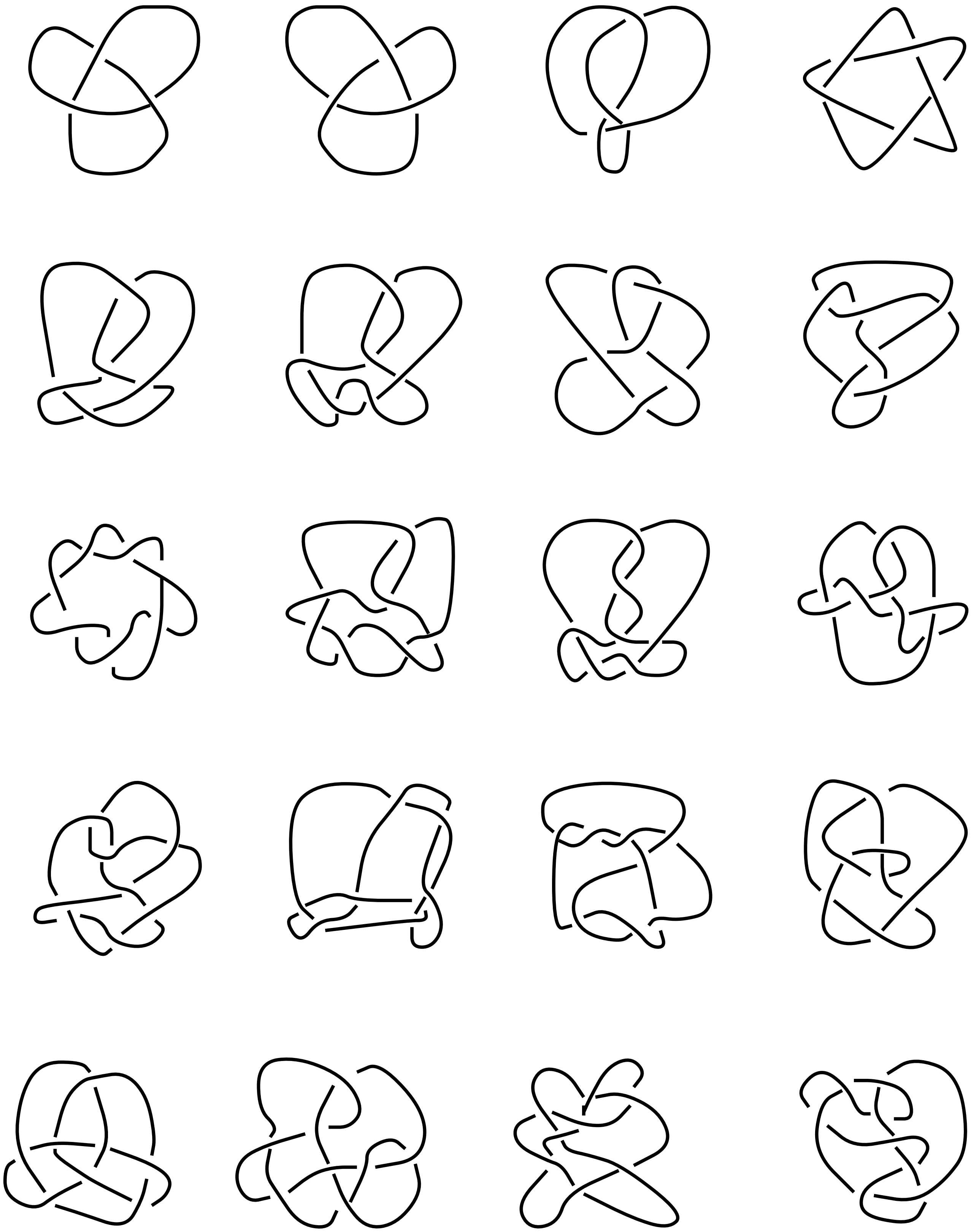


Figure 3.8: Diagrams of many knots to study and make. The broken lines indicate that the rope should cross under.

4. Using a large strand of rope, demonstrate to the students how to form a trefoil knot according to the diagram. It is easiest to make knots from diagrams on a flat surface, rather than holding them in your hand where they can flop around. Use a slanted board to demonstrate to the whole class, or form the knots on the table, the floor, or other flat surface and have the students gather round and watch. Show how you fasten the ends of the knot together with tape so that the closure is secure and smooth.

5. Have the students make the trefoil knot. Help them verify that the knots have been made correctly. Be especially careful that the students have not made the other trefoil knot—the one to the right of it in the knot diagram—by mistake. Encourage students to help one another, to explain the method or techniques that they used to make the knot and to check that it was correct.

6. When students understand how to make knots from the diagrams, tell them which other knots they should make. Choose four or five knots that everyone will make. Then allow students make two or three other knots of their own choosing from the sheet.

7. It is very important that the knots be produced exactly as they are in the diagrams. (If the knots aren't quite right now, this will pose a problem later when students will combine their knot collections and examine knots which are identical.) As the number of crossings increases, it is easy to make mistakes. Have students pair up and compare the knots that they have made to be sure they are indeed identical. If the knots appear different, have them study the knots and the diagram carefully to determine what the problem is. When one group has completed a set of knots, they should ask members of another group to verify that they have made the knots exactly as they are in the diagrams.

8. When the knots are finished and verified for accuracy, have students label them with their names. Also, be sure the knots are strong, and not likely to pull apart, because they will receive a lot of use in further activities.

Discussion

1. Debrief the activity with questions similar to the following:

    - How did you figure out the way to make the knot from the diagram?
    - The diagram doesn't show where the tape goes, is that a problem?
    - What techniques did you find most useful in putting the knot together?
    - What problems did you have when making the knots? How did you solve them?
    - How can someone look at your knots and know which knots from the sheet you had made?

2. Some of the knots, such as the right- and left-handed trefoil and the figure-eight knots have names. Students may want to name their other knots.

3.4 Are They the Same—or Knot?

Description

Students will work with a variety of knots and explore the idea of **equivalence**. They will understand that knots are not changed when they are pulled, wrenched or otherwise deformed. They will manipulatively experience the idea of equivalence.

Materials

For each student or group of students:

- Several copies of a variety of different knots.
- The sheet of knot diagrams (Figure 3.8).

NOTE: Students can use the knots that they made in the activity **A Classroom Menagerie of Mathematical Knots**.

Instructions

1. Explain to the students that they will be asking a question about knots that is a fundamental question which mathematicians are always asking about the things that they study. That question is: How can I tell if two of them are the same or different?

2. Group students and knots together so that each pair of students or small group has a batch of knots consisting of pairs of identical knots. Have the students pick out three pairs of identical knots that they want to work with.

3. Have the students begin by getting their knots as mixed up as they can. Pull the loops out, wrap them around, twist them, tuck them under—do as many things as they can to their knots to change their form, being careful, of course, not to break the knots apart.

4. When the knots have been well-deformed, have each group of students give their knots to a different group.

5. When the students receive their new group of knots, they must sort them back out, grouping the identical knots together. Explain that the way to show that two knots are the same is to take one of the knots and move the loops around until it is identical to the other one.

6. Have the students refer to the sheet of knot diagrams and identify the knots that they received.

7. Students will probably not be short of ideas for continuing and modifying this game. They may want to make more knots if it becomes too easy. Some suggestions are:

- Find out which of three knots is the “odd” one
- Sort out a collection of pairs of equivalent knots
- Verify if two knots are equivalent blindfolded
- Play 20 questions. A student selects a knot and hides it from the others. Students ask yes/no questions about its structure to try and determine which knot it is.

Discussion

1. Have students describe the techniques they used for determining if two knots were the same. List them. Explain the idea of an **invariant**—something that will be true about a knot regardless of how the knot has been deformed. Do any of their techniques for determining if two knots are the same rely on the concept of invariance?

2. Have students describe the ways that they could be sure that two knots were not the same. List them.

3. What techniques were reliable in finding possibilities for deforming knots? List them.

4. Two particularly tricky knots are the right- and left-handed trefoil knots which are not equivalent. Students should be allowed plenty of time to experiment with these to see if they believe it.

5. The right- and left-handed trefoil knots are called **mirror images** of one another because (as students can verify with the aid of a mirror) you can make either one of them look exactly like the other by looking at it in a mirror. Study the mirror images of other knots. It is not always the case that a knot and its mirror image are not equivalent (as is the case with the trefoil knots). Some knots can be transformed into their mirror images without untying. Select a group of knots and divide them into two categories: those which are equivalent to their mirror image and those which are not.

6. Encourage students to talk about what it takes for them to be satisfied that two knots aren't equivalent. Help them to make distinctions between being reasonably sure and absolutely certain. Explain how this fits into the context of mathematical proof.

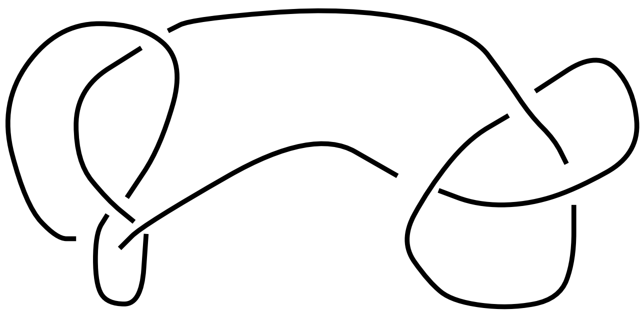


Figure 3.9: Knot addition. With a little surgery, two knots can be added. Compare this to figure 3.10.

3.5 Knot Arithmetic

Description

In this activity students will combine knots to see how they can be added together. They will manipulatively experience the concept of **identity** and explore some of the subtleties of the nature of mathematical truth.

Materials

- Knots from previous activities.

  NOTE: Since adding two knots together involves cutting the original knots, and "transforming" them into a third, students may want to make new knots for this activity so that they can still have the ones they have already made as samples.

- Ropes.

- Paper and pencil.

Instructions

1. Draw the knot in Figure 3.9 on the chalk board. Explain that knots are added together by doing a little bit of surgery. Each of the two knots that is added is cut, then carefully, without untying, each of the ends of rope from the first knot is attached to ends of rope from the second knot.

2. Help the students to generate questions about adding knots. Some possible questions are:

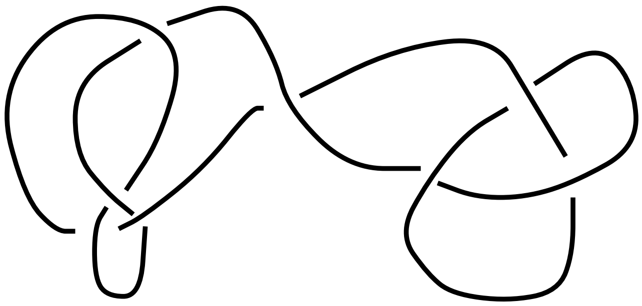


Figure 3.10: Knot addition with a twist. Does it make a different knot if you cross the strands before joining them? Compare this to Figure 3.9.

- Does it matter how you join the two ends of rope when you add two knots? For example, is the knot produced by the addition in Figure 3.10 the same as the addition produced in Figure 3.9?
- Can we make some of the knots on the sheet of knot diagrams by adding knots?
- Can we make the knots that we are most familiar with?
- Is there any knot that will be different when we add the zero knot to it?
- If we switch the position of the two knots (i.e., put the left one on the right and the right one on the left) how does that affect the result?
- If you add three knots together, does the order in which they are added affect the result?
- When we add two knots together, do we get a different result if we cut a different part of the knot when we do the surgery?
- When we add two knots together, will we get a different result if we use the **mirror image** of one of the knots instead?
- What if we use the mirror image of both knots?
- What happens if we flip one of the knots over when we add them a second time?
- What if we flip both knots over?
- Are there two knots that you can add together whose result will be the zero knot?
- Can you add a knot to the left- hand trefoil knot to turn it into the right-hand trefoil knot? What about other mirror image knots?
- If this is knot addition, what about knot subtraction, knot multiplication, and knot division? Could there be knot fractions? negative knots? odd knots and even knots? prime knots?

3. After the students have generated a list of interesting questions about knot addition, discuss how you could go about answering one of the questions. A suitable plan might look like the following:

    - Clearly state the question.
    - Decide which knots will be used for the investigation.

- Plan how to keep track of information, such as making lists and tables.
- Draw the knots that will be added.
- Draw the way in which they will be added (like Figure 3.9).
- Draw equivalent knots of the knot sum.
- Identify related questions. (Maybe someone else is working on them.)
- Locate other people doing similar work. Compare and discuss results.

4. Have the students work individually or in groups to answer one or more of the questions that they generated.

Discussion

1. Have the students present to the rest of the class information about the question(s) that they investigated, how they went about their investigation, and what they found out.

2. Encourage students to be active listeners to the other students presentations. Point out how such simple objects as knots become complicated once we start comparing them, adding them, and doing other things with them to understand more about them. It is easy to make mistakes, or to get half way through what you were explaining and find out you have confused yourself. This happens to mathematicians all of the time.

3. Explain that mathematicians rely on one another to listen carefully to their ideas and help them figure out if they have done as good a job as they possibly can when they investigate a question. When mathematicians present their discoveries to each other, they aren't thinking, "Oh, I hope they will be impressed by how smart I am to have thought about all this complicated stuff." Their attitude is very different. They are thinking, "I hope they will listen carefully to what I am explaining. When they don't understand, they will ask me questions so that I can help explain it better. If they think I have made a mistake, they will tell me. I want to make sure that this is all done right."

4. Encourage students to think about the **mathematical truth** of their conclusions. A **conjecture** is a statement that is not only plausible, but is borne out by much experimentation with examples. It turns out that without advanced mathematical techniques, students will not be able to demonstrate the mathematical truth of many of their conjectures. There is no shame in this. There exist today many conjectures in all fields of mathematics, some of which are hundreds of years old. Occasionally, when new mathematical techniques are developed, mathematicians can apply them to conjectures that have been made and either prove or disprove them.

3.6 Building Block Knots

Description

In this activity, students continue their experience with knots and knot addition by thinking about how the process of knot addition might work in reverse. Students will try to decompose knots into two simpler parts that can be added together to make the original knot. Students will be exposed to the notion of **prime knot**.

Materials

- Information collected when doing knot addition—tabulations, conjectures, etc.
- 24-inch lengths of rope 1/4 inch in diameter for making knot decomposition puzzles. (One per group of students.)
- Knot samples
- Pencil and paper

Instructions

1. Review the idea of knot addition. Ask students what will happen if this is done backwards. In other words, is it possible to take a knot apart or decompose it into two knots which can be added together to produce the original knot?
2. Have each student or group of students choose a knot to decompose and talk about what happens. It seems that some knots can be broken down into smaller components and others cannot.
3. Ask students to work in pairs or small groups to make knot decomposition puzzles. Have students draw two knots on paper and add them. Make the knot that results out of rope. It is important to do this carefully, because the slightest error might result in a knot that won't decompose. Give the knotted rope to another student or group of students. Their job is to find the two knots that the knot they receive is composed of. This may actually turn out to be quite difficult. Any knots that cannot be decomposed during the class period can stand as **open problems** in the classroom. Open problems are the lifeblood of mathematics.

   NOTE: You may want to place some restrictions on the size (i.e., number of crossings) of the knots that are used to make these puzzles so that the knots to be decomposed aren't outlandishly complex.
4. The knot decomposition puzzles (above) can be made even more challenging if more than two knots can be summed to make the puzzle.

5. If students try to decompose the knots on the knot diagram into smaller knots, they will meet with quite a bit of difficulty. All of these knots are believed to be **prime knots**—knots that cannot be decomposed into two simpler knots (unless, of course, one of them is the zero knot). This has not be conclusively proved, however. After much experimentation with the aid of computers, no one has succeeded in decomposing these knots. Recall, however, that the knots in Figure 3.10 were not known to be equivalent for 75 years, so it is possible that a decomposition for one of the knots in Figure 3.8 is yet to be discovered. If a student should discover one, contact the authors of this book right away!

Discussion

1. What do you look for in a knot to see if it will decompose? Are there any sure signs that it will? that it won't? If you think it will decompose, how do you go about trying?

2. The idea of **basic building blocks**—irreducible units that can be put together in a variety of ways to produce more complex entities—is one that runs all though mathematics and science. In fact, this notion is fundamental to the way that we think about and understand the world. Identification of the ultimate building blocks of the universe is one of the quests of particle physics. Amino acids are the building blocks of the proteins in all living things. The DNA molecules of which our genes and chromosomes are made are built up from 4 basic building blocks. Chemists study molecules made of atoms, as well as how certain clusters of atoms are intermediate size building blocks for complex molecules. Students may be familiar with the concept of **prime number**, and how prime numbers are the building blocks of multiplication.

3. Have students identify other areas of mathematics and science where they have experienced the idea of building blocks.

3.7 Knots in the World

Description

Students examine and learn how to tie some practical knots that are used for various purposes. This gives them further exposure to the structure of knots and provides a new context in which to apply their skills of observing and making conjectures about knots.

Materials

- Information on knot-tying for useful purposes. Such books, written at a level accessible to children, are readily available. Most libraries have them.
- Rope.

Instructions

1. Remind the students that knots have been useful in the world for years.
2. Review some of the purposes and uses of knots. Many books about knot tying classify knots according to some scheme based on their type and purpose (i.e., tying to a post, shortening a rope, etc.)
3. Have students examine and tie several knots and relate this experience to their mathematical study of knots.

Discussion

1. What are some of the differences between mathematical knots and practical knots? How are they the same?
2. Knots can be classified by the ways in which they are used for practical purposes. Do these classifications match the ways in which you have classified knots mathematically?
3. Can you find knots in the sheet of knot diagrams (Figure 3.8) that appear in the diagrams for practical knots?
4. Using several practical knots as a starting point, ask some of the questions you have asked about mathematical knots. Are some knots equivalent? Can they be decomposed? What happens when you add them?

3.8 Braids and Links

Description

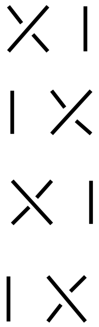
The mathematical study of knots includes the study of **braids**. Students will braid strands of rope and make these braids into knots. They will experience some of the questions, problems, and ideas that other mathematicians have encountered while studying braids.

Materials

- For each student:
  - Ropes.
  - Braid Sheet (Figure 3.11).
- Optional: Large braids for demonstration and display.

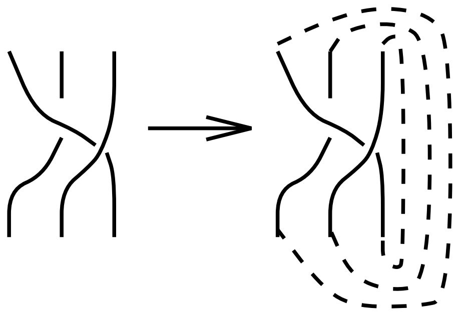
Instructions

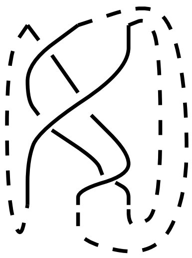
1. Introduce the idea of a **braid**, and connect it to what students already know about braids.
2. Referring to the Braid Sheet (Figure 3.11), show students how braids are built up from their components. Notice that there are four possible ways to cross three strands. These are called the **components** for building braids. An infinite number of braids can be made from these components, not just the familiar one that we know for braiding three strands of hair.
3. Point out on the Braid Sheet how a finished braid is turned into a knot by joining the ends. Notice, however, that you can join the ends in several different ways, and not all of them produce the kinds of knots that were studied in other activities. When two or more closed loops of rope are tangled and knotted, the resulting object is called a **link**. Counting the number of closed loops in the link is a good way to begin classifying links. The number of closed loops is called the **link number**.
4. Experiment with some three-strand braids and keep track of the results. The possibilities for interesting questions multiply fast. Some ideas include:
   - What are all the possible braids that can be made from two components? three? four?
   - Using the four components for braiding three strands, what familiar knots can you make?
   - There are no knots which, when added together will cancel each other out, leaving only the zero knot. Is the same thing true for braid components?



There are four possible ways to cross three strands of rope to make a braid. How many different ways can you join the ends to make knots or links?

The ends of this braid have been joined. Is it now a knot or a link?





What familiar knot does this braid make?

Mathematicians have shown that any knot can be rearranged into a "closed circular braid". Try it.

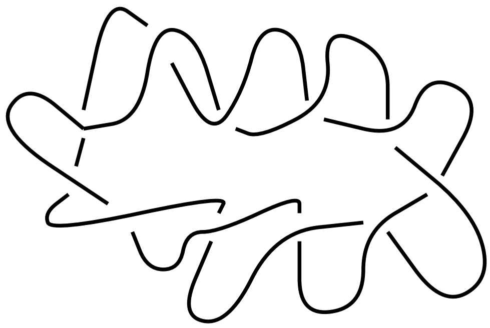


Figure 3.11: Analyzing how braids are made can help us figure out a way to tell them apart.

5. Ask students to figure out how many components they can use to build braids made of four strands. Using these components, experiment with braids that have four strands. What about more strands?

6. **Every knot is a closed circular braid.** This is a famous and surprising theorem of Knot Theory. It means that no matter how much a knot appears to be tangled, looped and twisted, it can be rearranged as a closed circular braid made of some number of strands. (See Figure 3.11.) Have students experiment with some knots to see if they can always rearrange them into closed circular braids. How might this be useful for writing down descriptions of knots?

Discussion

1. Comment on what the idea of braids adds to the study of knots.

2. Have students present the results of their investigations with braids to one another.

3. Does the idea of every knot being a closed circular braid help in the classification of knots?

3.9 Seifert Surfaces

Description

Using knot shapes made out of wire, students will see what happens when they dip them into bubble-making solutions. They will experience the idea of a **surface**, and they will experience how the study of knots can be taken in amazing directions.

Materials

- Trefoil knots (the knots in Figure 3.2 made out of wire that will be dipped to make bubbles.
- Pans or trays of detergent or other bubble solution. (Adding a little glycerin to ordinary dish detergent makes the bubbles more sturdy.)
- Additional wire for making other knot-shaped bubble-makers.

Instructions

1. Have students dip trefoil knots into the bubble solution and examine carefully the way the soap film clings to the wire. Ask students to describe what they see.
2. The form of the stretched and twisted soap film is called a **Seifert Surface**. Notice that it has two sides, a "front" and a "back". When you hold the surface up, which parts face you? You can draw an accurate representation of the surface by using two colors, one for each side.
3. Ask students to choose another knot that they would like to twist from the wire and examine the soap film that it will make. Before dipping the knot into the bubble mixture, have students predict the surface that they think will form.
4. Experiment with other knot-shaped bubble makers and catalog the surfaces that form.

Discussion

1. Describe what happens when various knots are dipped. Why do you think this occurs?
2. Does the same surface always form when knots are dipped?
3. How can you account for the things that you observe?

Chapter 4

Games on Graphs

4.1 Introduction

Overview

Graphs are mathematical objects that are made of dots connected by lines. Graph Theory is the branch of mathematics that involves the study of graphs. Graphs are very powerful tools for creating mathematical models of a wide variety of situations. Graph theory has been instrumental for analyzing and solving problems in areas as diverse as computer network design, urban planning, and molecular biology. Graph theory has been used to find the best way to route and schedule airplanes and invent a secret code that no one can crack.

Big Ideas and Key Concepts

• Graphs • Properties of mathematical objects • Modeling • Abstraction

NCTM Curriculum Strands

• Mathematics as Problem Solving • Mathematics as Communication • Mathematics as Reasoning • Mathematical Connections • Concepts of Whole Number Operations • Spatial Sense • Patterns and Relationships

Vocabulary

• graph • vertex • edge • degree • distance • diameter • mathematical object • property • isomorphic

Materials and Preparation

- Black line masters of graphs for various activities.
- Markers of some kind to use as game pieces.
- Any graph game board can be drawn on the classroom floor with tape and students can move about the graph themselves instead of moving game pieces. These games can also be played outdoors on graphs that are painted on the pavement or drawn with chalk.

Evaluation

The goal of these activities is to expose students to graphs and let them experience the kinds of problem-solving that is done by mathematicians who study graphs. They will have an opportunity to communicate ideas about patterns and relationships that they see in mathematical objects that are not numerical. Playing the games will allow them to develop strategies that involve thinking several steps ahead and considering multiple activities in various parts of a system.

No mathematician ever considers these skills entirely mastered; they work all their lives to develop them further. In evaluating students' performance, teachers should look for progress along a continuum. Students should be encouraged to participate in their own evaluation and recognize the progress that they make. When students recognize that these are skills that can be developed, they learn to become responsible for their own progress. One of the most valuable things that teachers can do is help students put their insights into both written and spoken language.

It is not necessary for students to learn the names of many graph properties, nor to learn which graphs have which properties. When students understand what a graph is, and that the strategies that you use to play the same game will differ depending on the graph that you play it on, they have learned very much about the nature and structure of graphs and the ways that mathematicians think about them.

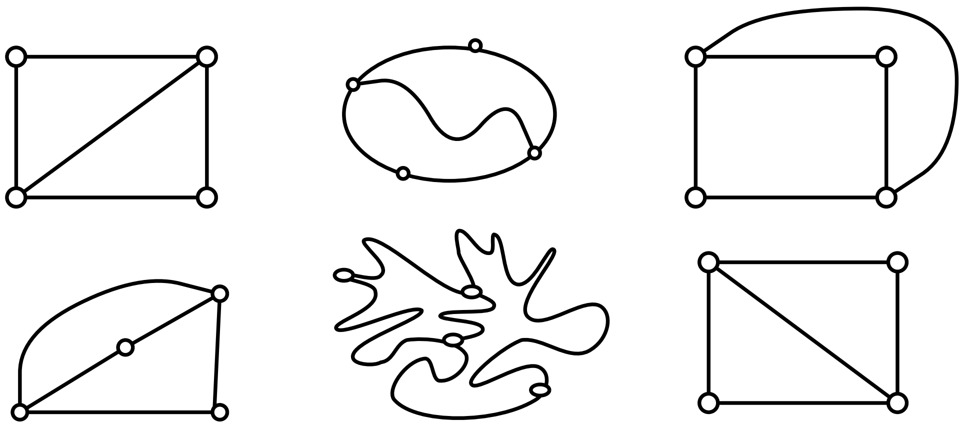


Figure 4.1: Isomorphic Graphs. No matter how the edges of a graph are stretched, shrunk, pulled or deformed, if the way that the edges are connected to the vertices does not change, it is the same graph.

The Mathematics of Graphs and their Games

Graph: a confusing term

When mathematicians talk about **graphs**, they are most likely to be thinking of the collections of dots and lines that you see in the illustrations of this section. Sometimes graphs are called **networks**, and a glance at the pictures in this section shows you why. These graphs are not the same as the graphs which are used to plot or chart statistical information. The potential for confusion between these two very different kinds of objects that have the same name is unfortunate. When mathematicians talk about graphs, it is always clear from the context which type of graph they are considering.

The Vocabulary of Graphs

Of course there is a vocabulary of terms for talking about graphs. The following discussion of graph terminology is not a prerequisite for understanding graphs, and in fact students will quickly find graphs quite boring if they have to learn all these terms before they can do anything interesting with graphs. The terms become quite useful when students try to talk about their observations or their strategies for playing games. Familiarize yourself with these terms so that when students find themselves using round-about awkward constructions to try to explain something, you can supply word that are useful to describe it.

Each dot is called a **vertex**. The plural of vertex is **vertices**. (Although many excited young students experiencing graph games and puzzles for the first time use the word *verticie* as the singular of vertices!) The lines that connect the vertices are called **edges**. Edges can connect any or all of the vertices. Edges in a graph can cross one another without there being a vertex at the place where they cross.

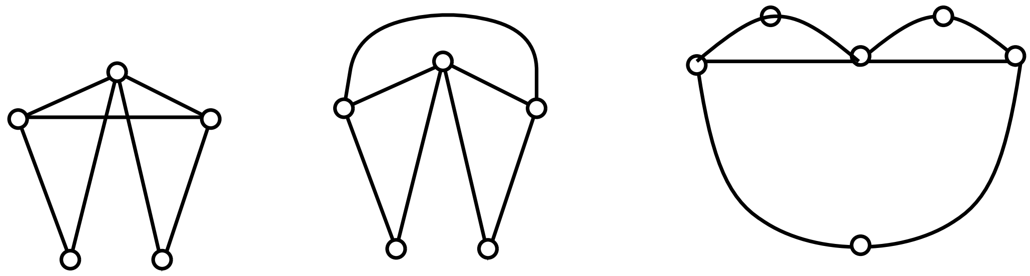


Figure 4.2: More Isomorphic Graphs.

The edges of a graph can be stretched, shrunk, pulled out of place, or deformed in any imaginable way and it is still the same graph. Therefore, all of the drawings in Figure 4.1 are pictures of the same graph. Graphs such as these, which are all representations of the same graph, are said to be **isomorphic**.

Sometimes it is not so easy to tell that two graphs are isomorphic. It takes quite a bit of moving edges around to see that the graphs in Figure 4.2 are isomorphic. In fact, no one knows a reliable way to quickly determine if two graphs, no matter how large they are, are isomorphic—although it is fun to try.

Mathematical Objects and their Properties

A graph is an example of a **mathematical object**. A number is the mathematical object that is most familiar to everyone. Other mathematical objects you will encounter in this book are knots, maps, and finite state machines.

One way mathematicians try to understand objects is in terms of their **properties**. The number 6, for instance, is a positive, even number divisible by 3 and less than 10. All these things are properties of the number 6. We could list many more. Depending on what you are trying to find out, some properties of the object you are looking at are more useful than others.

Some Properties of Graphs

Three properties of graphs that are important in many problems are **size**, **regularity**, and **diameter**. The **size** of a graph is the number of vertices it has. For example, the size of all the graphs in 4.1 is 4.

A graph is **regular** if all of the vertices have the same **degree**. The **degree** of a vertex is the number of edges that touch it. Each of the graphs in Figure 4.3 is regular. Mathematicians would use the term 3-regular to describe the graph in Figure 4.3 because its vertices all have three edges touching them. Figure 4.3 also has graphs that are 4-regular and 5-regular.

Just like the diameter of a circle measures the distance across a circle, **diameter** is

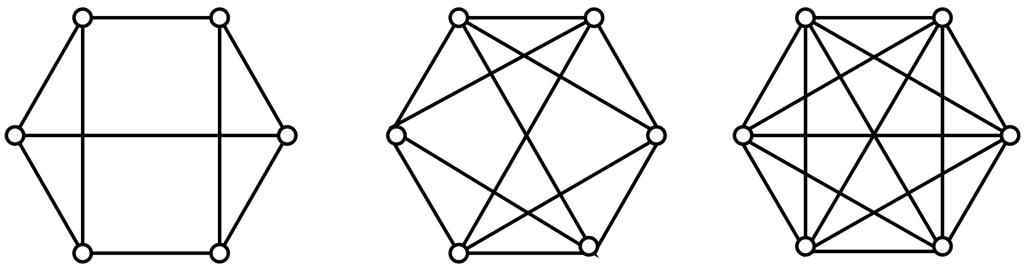


Figure 4.3: Regular graphs. From left to right, these graphs are 3-regular, 4-regular, and 5-regular.

a measure of **distance** between vertices in a graph. The idea of "distance" in a graph is different from the "distance" we commonly measure with a tape measure, however, because we can stretch or shrink the edges to be any length at all. Imagine traveling from vertex to vertex along the edges of a graph. Each trip along one edge, no matter how long or short the edge happens to be, counts as one move. The **distance** from one vertex to another is the smallest number of moves that it takes to get there.

The diameter of a circle measures the distance between two points on the circle that are farthest apart. Similarly, the diameter of a graph is the distance between the two vertices of the graph that are farthest apart. So the diameter of the graphs in Figure 4.1 is 2.

A Real-World Problem that Uses Graphs

The connecting dots and lines of graphs make the ideas in graph theory particularly useful to people who design computer systems. On the smallest end of the scale, a graph can be used to model the way that tiny pulses of electricity flow through the silicon chips that are built into electronic devices. In the big picture, a graph can model the ways that computing systems can be interconnected, even when the computers are located all over the world and connected by telephone wires and satellites. When graphs are used to model circuits for silicon chips or large computer systems, properties like size, regularity, and distance or diameter are important.

For example, imagine that you have been given the task of deciding how to link 12 computers together. Each one of these computers will use information that is generated by all the others, so that all of them will need to exchange information with each other. Unless each computer can be connected directly to all eleven of the other ones, sometimes the computers will have to interrupt the work that they are supposed to be doing just to relay messages. It would be nice if every computer could be connected to all of the others, but that can get expensive. Suppose that each computer is made to be connected to only three others. Suppose further, that someone has figured out that the whole system can handle the traffic only if messages don't have to pass through more than two computers on their way to their destination. If messages have to pass through three or more computers, it is likely that the whole system will get bogged down because it is doing to much work just passing messages around. How will you decide how to plug these computers together?

When you try and solve such a problem, a good way to show these connections is with a graph. Draw the computers as vertices or dots and the connecting wires are edges or lines. You will be trying to construct a graph that is 3-regular whose diameter is not greater than three. With a little bit of experimentation, you can figure out how to connect all 12 computers so that these criteria are met.

Now suppose you are given a similar task, but instead of connecting computers, you have been asked to lay out where the electric impulses should go on a silicon chip. You are allowed to use only one side of the chip, because the other side of the chip is being used for something else. If any of your connections cross, the whole thing will short-circuit and be useless. It is quite a bit harder to find a way to draw the graph that represents the right set of connections. This graph will be 3-regular and it's diameter will be 3, (just like the graph that you drew to connect the computers in the problem above) but it will have the additional property of being **planar**. That means that it can be drawn in a flat plane so that edges (lines) only touch each other at a vertex (dot). If you can draw this graph without peeking at Figure 4.4, you should be triumphant!

The graph in Figure 4.4 is a 3-regular planar graph with diameter 3. It is easy to verify that it is 3-regular and planar. It may take some experimentation to verify that the diameter is also 3. The relationship between degree and diameter in graphs occurs over and over again in questions that come up in computer system design.

Understanding Properties by Playing Games

It is difficult and confusing to pour over pictures of dots and lines and try to apply what you have memorized about graph properties to the graph that may happen to be in front of you. If you use a graph as a game board, however, and invent games with game pieces that move about it in a variety of ways, many of the properties of the graph will emerge as you develop strategies for playing the games.

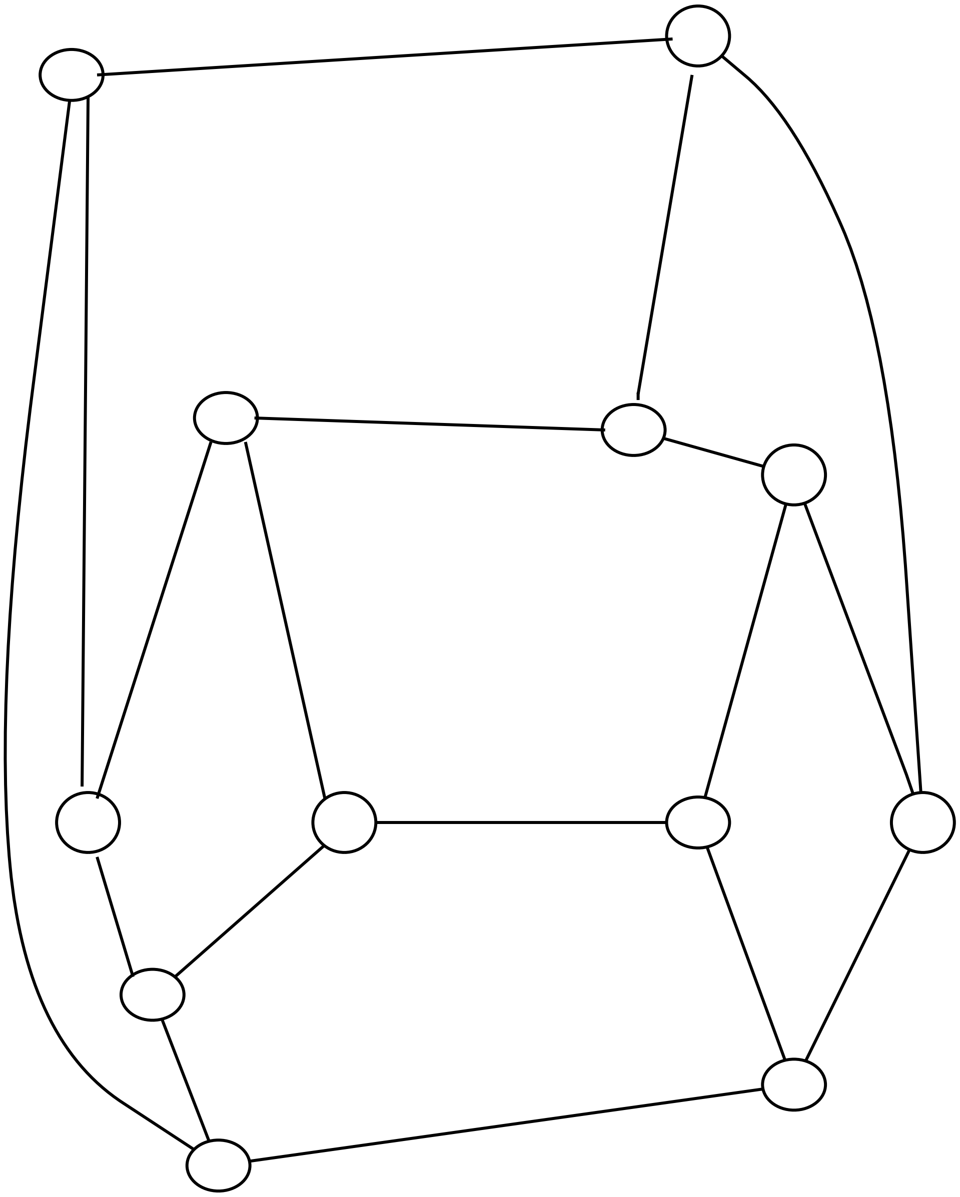


Figure 4.4: The largest known planar graph of degree three and diameter 3.

4.2 A Graph, A Story, A Game

Description

This story provides the scenario for a game that students can play. It helps them to explore the concept of **distance** in a graph. The game can be played using the blackline master of Figure 4.4 as a game board, but it is particularly effective if Figure 4.4 is drawn on the ground so that the circles are large enough for a student to stand in. Students then take the part of the characters in the game and move about the graph.

Materials

- A large copy of Figure 4.4 drawn on the classroom floor with tape. The circles should be large enough for a person to stand in. It should be in an area where all the students can gather round and watch what is happening.
- For each small group of students:
  - Copy of Figure 4.4 to use as a game board
  - 3 different markers to move on the game board

Instructions

1. Explain to the students that the figure on the floor is a game board and that they are going to play a game which is best explained by a story. The story has three characters in it: Gertrude (a goose), Superperson, and a Monster. Select a student to play the role of each of these characters.
2. Tell the students that the circles represent ponds in the land where Gertrude, Superperson and the Monster live. To begin the game, the three characters must select the ponds where they will start out. During the course of the game they will see that Gertrude has the least amount of moving power and she runs away from the Monster. So to give her as much advantage as possible, the Monster should select his place before Gertrude selects hers. Superperson chooses her place last.
3. Read the story **Superperson Saves the Monster** and have the students act out their parts as the story moves along. The story serves as script and rule book for the game. Encourage the students to act out their parts with as much drama as possible.
4. The game doesn't end when the story ends. The story simply provides enough information for the game to continue.

5. After Superperson has saved the Monster from Gertrude, have a different group of students play the roles of the three characters. You can continue to read the story as they play, or a student might want to read the story. Once the students know the roles of all the characters, they might want to play without reading the whole story each time.

6. Depending on the age and experience of the students, they will be able to figure out, after some number of rounds, that the Monster always catches up with Gertrude and that Superperson always saves the day. Help them to explore what difference various starting points for the three characters make. Help the students to articulate why it is easy to figure out what will happen. The concept of distance and diameter in a graph figure largely into the reason why. Provide them with that vocabulary when it will be useful to them in talking about their experience.

Varying the Game

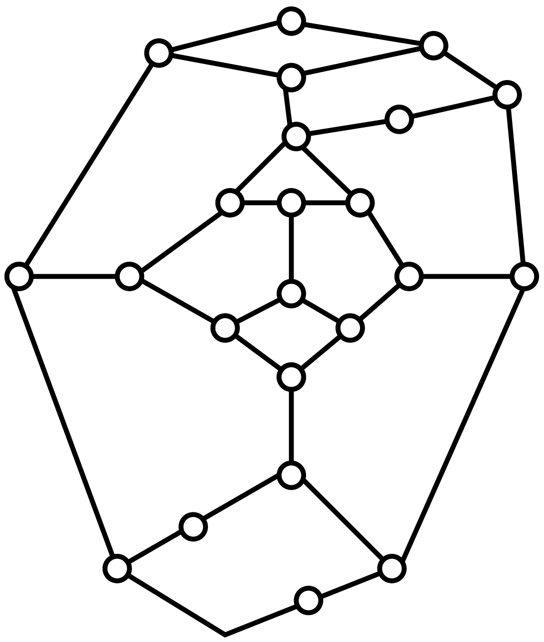
Once the Monster can land in a pond that is right next to the pond that Gertrude is in, he is certain to be able to catch up with Gertrude on her next turn. This is because Gertrude can only move a distance of one. No matter where she goes she is certain to be two steps away from the Monster when it is the Monster's turn to move next. Notice, however, that at this point Gertrude can always assure that Superperson will be able to arrive in the pond where she and the monster are. Once Gertrude knows she is going to be "caught" she can look around and be sure that the pond she is caught in will be exactly three steps away from where Superperson is. When the game ends, the players have to give up their turn, sit down and let someone else play. Perhaps they don't want to do that. Let the students try to find a way to make the game last as long as possible.

Students can decide on any rules at all to change the game to make it more challenging. Some suggestions are:

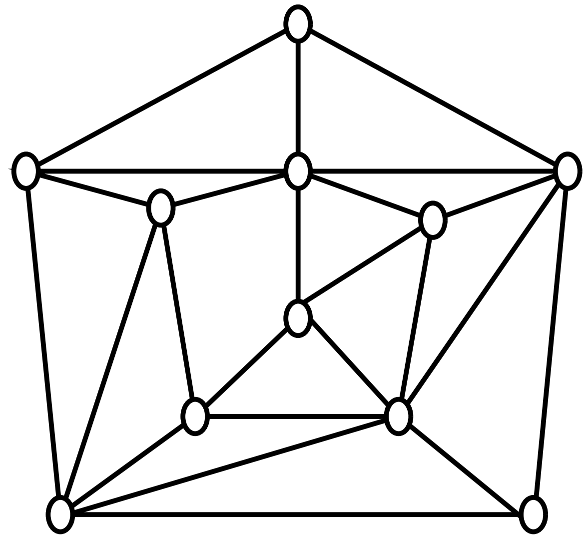
- Let all the players move a distance of exactly two at each turn.
- Use a different game board, such as any of the ones in Figure 4.5. to play the game. Because the distances in these graphs are different from the graph in Figure 4.4, the students need more complex strategies to play that they had to use with Figure 4.4. This will help them to gain a better understanding of the concept of distance in a graph, and to see how these configurations of dots and lines have considerable structure.

Superperson Saves the Monster

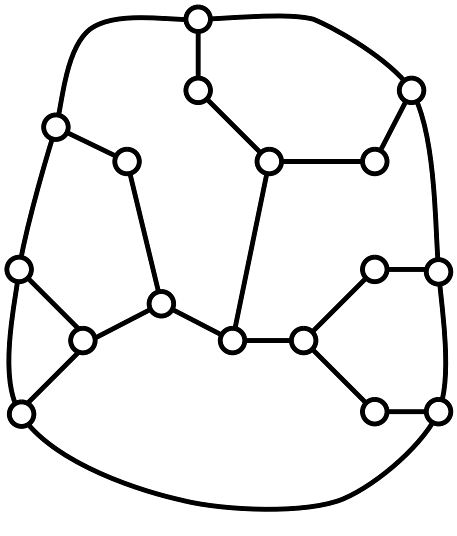
The Land of Many Ponds where Gertrude the goose lives is a beautiful place, but she doesn't have anyone to play with. One afternoon, tired of swimming in circles and diving all by herself, she decides to look for some friends.



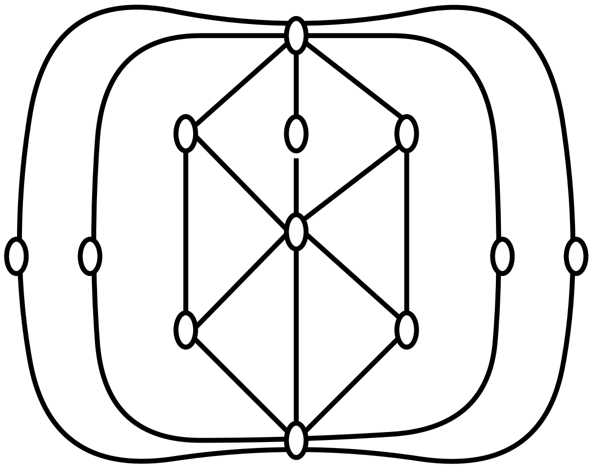
max degree = 3, diameter = 5



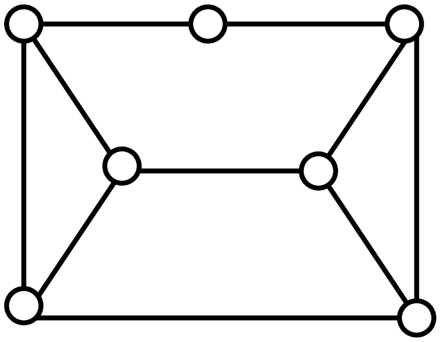
max degree = 6, diameter = 2



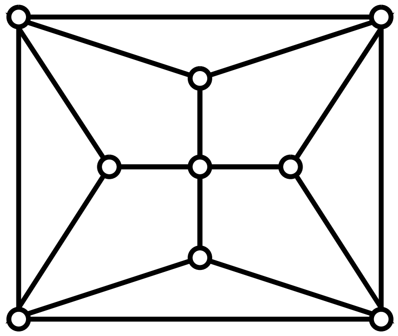
max degree = 3, diameter = 4



max degree = 7, diameter = 2



max degree = 3, diameter = 2



max degree = 4, diameter = 2

Figure 4.5: More planar graphs with small degree and diameter.

"Someone else has to live here somewhere. I can't be the only creature in this whole Land. I'm bored. I want a pal to play with." She decides to fly off and find one.

The trees are tall in this land, and it is hard to find the other ponds. Gertrude flies along one of the narrow flyways that lead out of her pond. She is such a well-fed fat goose that by the time she gets to the next pond, she can't fly any further

"Well, no one lives here," she pants. "But I'll have to rest before I can go on."

In another pond, a Monster was swimming by himself. Most folks didn't like his looks, so he always swam alone, but today he was feeling particularly brave.

"Did I hear splashing in a distant pond?" he asked himself. "Then it is true that a goose named Gertrude lives in this Land! I am going to find her so we can be friends."

The Monster is a huge creature. It takes tremendous wing-flapping power just to get him out of the water and into the air. "Feathers flying!" he cried. "Horns up! Claws aloft! Tail high! I'm gonna make it!" He flew past one pond while he was still on the way up. When he got to the next pond, he was so exhausted that he crashed into the water without even noticing if Gertrude was there or not. (She wasn't.) All he could say was, "Glug, glug, glug..."

There is one other person who lives in this Land. Her name is Superperson. In another pond she is enjoying a nice relaxing swim. "Oh," she said, "It's probably time for my award-winning Superperson Patrol. I'd better go fly around a bit just to be sure everything is okay. That's my job, after all." She stretched and got up.

Superperson's flying power is so great that she flies Zip!Zip!Zoom! in a blinding flash. When she takes off, she goes up over two ponds and lands in the third one without even working very hard. She lifts off, splashes into the water three ponds away and congratulates herself on her fine flying power.

By now Gertrude has had her rest and she is ready to keep looking for some company. She takes off, but she just isn't much of a flier. "Tired already," she huffs. "There's the next pond up ahead. I'm going to land and rest a bit."

The Monster tries again to find her. "Where are you Gertrude? Where are you Gertrude?" he cries, on the way up, passing over one pond. "Gertrude! Gertrude! Glug, glug, glug..." He splashed down in the second pond.

Superperson has had a nice swim, but it is time for her patrol. "I can't just have fun all day. I am a person with responsibilities," she says, and she is off. Zip! Zip! Zoom! "Wow, look at me! I'm so powerful, I couldn't fly less than three ponds if I wanted to!"

In her pond Gertrude is starting to get a little worried. She has heard the loud crashing sound of the Monster's landing in a nearby pond. "That doesn't sound like somebody I want to play with. The first thing he'll probably want is a snack! I'm getting out of here." She flies off to the pond that she hopes is the farthest one away from where the Monster is.

The Monster hears some splashing and is even more determined to find Gertrude. This will be a wonderful land to live in once he makes friends with a playmate. “Here I am Gertrude!” he cries on the way up, passing the first pond. “Where are you Gertrude?” he shouts on the way down, plummeting towards the second pond. “Glug, glug, glug...” is all you hear as he splashes into it.

Superperson wakes up from a restful nap. “It’s that time again. I better go see how everything is.” Zip!Zip!Zoom! Her three-pond flying tour is finished in no time.

All three of them, Gertrude, Superperson and the Monster, continue flying from pond to pond like this, until the monster surprises both Gertrude and himself by actually crashing into the pond where Gertrude is swimming nervously.

He is still saying “Glug, glug, glug....” when Gertrude attacks him wildly.

“I’ve got to drown this thing!” she cries. “Save me! Save me! Someone save me before it is too late!”

“Glug, glug, glug...” is all the monster can say. He is surprised that this new friend of his plays so rough.

Superperson hears Gertrude’s call. “Silly Gertrude. I better go help her out before someone get’s hurt.”

What will happen now? Will Zip!Zip!Zoom! take Superperson into the pond where the Monster must be saved from Gertrude? Or will she whoosh right past?

If Superperson doesn’t get to the pond this time, Gertrude will stop trying to hurt the monster and fly away to a different pond to escape. But the Monster will keep looking for her because he wants a friend. He thinks they already had a fun round of playing together already. If he keeps this up, he might get hurt. Will Superperson ever be able to get to their pond when they are together and show Gertrude that she really has found a friend?

4.3 Old Friends, New Game

Description

This story provides the scenario for playing a game on a graph. It can be played with the graph in Figure 4.4, or with any other graph in this section, or graphs that students draw themselves. At the end of the story the Monster and Superperson have begun to solve their problem, but it is not completely clear what they should do next. The students who play the role of Superperson and the Monster will have to decide what to do. The planning and strategies that the characters will need to do will change when the game is played on different graphs. Playing the same game on different graphs helps students understand the structure of graphs in general, and to make inferences about properties that can be used to distinguish one from another.

Materials

- Large copy of Figure 4.4 drawn on the classroom floor
- (Optional) Copies of Figure 4.4 and other graphs with unifix cubes or other markers that students can use to play the game.

Instructions

1. Tell the students that the adventures of Gertrude, Superperson and the Monster continue, and that you will read them a different story that explains a game that can be played by the three characters.
2. Select students to play the roles of Gertrude, Superperson, and the Monster. Encourage them to play their parts with as much drama as possible.
3. Read the story and help the students who play the parts of each character act out their part.
4. When the story ends, Superperson and the Monster have managed to clear one pond of pollution. Now they must do the rest. The precise strategy that they used to clear the first pond will not work in exactly the same way again. The students will have to figure out what to do to force Gertrude to fly into all of the ponds one at a time.
5. Remind Superperson and the Monster that they can only talk to each other and make plans about where they will go next when they are in the same pond together. Before they set out each time they have to agree the following:
   - the route by which they will each return to Gertrude's pond so that they can force her to fly into the next pond they want to clear out.

- where they will have to fly in order to return to Gertrude's pond by that route
- how to time their flight so that they both arrive in Gertrude's pond at exactly the same time.

Discussion

1. Help the students to articulate the strategies that they want to use.
2. Ask the students what will happen if they play the game on different graphs. Have them experiment with different graphs to see if their predictions are correct.

4.4 Our Friends Battle the Horrid Pollution

Gertrude, Superperson and the Monster live in a Land of Many Ponds, deep in a forest of tall, tall trees. The trees are too tall to fly over, too dense to fly through, but between certain ponds are flyways through the trees that they use to get from one pond to another.

No matter how hard she tries, Gertrude can't ever seem to fly any farther than one pond at a time. Then she has to rest.

When the Monster flies, he gets so excited about going *up*, that he always flies past the first pond he comes to. But he never makes it any farther than the next pond before he crashes.

The flying power of Superperson sends her Zip!Zip!Zoom! She flies over two ponds and lands in the third without even trying very hard.

Because of their flying habits, Gertrude, Superperson and the Monster sometimes have to do some careful planning if they want to go anywhere together.

There came a time when a terrible pollution invaded the ponds of the Land. The waters became cloudy and foul, and being in these waters so sapped the strength of Superperson and the Monster that they were dying.

The mysterious pollution did not harm Gertrude. She did not know it, but she possessed the amazing power to cure it. Whenever Gertrude's feathery wings kissed the surface of polluted pond, the waters cleared, the air freshened, and the birds, plants, and pond flowers returned to health.

Gertrude didn't even know she could do this. All she knew was what she heard from Superperson and the Monster—that the deadly pollution had invaded all of the other ponds. Only the pond she was in remained safe. She was terribly frightened—too afraid to move. The single clean pond where Gertrude was swimming was the only clean one left in the Land.

Superperson and the Monster knew that only Gertrude could save them. But it wasn't going to be easy to convince her to fly from her safe, clean place and into the blackened stench of a foul polluted pond. They swam away from Gertrude to the edge of the water and devised a plan.

"We have to scare her out," whispered Superperson. "We will make her so afraid to stay in this pond that she'll forget how scared she is to fly to another one."

"Scaring isn't nice," mumbled the Monster, hanging his head very low. He always had trouble with people being afraid of him because he looked scary just when he said, "Hello".

"We're not *hurting*," hissed Superperson, "we're *scaring*. There's a difference. We'll explain in the end. She'll understand."

"Well, OK," agreed the Monster, brightening up. "I am pretty good at being scary, you know. How do we do it?"

"First, we fly away," said Superperson.

"Into the *pollution*???" The Monster's eyes were wide with fear.

"It will weaken us, but we can make it," explained Superperson.

"Huh-*uh*." The Monster crossed his wings and pointed his nose horn high into the air.

"You can't expect Gertrude to be brave if you aren't brave, too," replied Superperson in her stern, reminding voice.

The Monster knew she was right. But he still didn't like it. "Well, OK. But I can't be in the pollution long."

"Neither can I. So we have to plan carefully—or we're sunk." Superperson glanced over her shoulder. Gertrude was swimming and diving. "You fly two ponds when you're excited, right?"

The Monster nodded. "If I'm going to be scary, I'm going to be excited. I can never land in the first pond when I'm excited. When I get to the first pond, I'm still going up. I could try to—"

"That's fine," interrupted Superperson. "We have to plan for what we do best. If I'm excited (and I know I will be, too) I can't depend on myself to do anything except Zip! Zip! Zoom!, three ponds in no time flat. So lets plan for that."

Superperson continued, "You fly two ponds away, I'll fly three, then what?"

"Gertrude will freak out," declared the Monster. "You know how she's been lately, never letting us go anywhere without her. She'll think we're going to die." He looked up. "She may be right."

"We're *not* going to die," corrected Superperson. It was her reminding voice again. "But if she thinks we are, that's great. We can be ghosts."

"Huh???" The Monster was really confused.

"Yes! We fly away. You fly two, I fly three. We land in one of those awful ponds, but just for a second. I'll transform myself into some disgusting shape. You get some slime on your horns or something. Then we race back to here with a terrible howl."

"You howl," snorted the Monster in his reminding voice. "I'm going to ROAR! She won't have a heart attack, will she?"

"No," Superperson assured him. "She'll just be so scared she'll fly down the flyway to another pond—even though it's foul."

"And she'll make it clean!" Now the Monster was really excited.

"Right!" grinned Superperson.

"And we'll do it again! And again! And again! And again!" shouted the monster, splashing the water with his wings.

Gertrude stopped her diving and swam over nervously. This whole pollution crisis had her on edge. "Do what again, and again, and again?" she asked. She twisted her head from side to side, looking at the Monster first out of one eye, then the other.

"I can't stand it another minute!" shrieked Superperson. I'm getting out of here!" She zipped into the air. The Monster flapped his wings and began his takeoff, too.

Everything went as planned.

Gertrude freaked out.

The Monster flew two ponds away and doused himself with slime.

Superperson flew Zip! Zip! Zoom! three ponds away and shifted her shape as soon as she landed. She was hideous. She turned and began to fly back. She zoomed in from the flyway on the left just as the Monster, whose roar shook every tree, was crashing in from the right.

In a terrible panic, Gertrude flew off. She crashed, exhausted, into the next pond where the water was slimy and foul, certain that she was doomed.

When Gertrude's feathery wings splashed on the surface of the polluted pond, the waters cleared, the lily pads rinsed their leaves, and the black fog dissolved into a few white wisps. Bird song rose in the crystal air.

In the pond where they had landed, Superperson and the Monster freshened up after their successful adventure.

"Let's do it again!" shouted the Monster.

"Right away!" cried Superperson. "We have to get her to clean up all the other ponds before the pollution can leak back in here."

"We'll make her think the whole place is haunted and get her to fly everywhere!" roared the Monster. "Let's go! I'll fly two!"

"And I'll fly three!" howled Superperson.

"We'll fly fierce in the flyways toward Gertrude!" yelled the Monster.

"And scare her off to clean up a new pond!" rejoiced Superperson.

"We'll save our land!" roared the Monster.

"And save ourselves!" clapped Superperson.

Suddenly it was very quiet.

Slowly they turned to face each other.

They looked a little worried.

"I think we better do a little more planning," they whispered to each other, both at exactly the same time.

4.5 Three for the Money: The Degree/Diameter Problem

Description

Students can understand and work on an unsolved problem in mathematics. There is a good chance that this problem can be solved by someone who spends enough time experimenting with it. The only skills required to work on it are the ability to draw dots and connect them with lines, and the understanding of four ideas related to graphs: **degree**, **diameter**, **planarity**, and **size**.

Materials

- One large copy of the graph in 4.4 that all students can see, or a copy for each student or group of students.
- Paper and pencil

Instructions

1. Explain to students that you are going to tell them about a problem in mathematics that is easy to understand that no one has solved yet. There is no reason why one of them could not be the person to solve it. A famous mathematician, Paul Erdos, has offered a large cash prize for the person who succeeds in drawing the graph that solves this problem. They could win it.

2. Have the students look at the graph in Figure 4.4 carefully. If they have played the games with Gertrude, Superperson and the Monster, they have already learned a lot about the structure of this graph. Explain that it has four properties or characteristics that mathematicians have found interesting:

   - Its **size** is 12, because it has 12 **vertices**.
   - It is **planar** because it's edges can be drawn in a flat plane without crossing each other. That is why Gertrude, Superperson and the Monster can fly around their Land without needing traffic lights or underpasses.
   - The **diameter** is 3. If Superperson can't get to where Gertrude is trying to drown the Monster, it is because she overshoots the pond by flying too far. She would never have to fly more than three ponds to get to them.
   - It is **3-regular**. When Superperson and the Monster were trying to get Gertrude to fly to all of the other ponds one by one, they flew into the pond where she was along two of the flyways. This left Gertrude exactly one choice for flying out. This

can be done from any pond because every pond has exactly three flyways touching it.

3. Tell the students that no one has ever figured out a way to draw a graph whose **size** is greater than 12 (a graph with more than 12 vertices) that has all of the other properties. Nor has anyone been able to figure out a good reason why it can't be done. Invite them to make mathematical history—and win a cash prize—by being the one who does it.

4. Have students experiment with this problem and try to draw the graph. If someone succeeds, contact the authors of this book immediately!

5. When they begin working on the problem, students usually produce graphs that have most of the properties, but not quite all of them. Help students to work together and carefully each others' graphs.

6. Draw large copies of graphs that *almost* meet the criteria and display them. This is a problem to let rest for a while and return to, perhaps weeks later. Give students plenty of time to experiment and mull it over. Eventually after much trial and error someone may come up with a graph that meets the criteria. Or perhaps after enough experimentation with various possibilities, someone will be able to explain why it is that size 12 is the largest that such a graph can be. If time and perseverance are what is needed to solve this problem, there is no reason why the person to solve it cannot be a child.

Discussion

1. What strategies did you use to try and construct the graph? Where did you get stuck? What do you plan to try next?

2. Have students look carefully at the graphs in Figure 4.5. These are graphs that have been drawn by mathematicians trying to understand and solve the degree-diameter problem. It shows some of the largest known graphs with certain degrees and diameters. The numbers down the column to the left show the maximum degree of every vertex in the graph. The numbers across the row on the top show the diameter. The graph that is marked with * is the only one that mathematicians know for sure to be the largest one that meets the degree and diameter criteria for that slot on the chart. If you can find any larger ones for the slots (most of them!) for which the largest possible graph is still unknown, be sure to contact the authors immediately!

3. If students become convinced that it is impossible to draw a larger graph that is 3-regular, planar and has diameter 3, explain to them that "I tried for hours to do this and couldn't," is not a statement that proves that whatever they were trying to do is impossible.

4. You can show that it is impossible to make a 3-regular, planar, graph of diameter 3 that has 13 vertices. You can demonstrate as follows that there is no sense in trying to do this because it can't be done. Here's how:

- Draw the 13 vertices.
- Make each vertex degree 3 by drawing the all three edges that will touch each vertex. Now you have 13 vertices, each with half an edge touching it.
- In order to make the graph that we are trying to make, you will have to connect pairs of half-edges to make single whole edges. But count the number of half-edges—there are 13 x 3 = 39 of them!
- Every time you connect two vertices together you will use up two of the half-edges. There is no way to connect the vertices together without having one half-edge left over. Therefore, you can be sure that you are wasting your time trying to make a 3-regular graph of size 13, much less one that is planar and has diameter 3. What other sizes of graph are certain not to work?

4.6 Other Games on Graphs

Description

The source of new problems, discoveries, and theories in mathematics is mathematicians own inventiveness and their propensity for asking "what if" kind of questions. Students who puzzle over existing questions or ask new questions of their own are *doing mathematics.* When students play games on graphs it is natural for them to ask two types of questions that get to the core of what professional mathematicians who are graph theorists do. These questions are:

- How will this game change if we play it on a different graph?
- What if we changed the rules or invented a different, but similar game?

A series of games are presented here to give students some ideas for getting started, but it is by no means exhaustive. The games can be played using any graphs in this section as game boards. Students can draw their own game boards. Students will probably invent their own games. The possibilities are endless.

Instructions

1. Invite students to play games, invent new ones, and to discuss what they learn as they do so.
2. Using the stories about Gertrude, Superperson and the Monster as a model, invite students to invent other stories with other characters and situations that they can use to present and act out their games.

Some Games

Mail Carrier's Tour

Imagine that the graph is the map of a neighborhood. The edges are streets and at every vertex is a house. The mail carrier doesn't want to walk down any more streets than necessary to pick up and deliver mail to all of the houses. What is the best route to take?

The Painter of Lines in the Street

The person who paints the lines in the streets works hard and doesn't want to have to spend time lugging painting equipment when it is not necessary. What is the best route to take as you paint the streets so that all the streets get painted, and the only time that you are on a street is when you are painting it?

The Colorful Houses of Tourist Town

In Tourist Town there is something interesting to see in every house, and the people who live there want the tourists to visit them all. The City Council has decided to buy enough paint to repaint all of the houses. If tourists leave a house that they have just visited and look down a street and see a house that is the same color, they might think that it is just like the house that they just left and not visit it. So the City Council wants to repaint the houses so that no house is the same color as the next house down the street in any direction. How many different colors of paint with the City Council need to buy?

The Colorful Sidewalks of Tourist Town

Since all the grownups got to paint their houses, the City Council decided that the children should be able to paint the sidewalks in bright and interesting colors. They don't want the tourists to ever get bored by walking down two streets in a row whose sidewalks are the same color. How many different colors of sidewalk paint must the City Council buy?

4.7 Our Friends Recover From a Disaster

HORRORS! The Land of Many Ponds has been clearcut! Where there were once tall, tall trees and flyways, there is nothing but stumps and empty space. The land is parched, the soil is dust.

Gertrude, Superperson and the Monster met in one of the ponds to talk over what they should do.

Gertrude swam round and round in furious circles. "Oh we are doomed, just doomed!" she wailed. "As soon as it rains, the soil will begin washing away. WHOOSH! It will go into the ponds! The water will get silty and brown, and soon the ponds will dry up. Oh it's awful! We are doomed! So this is what it feels like to be doomed! I don't want to be doomed! I HATE being doomed!"

"Settle down, Gertrude," said Superperson in her reminding voice. Gertrude hated it when Superperson said that. Gertrude also hated Superperson's reminding voice.

"We'll plant more trees," declared the Monster. "Simple as that."

"Simple!" shrieked Gertrude. "And just how do you propose to do the *simple* job of replacing hundred-year-old trees in an afternoon?"

Superperson flew to Gertrude's side and began swimming next to her. "Settle down, Gertrude." she said quietly. "You can't think well when your mind is wild."

"We can't plant trees. We can't do anything!" sobbed Gertrude.

"Gertrude, please try," urged Superperson, "Please settle down." Her reminding voice softened. "What else could we plant?"

"We could plant seeds," suggested the Monster. "Seeds of things that grow THICK instead of tall."

"Only at first," murmured Gertrude. "We still have to have trees again someday." She was still upset, but she was settling down.

"First we'll save the soil. Then we'll plant the trees. Does that sound like a plan?" asked Superperson.

"YES!" shouted the Monster. "We'll travel out in to the rest of the world and look for plants that would be good to have."

"We'll find beautiful things!" cried Gertrude, "Plants that grow in thick, lovely carpets that hug the soil!"

"We're OFF"! they yelled, and flew in three directions away from the Land of Many Ponds and out into the world.

They were gone for many, many days.

The Monster was the first to return. He was lugging a bulky sack. Gertrude flew home with a mass in her beak that looked like straw. When Superperson whooshed in, she carried a ball of white fur.

"This," announced Superperson, "is dandelion fuzz. Each little white speck has a seed inside. Dandelion roots grow long and deep and the leaves are tasty to eat. But we have to watch this stuff so it doesn't blow away."

She sat the fuzz-ball down gently and sat on it. "What's in the bag, Monster?"

"I came to a meadow that looked like it had been painted purple with violets," he reported. "Everywhere these little flowers grew, packed in tight as can be. I said to myself, 'Now *this* is what Gertrude meant by a carpet!' I couldn't find any seeds, so I dug up some clumps of them that we can transplant."

He set the sack down and nestled it under his feathery, scaly wing. "What's with the sticks, Gertrude?"

"They're not sticks," she snapped. "They're seed stalks."

She laid them on the ground and settled down alongside the pile. "I came to a grassy plain, emerald green in all directions. These seedy stalks were standing high. We'll shake the seeds loose and sow them for a grassy carpet."

The Monster jumped up and began to sing. "Yellow! Yellow! Purple and green! The finest carpets you have ever seen!" He began to wave all three of his horns and stamp his webbed feet.

"Settle down, Monster," said Gertrude. "Your work doesn't get done well when you are being wild."

"When we plant these seeds, let's not mix them up," suggested Superperson. "We want our carpets to be separate colors."

"Not only that," added the Monster, trying to look calm, "we don't know if one kind of plant will choke out the other kind."

"We can use the old flyways for boundaries," said Gertrude. "But we better not plant any two things next to each other. When the carpets grow thick and join, we'll never remember where the flyways belong."

"OK," agreed the Monster, grabbing his sack. "Yellow, green, and purple. No two colors touch. Let's go!"

"Wait a minute," called Superperson. "What about at the ponds? Can colors touch at the ponds? We'll never be able to do this if the colors can't touch at the ponds."

The Monster fidgeted and dug into his sack.

Gertrude took charge. “Colors can meet at the ponds,” she declared. “That’s OK. When you step across a flyway, you don’t want to step into the same color you stepped out of. But it’s OK to swim across a pond and get to the same color.”

The monster was impatient. “OK, OK,” he said rolling all of his eyes. “Too much talking. These plants need to be in the ground. They’ll die if they stay much longer in this sack. I’m going to start.” He began poking his nose-horn into the soil and dropping the tiny violet plants into the holes.

“Monster, WAIT!” shrieked Superperson. “If you go rushing off ahead and I end up planting this dandelion fuzz in the wrong spot, YOU’RE the one who has to figure out how to unplant it!”

“Settle down, Superperson,” huffed the Monster. “I’ll only plant this patch.”

“Yes, Superperson,” said Gertrude in her reminding voice, “please try to settle down. You won’t be able to help us think if you are too wild inside.”

Gertrude continued, “Let Monster plant that patch. Then you plant some fuzz. Then I’ll sow some grass seed, and it will be Monster’s turn again.”

Superperson sighed. She had settled down. “I suppose that will work.”

“I sure hope so,” grinned the Monster, “because I have no idea how to un-plant dandelion fuzz!”

Chapter 5

Machines that Eat Your Words

5.1 Introduction

Overview

A **finite state machine** is an imaginary (or **abstract**) machine that is used to study and design systems that recognize and identify patterns. The idea of a finite state machine is a powerful one that has many applications in computer science.

The main activity in this chapter is **Learn My Language** in which students act out the operation of a finite state machine that has been drawn with masking tape on the classroom floor. Students receive (or make) tickets that serve as inputs to the machine. The tickets tell them which lines on the floor to follow and which circles or **states** to pass through. If they end up in an **accept state** they receive a prize. This means that what was printed on their ticket was recognized by the machine as a "word" in its "language".

Big Ideas and Key Concepts

• Finite state machines • Pattern recognition • Infinity

NCTM Curriculum Strands

• Mathematics as Communication • Mathematics as Reasoning • Mathematical Connections • Spatial Sense • Patterns and Relationships

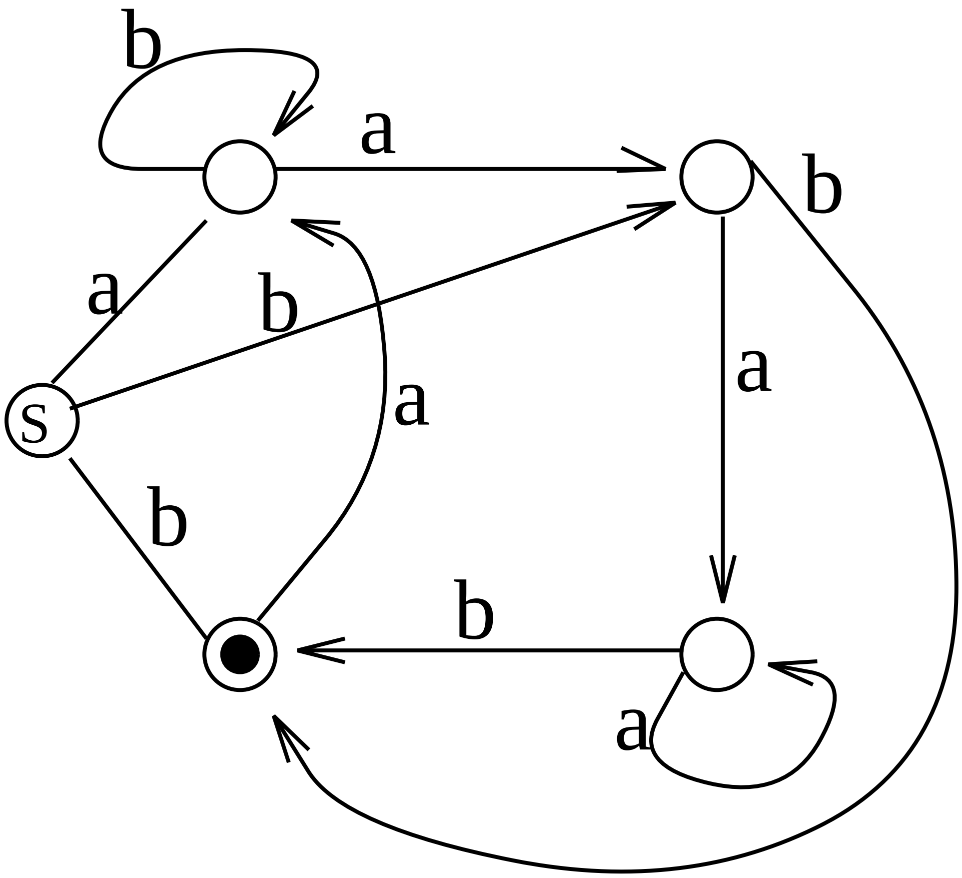


Figure 5.1: A Finite State Machine

Vocabulary

• finite • infinite • state • finite state machine • transition • accept state • “no” state • word • alphabet • vocabulary • abstract • string

Materials and Preparation

The activity **Learn My Language** requires a finite state machine (Figure 5.1) to be drawn with masking tape on the floor. A large quantity (at least 3 per student) of small prizes is also required. The other activities only require paper and pencil.

Evaluation

Students have been exposed to the workings of a finite state machine when they can:

- Follow the letters printed on a ticket and walk along the lines to the correct places in the machine.
- Recognize that certain words are accepted by the machine and other words are not, and that whether or not a word is accepted depends on the circle or "state" you are in when you have reached the end of your word.
- Recognize that there are patterns in the words that are accepted by the machine.

Depending on the age and experience of the students, teachers may also want to note the extent to which students are able to:

- Predict whether a word will be accepted or not.
- Devise words that would be accepted.
- Articulate the strategies that they used to devise words that will be accepted.
- Predict whether a word would be accepted by looking at the machine but not walking through it.
- Predict whether a word would be accepted by looking at the notation for words that would be accepted.
- Articulate the patterns that they discovered in words that were and were not accepted.
- Understand the notation for generalizing which words are accepted.
- Use the notation for generalizing which words are accepted.
- Draw conclusions, make predictions, generate questions about finite state machines and follow up on them.

The Alphabets, Words, and Languages of Finite State Machines

The idea of a **finite state machine** comes from an interdisciplinary branch of mathematics (with roots also in computer science and linguistics) called formal language theory. Looking at the structure of our own written English language, we can describe it as being made up of building blocks called words. Words, in turn, are made up **sequences** or **strings** of the 26 letters in the alphabet when they are written down. Not every sequence or string of letters that we put together forms a word that we recognize or **accept** as part of the English language. The Finite State machine can be used to check strings of letters and determine whether they belong to a language or not. English, like all other natural languages that people speak, is very complex. Some other languages are simpler to think about.

When we sit at a computer keyboard and type strings of letters such as "copy"or "save", the reason that the computer complies with these requests is not because they make any kind of “sense”. The computer has been programmed to recognize certain strings of letters as part of its language, and to perform certain electronic functions on the basis of which string was typed in. If we misspell “save” and type “saev” or type in “keep” instead, the computer cannot guess what we mean. Instead it puts out a sometimes frustrating message to the effect of, “That word is not in my language.”

Languages that are made of “words” which are “strings” of “letters” from an “alphabet” are found in many areas of science. Biologists, for example, know that proteins are made up of discrete building blocks called amino acids that can only occur in certain combinations. A DNA molecule is a long chain made up of only four building blocks, but the patterns and ordering of the elements of this alphabet is used to write out the “words” that describe the genetic material of all living things.

Formal language theory is a truly interdisciplinary area of science. Whenever pattern recognition is vital to scientific inquiry, finite state machines may play an important role. Computer scientists team up with scientists from other disciplines and design ways to use the computer to analyze their data to recognize and identify patterns. The diagrams that depict the finite state machine are mathematical models that the scientists use to figure out how to design a system that will tell them what they need to know. Then these systems are turned into computer programs that analyze the data for the scientists.

Although a finite state machine cannot identify and recognize all types of patterns, it is still very powerful. It is the simplest and most basic pattern-recognizer and pattern-describer that computer scientists use.

The input to a finite state machine is always a string of symbols from the alphabet that the machine recognizes. We usually represent these symbols by *a, b, c,* etc, although any other symbols could be used. The finite state machine consists of a finite number of **states**. The states are represented by the circles in the pictures of Figures 5.1 -6. The lines with arrows are called **transitions**. They show what state to change to depending on the next letter in the input. In the pictures of Figures FDA1-4, the states or circles marked with a large dot are **accept states**. If this is the final state after all the letters of the input have been read, then the input has been accepted. This means that this particular string of symbols is a word in the language that this particular machine recognizes.

The best way to experience the workings and power of finite state machines is to experiment with them, trying a variety of inputs on a variety of machines and seeing what happens. When the finite state machine is drawn on the floor large enough so children walk around on it, they can experience the abstract concept of transitions from state to state as moving from circle to circle.

After experimenting for a while, it will become evident that for any finite state machine, the symbols of the alphabet, the number of states, and the possible transitions are fixed and finite. Even so, an infinite number of words can be tried as input. Not only that, for some finite state machines, of the infinite number of input words you can try, an infinite number of those will be accepted. This is one aspect of the power and versatility of finite state machines.

5.2 Learn My Language

Description

Students act out the operation of a finite state machine that has been drawn with masking tape on the classroom floor. Students receive (or make) tickets that serve as inputs to the machine. The tickets tell them which lines on the floor to follow and which circles or **states** to pass through. If they end up in an **accept state** they receive a prize. This means that what was printed on their ticket was recognized by the machine as a **word** in its **language**.

Materials

- Finite State Machine (Figure FDA1 drawn on the floor as described below.)
- For each student:
  - Several strips of paper, approximately 2" × 8".
  - Pencils, pens, or markers for writing.
  - A large quantity of small prizes, at least 2 or 3 per student.

Preparation and Set-Up

Draw a large copy of Figure 5.1 on the classroom floor with masking tape.**NOTE:** For younger or less experienced students, you could use Figure 5.3 which is smaller and less complex.

The circles in the figure should be large enough for one child to stand in. Place the prize containers in or alongside the circles which have the large dots inside. These are the circles for the "accept" state. The labels "**a**" and "**b**" can be written on pieces of paper and taped alongside the lines. Be sure to include the arrows that are on the lines in the drawing as well. Use an area of the classroom floor where students can bring chairs to sit in a circle around the drawing. It is ideal if this is an area that will not need to be covered with furniture as soon as the activity is finished and the tape can remain on the floor for several days.

NOTE: For very young students who have difficulty recognizing and distinguishing among letters, you can use colored dots or pictures of familiar objects in place of the letters throughout the activity. If colored dots are used, children could construct their input tickets by connecting strings of colored unifix cubes.

Instructions

1. Tell the students that the diagram on the floor is what computer scientists call a finite state machine. This machine has a language of its very own. It's kind of a silly language

by our standards. It doesn't have sentences, just words, and the only letters it uses to make it's words are **a** and **b**. Our language, English, can't make very many words with just **a** and **b** in them, but the words in the language of this Finite State Machine are different from our **a** and **b** words. When you come to the machine with a word, the machine can check to see if the word is in it's language. If it's not, the machine isn't really very interested in you. But if you bring the machine a word that is in it's language, it accepts that word and you win a prize.

2. Each of you has some strips of paper. Take one of those strips of paper and write down a word made up of **a**'s and **b**'s that you want to test to see if it is in the machine's language. There are two important things to remember while you are doing this:

3. First, the machine *only* understands words that are made up of the letters **a** and **b**. If you use any other letters, the machine won't understand them for sure.

4. Secondly, the machine doesn't follow rules that we have about spelling, so a word like **babb** or **baaaabb** might work—even though these don't make much sense as English words.

5. For now, don't make your words any longer than eight letters.

   NOTE For the first round you may want to supply students with words to try, and then let students make up their own words afterwards.

6. Invite a student to try a word by walking through the machine. Have the student read off the letters in the word to the other students.

7. Have the student stand in the circle marked **Start** and ask. "What is the first letter of your word?" Show the student how to walk down the line that is marked with that letter and stop in the next circle. Then ask, "What is the next letter of your word?" Again, the student must walk down the line labeled with that letter and stop in the next circle. Have the student continue in this manner until the end of the word on the ticket is reached.

8. Ask the other members of the class to note where the student has ended up. Is the student in the circle where the prize bucket is? That circle is called the **accept state**. If the student has landed in the **accept** state when the word is all finished, then we know that this word is in the machine's language, and the student receives a prize. If it isn't, the student returns to his place. In either case, the student can write down a new word to try next time.

   NOTE: If the student happens to *pass through* the **accept state** (the circle with the prizes) before getting to the last letter of the word on the paper, this does not count as having the word accepted. The word is accepted only when the student lands in the **accept state** after processing the last letter of the word.

9. Invite a few more volunteers to walk through the machine slowly with their tickets in hand. When all of the students understand what to do, they can begin walking though the machine more quickly.

10. Allow each student to try several words. When students catch on to how to make a winning word, suggest that they try making a word that is very long—a dozen letters, or even more.

11. Have students keep playing until many of them can put together words that will win.

Discussion

The debriefing and discussion can proceed through the following topics:

- How the finite state machine works.
- Strategies for discovering words that the machine will accept.
- Notation that acts as a shortcut for writing long words.
- How to use that notation to show the patterns in the words that get accepted.

How the finite state machine works

1. Students can think about and discuss the following questions:
   - How do you know where to go next?
   - When do you get to take a prize?
   - If I use a word like **aaba**, I pass the place where the prizes are stored, but I can't have one. Why?
   - Without actually walking on the machine, tell me if these words will be accepted:

     aabb **baaaaaab** **abbbaab** **bbbaa**

   - How can you figure out before you try whether a word will work?

2. Make two columns on the board and labeled **Accepted** and **Not Accepted**. List words that the students tried in the appropriate columns. You can use a shorthand notation for writing down words that have the same letter repeated by writing **baaa** as $\mathbf{ba^3}$. This notation will be useful in recording words that were tried and in talking about them.

Strategies for discovering words that the machine will accept

1. What strategies did you use to come up with words to try?

2. Look for patterns in the words in the two lists of words that were accepted and not accepted. Label a third column **Patterns** and list the patterns that the students describe. Since the words describe ways to move through the machine, help students to continue referring to the machine as well as the lists of words when they talk about patterns. Have all the students agree on the description and the wording of the patterns as they are written down. If agreement can't be quickly reached for a statement, put question marks by it as reminder to think about it some more and return to it later.

3. How does understanding patterns in the words that are accepted help you to think up more words that will be accepted?

Shortcut notation for writing long words

NOTE: You can use examples from the list that students compiled of accepted words for this discussion.

1. What do the following words, all of which are in the finite state machine's language, have in common?

   abab **abbab** **abbbbbab** **abbbbbbbbbbbbbbbbbbbbbbab**

2. We could think of hundreds of words that followed the same pattern that is in these words. If we want to talk about all of them, we can write: **ab*ab** The * after the first **b** means "Put in as many **b**'s as you want. It won't change anything." (This can even mean putting in zero **b**'s.)

3. Can you write a similar expression for winning words that has a * in it? Collect examples from students and write them down. Verify with the students that these are words that will be accepted.

4. Sometimes part of a word repeats a **string** or **sequence** of letters over and over again. For instance, the machine accepts of all the following words:

   aaab **aaabaaab** **aaabaaabaaab**

   so you can show this by writing **(aaab)***. The parentheses are used so that we don't get confused about what to repeat and what not to repeat.

5. An even shorter way to write **(aaab)*** is $(\mathbf{a}^3\mathbf{b})*$. Can you explain what all the parts of this expression mean?

More Experimentation with Finite State Machines

After the discussion, give students time to experiment some more with the finite state machine on the floor. If it is well-understood by all students, the activity could be modified in any of the following ways:

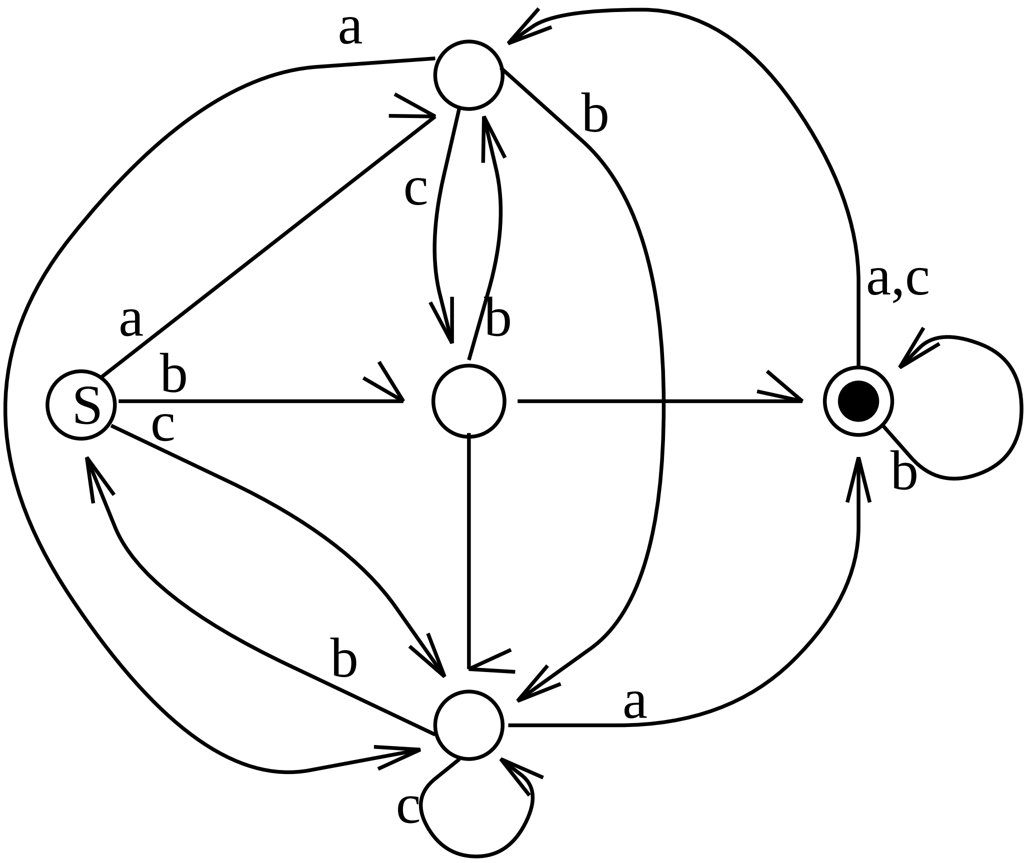


Figure 5.2: A finite state machine that uses an alphabet of 3 letters.

- Move the accept state to a different circle.
- Add a second accept state.
- Require the students to use words that have * in them.

For Further Investigation

Extending the Original Activity

- Repeat the activity with a different finite state machine. The machine in Figure 5.2 uses an alphabet of three letters instead of just two.

  NOTE As you continue to experiment with finite state machines, you may want to move to a larger area, such as a gym or multipurpose room so that you can draw new finite state machines on the floor. You can also photocopy diagrams of a finite state machine and let the students work with them individually or in groups. Markers such as unifix cubes, poker chips, or beans can be used to show the **state transitions** (how the player moves) on the paper.

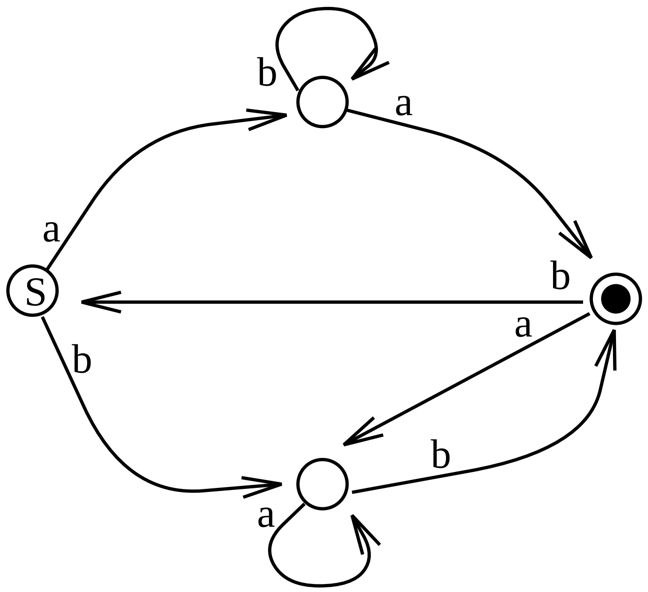


Figure 5.3: A finite state machine which can be described by the expression: **ab*a + ba*b**

- Students can draw their own finite state machines and experiment with which words will be accepted and which ones won't.

More About Notation

The notation for describing the words that a finite state machine will accept has one remaining element.

The machine in Figure 5.1 will accept either of the following: **aaab** **bab** In the notation, that can be written **(aa + b)ab**. The + means "Pick one of these, but not both." With the addition of this final element, the notation can be used to describe the inputs that would be accepted by any finite state machine in a single expression. Such an expression is called a **regular expression**. Looking first at a simple example, the finite state machine in Figure 5.3 can be described by the expression: **ab*a + ba*b**

The expression for the finite state machine in Figure 5.1 is considerably more complex:

(aba* + b)(aa*b + b)(ab*a(aa*b + b))*

Verify that the above is indeed the expression for Figure 5.1. Have students experiment with writing the notation for other finite state machines.

Mr.MacBuff

Every morning Mr. MacBuff gets up and prepares his breakfast in a certain way. The following expression shows what he does:

$$\mathbf{o(m + r + s)^*t(p + j)}$$

Each of the letters stands for the following:

o = a bowl of oatmeal

m = milk

r = a handful of raisins

s = a spoonful of sugar

t = a piece of toast

p = peanut butter

j = jam

- Act out Mr. McBuff as he prepares and eats his breakfast.
- Draw the finite state machine that shows how Mr. McBuff prepares his breakfast.

Finite State Machines in Parallel

Two finite state machines can be run **in parallel**. The same input is fed to both machines, and the input is not considered accepted unless the input produces the accept state in *both* machines. Try running 5.1 and 5.4 in parallel. Figuring out what will be accepted by two machines is more complicated than figuring out the same thing for one machine. What inputs are accepted by both machines? What are some of the strategies that you used? Can you draw a single finite state machine that does the same thing as 5.1 and 5.4 running in parallel? Experiment with other finite state machines running in parallel, and note your observations.

Same or Different Machine?

It is not unusual to be able to draw two different finite state machines that both accept and reject the same input words. See for example 5.5 and 5.6. Can you draw other finite state machines that behave the same but look different?

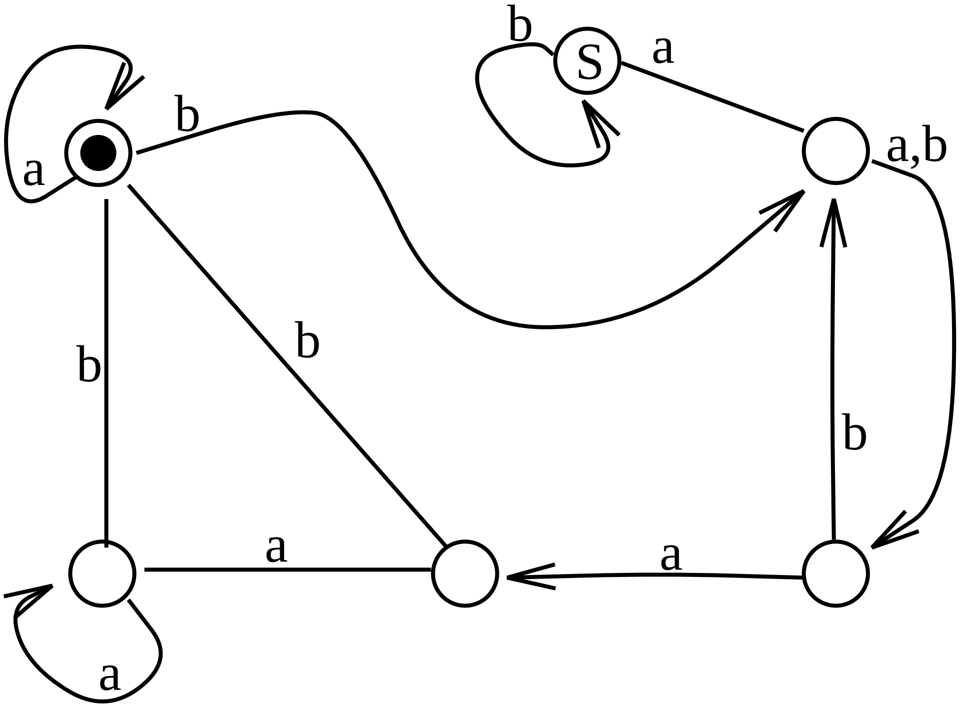


Figure 5.4: A finite state machine

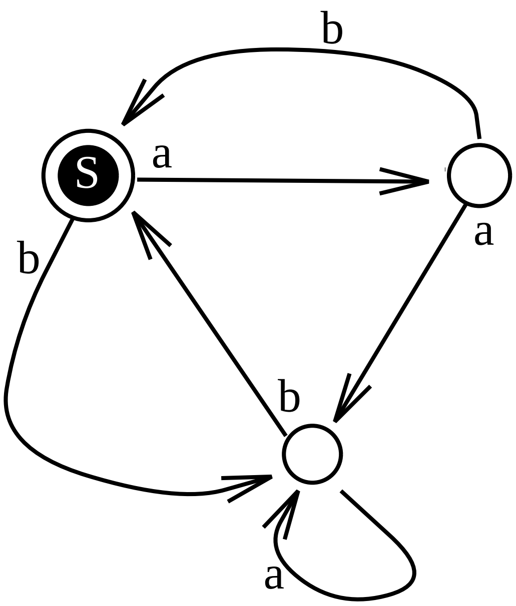


Figure 5.5: This finite state machine accepts and rejects the same inputs as the finite state machine in Figure 5.6.

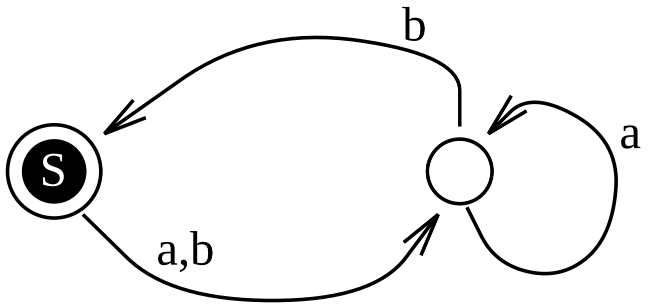


Figure 5.6: This finite state machine accepts and rejects the same inputs as the finite state machine in Figure 5.5.

Finite and Infinite

The finite state machine gets its name because it has a **finite** number of **states** (circles). Even if there are very many of them, you can count them, and you can say exactly how many there are. The same thing is true for the number of letters in its alphabet. The finite state machine in Figure 5.1 will accept an **infinite** number of words. We could work for a long time, get very tired and still never finish the rather silly task of completing just this part of the list of words that this machine will accept:

ababb, abaabb, abaaabb, abaaaabb etc.

Not only that, but there is an infinite number of inputs that you can invent that the machine is sure to reject! The concept of infinity is explored further in the chapter **Welcome to the Hotel Infinity**.

5.3 Guess My Language

Description

Students try various inputs for a finite state machine that they do not see. They are told whether their inputs are accepted or not, and from this must deduce what the finite state machine they are dealing with must look like.

Materials

- Paper and pencil

Preparation and Set-Up

Optional: Instead of having students make the finite state machines that they will use for guessing, you can make them up ahead of time.

Instructions

1. Divide the students into an even number of groups with about 4 or 5 students in each group.

2. Have each group draw a finite state machine of their own design. Tell the students not to let people in another group see their machine or overhear them discuss how it is made. To make sure that the finite state machines are not too elaborate or difficult to guess at first, it is a good idea to place some restrictions, such as a machine with four or fewer states whose alphabet has two letters.

3. Have each group pair up with a different group.

4. One group will be the guessing group and one group will be the machine group. The machine group tells the guessing group what the letters are in the alphabet of their machine, but do not show the machine to the people who are guessing. The guessing group writes down words which it passes to the machine group. The students in the machine group tell the students who are guessing if the word they submitted was accepted or not. The students in the guessing group try to figure out the finite state machine without being able to see it.

5. When the students in the guessing group think that they know what the finite state machine is that the others have made, they might draw a machine that looks different from the one that the students in the machine group had in mind. The students in the machine group cannot reject their guess out of hand. Sometimes two different machines

will accept all the same inputs. The people in the machine group will have to study the machine that the other students drew carefully. If it will always accept and reject the exact words as their machine, the guessing group has guessed successfully.

6. When one group has guessed the machine of the other, the groups reverse roles.

   NOTE: Students can also reverse roles if the guessing group gets too tired or frustrated and needs a break. Sometimes it is very hard to guess what the machine should be. Machines that haven't been guessed can remain as open or unsolved problems for the whole class to work on.

Discussion

1. What strategies did you use to try and guess the machine?
2. What things are easiest to figure out?
3. How did you figure out what the machine was like?
4. What did you have to do if the guessing group guessed a machine that didn't look like yours?

5.4 Babette and Babuie, Pizza Producers

In this story, the actions of the character Babuie can be explained in terms of a finite state machine.

It is early morning in kitchen of Babette and Babuie's Pizza Parlor. Since he woke up, Babuie has been busy preparing the kitchen for pizza production. He has scrubbed the floor, arranged the containers of pizza ingredients in a neat even row on the counter which he has polished to a gleaming shine. All the dishes are washed, and the pizza crusts have been placed in their pans and stacked high, waiting for Babuie to take them off the stack one by one, pile them high with pizza ingredients, and bake them in the Pizza Parlor oven.

Babette sat at the little table in the corner, finishing up reading the morning paper. "Babuie," she said, "I've been thinking...."

Babuie glanced in her direction as he reached for the first pizza crust. He dipped into the vat of tomato sauce with a ladle and spread the thick red paste over the top of the crust.

"If it rains today, Babuie, I'm not going out, and that's that!"

Babuie sprinkled cheese over the top of the pizza and popped it into the oven.

"Babuie, I have been thinking..." Babette began again. Babuie reached for the next pizza crust and smeared it evenly with tomato sauce. "Whatever are you doing?" she cried. "Why are you making another pizza? No one has even come to eat the first one yet."

"People will come. Someone will eat them," responded Babuie, reaching for the pepperoni and arranging the small flat circles neatly on the surface of the pizza.

"I suppose. Do you want some orange juice?"

"Sure," said Babuie, sprinkling cheese over the top of the pizza.

"Oops! Well, that's that," said Babette, as the last drops of orange juice fell from the rim of the pitcher. "It's all gone. One of us will have to go to the store." Babuie slid the finished pizza in the oven.

Babette folded up the newspaper she had been reading and placed it on the table, "Babuie, I've been thinking..." Babuie glanced at her, but didn't say anything. He just reached for the next crust and spread tomato sauce on it as Babette continued. "You are right. It probably is a good idea to get lots of pizzas made in advance, even in nobody is here yet. Look, the sky is already clearing up. If it doesn't rain, business will be good."

Without looking up or saying anything, Babuie reached for the cheese and sprinkled an even layer over the tomato sauce.

"Of course," Babette continued, "if it doesn't clear up and rains a lot, people won't want to be out walking on the street, and they will come inside. You're right, make lots of pizzas today."

Babuie sprinkled another layer of cheese onto the pizza.

"Except, well, maybe not. People who come in out of the rain aren't here to eat, they might just drink coffee and pop all afternoon. Then we won't need so many pizzas."

Babuie glanced at her, but didn't say anything, just heaped another layer of cheese on the huge mound already on the pizza.

"Well, it's not raining now," said Babette, as Babuie added cheese so that the pizza looked like a yellow mountain, "so I'd better run off to the store for more orange juice."

Babuie turned, walked to the other end of the counter and began to sprinkle chocolate chips on the pizza.

"I'll pick up whatever else we need, too. What would that be?" Babuie was carefully arranging pepperoni on top of the chocolate chips. Babette ticked off the rest of her list on her fingers, "Pickles, marmalade, shoelaces, soap, and that's that."

Babuie stepped back and looked at the pizza he had just made. "This is unacceptable!" he cried, and tossed it into the garbage. While Babette was gone, he tidied up the kitchen. The empty pizza pans were all washed up and the counters shining clean when the door slammed and he could hear the rustle of grocery bags.

"Babuie, I've been thinking," called Babette from the vestibule.

"Here we go again," murmured Babuie, reaching for a pizza crust and spreading the sauce neatly and evenly all over it.

Babette popped her head into the kitchen door. "Care for that orange juice now?" she asked.

"Not right now," said Babuie, adding a layer of chocolate chips to the pizza.

"I don't know how you can pass up a good glass of orange juice," marveled Babette. "I could drink it all day. That's not such a bad habit, though. It's good for you."

Babuie nodded and sprinkled cheese on the pizza.

"That's what was on my mind. Maybe we should open an orange juice shop." Babuie looked puzzled, he arranged pepperoni on the pizza and waited for her to continue. "You don't even like pizza. You just make them because that's how we earn our living. But orange juice..."

Babuie nodded. It was true that he never ate the pizzas he made. He sprinkled cheese on top of the one in front of him.

Babette tapped the glass of orange juice that she was holding. “I tell you. Rain or shine, this is my favorite drink.” Babuie sprinkled on another layer of cheese. “If it’s my favorite drink, other people probably find it irresistible, too. They’d all come here to get it. They’d order refill after refill. You wouldn’t have to make all those pizzas, and we’d still earn our living. It’s a great idea, and that’s that.”

Babuie looked at the pizza, gently lifted it up and carried it to the oven. He opened the door and popped it in.

It was quiet in the kitchen for a long time. Babuie sat down to rest at the table with Babette. In the silence he daydreamed about a life where he didn’t have to make pizzas ever again. But serving orange juice all day—would that be any better? He was dreaming of all the kinds of things he would rather do, when Babette spoke.

“Babuie, I’ve been thinking...” Babuie jumped up, grabbed a pizza crust and ladled tomato sauce onto it with a splat.

“I have an idea about how we can advertise our Orange Juice Shop.” Quickly, he tossed some chocolate chips over the tomato sauce.

“We can make up little jingles and rhymes. Like this: Whatever people think. It’s my favorite drink.”

“I don’t know, Babette,” began Babuie, arranging neat circles of pepperoni on the pizza.

“No, listen, it’s great,” she insisted. “How’s this: Whatever people say, I drink this stuff all day.” Babuie was not impressed. He put more pepperoni on the pizza.

“Or this,” she said excitedly. “Rain or shine, I’m gonna drink it all the time.” Babuie did not look pleased. He put chocolate chips on the pizza.

“Whatever you want to say is fine...” began Babette in a sing-song voice. Babuie glared at her and dealt out rings of pepperoni like he was dealing cards on the top of the pizza.

“...orange juice is the drink that’s mine!” she finished triumphantly. Babuie growled and threw a handful of cheese on the pizza.

“Whatever others rant and rave about....” Babuie picked up the ladle and put a glop of tomato sauce on top so that now the pizza looked like a bloody volcano. “...orange juice is what will make me jump and shout!” Babette clapped her hands. Babuie scooped chocolate chips onto the pizza and glared at her.

“Enough, Babette. No more. That’s that. Look at this pizza! It is absolutely unacceptable!” He threw the pizza into the garbage. “Talk, talk, talk. All this crazy talking that you do distracts me from my work. Just be quiet, will you?” he shouted. “Be quiet and let me make pizzas.”

Babette folded her hands and sat still. Babuie stared at her like he didn’t really believe she wasn't going to move. Babette didn't make a sound. Babuie grabbed his dishcloth and wiped off the counters, then whirled to took at her. She hadn't moved a muscle. She looked at Babuie and began to pick at a fingernail. Babuie shined up one more spot on the counter, then started to sweep the floor, slowly, slowly. As he passed the window, he looked out. He watched the street where no one passed by for a very long time.

Finally Babette took a deep breath and said, "Babuie, I've been thinking." This startled Babuie. He rushed back to the counter, pulled a pizza crust off the stack and spread it with tomato sauce.

"Well, don't think," he snarled. "It makes you talk. You talk and I can't concentrate on making pizzas."

"Babuie, if I don't talk, you don't make pizzas."

"This is nonsense. You rattle one and on, making up rhymes about orange juice, and you think this helps me make pizzas?" He put a scoop of chocolate chips into the center on top of the tomato sauce. "Then you get into some kind of a dither because it's going to rain," he said, scooping up a handful of cheese. "Ker-blamm! That's the thunder," he shouted, "and here's a cloud of cheese about to make a storm!" He sprinkled it like rain on top of the pizza. He scooped up more cheese, "Perhaps you are worried about tomorrow's rain. Or yesterday's rain. Maybe it will rain next year. Maybe we have forgotten how many times it has rained already this year. Nonsense!' When he had finished speaking five huge handfuls of cheese had rained down on top of the pizza.

"Whatever it is you are thinking about, I wish you would keep it to yourself," he cried, then began spreading tomato sauce over the mountain of cheese.

He picked up the pizza, looked straight over to where she was sitting motionless, and whispered fiercely, "And that's that!"

He looked at the pizza in his hand and said, "This is unacceptable." He threw the pizza into the garbage and slumped into his chair.

After a long silence, Babette spoke quietly, "Babuie, I've lived here a long time with you, and when I haven't been talking, I've been thinking..."

Slowly, Babuie pushed himself up from the chair, got a pizza crust and spread tomato sauce all over the surface.

"I've been paying attention to how my talking affects your work. My talking doesn't distract you, it inspires you—provided I talk about the right things. Rain for instance." Babuie continued working on his pizza, putting cheese on it next. "If I talk about rain, you work on making pizzas." He added more cheese to the pizza. "And you work on making pizzas when I talk about orange juice. It's a good thing I like it so much." He put a layer of pepperoni on the pizza next. "Whatever else I talk about doesn't matter." He put more pepperoni on. "Grape juice, thunderstorms, arithmetic, politics, snow, sleet, hail, cardboard, wheelbarrows, I've tried them all."

Babuie stopped working and turned to face Babette. "Street lights!" she cried, and he puts his hands on his hips. "Elephants!" hissed Babette. Babuie just glared at her. "Macaroni! Rumbas! Polkas! Computers! Hairdos!" Babuie was so angry he looked like he was going to explode.

"Orange juice," she whispered, and he sprinkled a layer of cheese on the pizza. "It gets the job done. And that's that." Babuie looked a little confused, but he put the pizza in the oven.

"Babuie, I have been thinking," continued Babette, "about why you ruin so many pizzas." Babuie was spreading tomato sauce on the next crust, but he was listening. "I don't know if your ruining pizzas has anything to do with what I say or not. But I'm sure it must. I talk about orange juice, rain, and whatever else I might be able to slip in to keep it from being completely boring. And sometimes you decide the pizza is unacceptable." Quickly, Babuie put two scoops of chocolate chips on the pizza, and added pepperoni. "And once you decide a pizza is unacceptable, there is no changing your mind. You throw it in the garbage and that's that!"

"Unacceptable!" cried Babuie. "Just like this one. All this nonsense. Look what you have made me do! Into the garbage with you!" And he tossed the pizza into the trash can, which was getting rather full that day.

"Well, Babuie, that's what I've been thinking about," said Babette.

"Well stop thinking! Get out of here and leave me alone."

Babuie spread tomato sauce on the next pizza crust. When he heard Babette close the door, he marched over and locked it behind her. He came back into the kitchen, sat down at the table, and started to read the newspaper.

"Well, I'm not used to it being this quiet," he muttered. "I'll listen to something normal for a change." He flips on the radio, picks up his dish cloth and begins wiping the counter. He has tuned into the middle of a news report.

"...and in the south, citrus producers are alarmed about the heavy rains that have been falling with no end in sight..."

Babuie whistles cheerfully and sprinkles a generous portion of chocolate chips on the unfinished pizza on the counter.

"...if the blossoms are damaged, the crop will be smaller than expected. We will experience this as a steep rise in the price of orange juice in about six months..."

Babuie danced happily over to the cheese container, and sprinkled a nice even layer all over the chocolate chips.

"...and that's that for the news. Now, on to more music..."

Humming to himself, Babuie picked up the pizza and put it in the oven.

"Ha, ha, Babuie. You do your best work when you are all alone, don't you. Yes, Babuie, old friend, I've been thinking." He snapped his fingers and pulled the top pizza crust off the stack. As he spread it with tomato sauce, he said out loud, "Other folks have plenty of ideas about what you do, but you're the one who knows the most about it. Heh, heh, heh."

He picked up the cloth and shined up a couple of spots on the counter. Then he straightened the containers of tomato sauce, cheese and chocolate chips so they were in a perfect straight line. He washed all the dishes. Then he washed all the windows, and the floor. He shined the water faucets, and the cupboard doors. He dusted off the chair at the table by the window and sat down in it. He is probably sitting there still.

Chapter 6

A Logical Look at Brights and Braves

6.1 Introduction

Overview

Sometimes we don't think we have enough information to solve a problem, but in reality we do. We can use logical analysis to pin down information which, although not explicitly stated, can be inferred from what we know.

Students perform a play which takes place in a school where some of the students always lie and the rest always tell the truth. Terry, the protagonist, is trying to find out which students are which, but at the beginning, there is no way of knowing whom to believe. At first it seems like there is never enough information, but in each case, it is possible for Terry figure out more about the students that might have seemed possible.

Big Ideas and and Key Concepts

• Logic • Proof • Metacognition

NCTM Curriculum Strands

• Mathematics as Problem Solving • Mathematics as Reasoning • Mathematical Communication

Vocabulary

• true • false • conjecture • logic • deduction • proof

Materials and Preparation

- The play can be performed in the classroom with the following props:
  - Desks and chairs.
  - Secretary's desk, located near the door to the principal's office.
  - Signs with the characters names that they will pin to their shirts (members of the audience must be able to read them).
  - Notebook for Terry, and a pen or pencil.
- The names of the characters can be changed if students would like to use their own names. The role of "Terry", the student who is seeking information about the other students can be divided up and played by several students.
- For younger or more inexperienced students, or as an introductory activity, a single interaction between Terry and two classmates can be excerpted, examined and discussed in detail.

Evaluation

The goal of this activity is for students to see that often you can find things out when it seems that there is insufficient information, but to do so requires very careful thinking. Knowing correct answers (in this case, the identity of the Brights and Braves) is not nearly as important as having some idea about how to go about how to start looking for them. It is more important for students to experience the delight in the flash of insight that reveals a solution, than to be pressed to "get it". Remember that the goal is for students to appreciate and enjoy the fruits of logical deduction, not to be frustrated by the structures that they must impose on their thinking.

Often, it takes a long time for students to catch on to the tricks that reveal the reasoning processes they need. Students who have trouble should be encouraged, because they *will* eventually be able to follow and discover the argument. But everyone should recognize that insight arrives on its own timetable, and sometimes the best way to invite it is through a combination of plodding and patience.

True, False, and the Mathematician's Logic

In many of the other activities of this book, students make mathematical statements which, after numerous trials or examples, seem entirely plausible. Even so, these statements often cannot be considered **true** from a mathematical standpoint because they have not been rigorously proved. A rigorous proof does not have to be a series of inscrutable statements filled with arcane terminology and unfamiliar symbols. It need only be a clear description of why something has to be true. A proof must take into account all possibilities.

The experience of thinking that something must be true, but not being able to actually prove it "beyond the shadow of a doubt", may lead students to believe that mathematical thinking places unnecessary restrictions on drawing conclusions about the world. The contrary is actually true.

Because of such stringent criteria for establishing and accepting something as true, there is no danger that a revolutionary new idea will appear that shakes the foundations of mathematics, causing mathematicians to have to reject things that they once though were true and start over. New discoveries in mathematics—new truths—are added to what is already known. New ideas may open up whole new fields of inquiry, but they will never cause anything to be removed from the body of knowledge that mathematicians recognize as true.

The play in this section illustrates another aspect of the power of mathematical rigor. We fill our everyday conversation with plenty of information about what we are describing so that our listeners can understand it the way we do. But in mathematics, when we begin asking questions about a collection of objects that we have invented, there are no rules of ettiquite that say these objects should be polite and generous about giving information the way that people are. The situation is more like the one at Unusual School.

Mathematicians have to make the most of every little piece of information that they have. A common strategy is to try to find out exactly what the possibilities are, and to try to eliminate the ones that cannot be solutions one by one. Sometimes the possibilities are infinite, or not all of them can be eliminated, and mathematicians must use other strategies. But the strategy of listing possibilities and eliminating them one by one will work well for figuring out who's who at Unusual School.

6.2 Unusual School: A Logical Place to Make Friends

Description

Students rehearse and learn their parts in a play where the characters' statments can only be interpreted if you think logically about them. By playing the roles of the people making the statements, students explore the logical possibilities of the statements that they make. Students will appreciate and enjoy the power of logical deduction.

Materials

- A copy of the play for each student
- The props that the play requires for the performance

Instructions

1. Explain that the class is going to perform a play about people called Brights, who always tell the truth and people called Braves, who always lie. Everyone in the play is either a Bright or a Brave, and either *always* tells the truth or *always* lies. When we start out, however, not even the people in the play will know which kind of person they are.

2. Distribute scripts, assign parts and have students begin rehearsing.

3. When students rehearse, help them to recite their lines accurately and not leave out any details. Sometimes a particular twist of wording is essential. Embellishments must be added with caution. Students can add them, but only after they can analyze them to be sure the changes don't alter whether people are Brights or Braves.

4. Have students perform parts of the play for one another so that they can discuss and analyze them together.

5. Help students see that in each interaction there are only four possible cases—both students are Brights, both students are Braves, the first is a Bright and the second is a Brave, or vice-versa. Students can then try out the assumptions one at a time. For example, begin by saying, "OK, let's imagine we are both Brights...". Then examine the content of each statement—was that true? Could a Bright have said that? Sometimes it is useful to make a chart or table of the possibilities.

6. It make take quite a few days for the students to catch on to ways to find out who the Brights and Braves are. Give them time, and try to assist them in structuring their thinking if they become frustrated.

7. It's not uncommon for students to lose solutions as fast as they find them. Help students to make tables, write down statements or use other tricks to remember how they figured something out. Logical structures are often intricate and hard to hold in our minds all at once. Encourage students by assuring them that it gets easier with practice. If you discover an answer and then lose it again, it will be much easier to discover the second time.

8. We often think of people who lie as being bad people. We may expect them to be rude, unfriendly, or disobedient. This is not the case at Unusual School, however. Make sure that students realize that they can only draw conclusions about whether people are Brights or Braves based on **statements** that are made and not any other part of their behavior.

9. When the need arises, help the students make the distinction between a true statement and a false statement that allows us to conclude something that we know to be true. Braves do not give bad information or try deliberately to mislead you. Braves give direct answers to the questions they were asked, and the statement that they use to answer the question is always false. For instance, if you know that someone is a Brave and that person points to someone else and says, "She's a Bright," you can conclude that the person she has pointed to person is a Brave. This is an example of how true conclusions can be drawn from a Brave's false statement.

10. Sometimes when students are trying to help one another understand how they drew a conclusion they can be frustrated when another student doesn't follow their chain of reasoning. Help students realize that "argue" has a special meaning in mathematics. They are not arguing about opinions like we often do in discussions and conversations. They are presenting logical arguments that are either true, false, or inconclusive, based on the logic of the statements. When someone does not accept your conclusions, it is either because they have found a flaw in your in your reasoning, or because they are having trouble following it. This happens often when mathematicians present logical arguments to one another, and there is nothing wrong with not understanding—even though the person presenting may find it frustrating at the moment. Sometimes the person who does not understand needs to have it explained in a different way, and sometimes they just need time to think about it.

Discussion

Discussing the logic puzzles embedded in the play is inseparable from the preparation for the play, because the students will need to talk to one another, ask questions and explain their interpretations of what is happening. Unlike other activities where discussion debriefs and reviews what students have learned, the performance of the play is the culmination and celebration of the fruits of all the discussion that went into its preparation.

The following suggestions for discussion will help students extend some of the logical deductive skills that they developed by performing the play, and also think about and gain insights into their own thinking and problem solving process.

1. Did you think at first that you would be able to figure out what was going on in the play? How did you start to get ideas for how to figure it out? Did you have one strategy that worked better than others. Which conversation was the one that you figured out first? Was that the easiest one?

2. Can you think of questions that Terry could have asked to make the job much easier?

3. Invent some other conversations that Terry could have with students that would reveal whether they were Brights or Braves. Add them to the play for future performances.

4. Each of Terry's conversations was with two other people. Invent some conversations that Terry could have had with three people that would have revealed their identity.

5. An adaptation of a famous puzzle in logic to Unusual School would have Terry lost in the hall, and not knowing if any of the passersby were Brights or Braves. What is *one* question that Terry could ask to anyone to find out whether to go to the right or to the left to find the cafeteria?

6. The play would be quite a bit more complicated if the students names were not pinned on their clothing. How would you change the script so that another thing that Terry has to find out from all of the Brights and Braves is what their names are? Try not to make it too easy, or to provide any more information than you need to.

7. Perform the play for a different group of students. As a group, plan ahead of time how to talk over the play afterwards with the audience. Simply telling them who's who at the end of the play will not be satisfying for them, they will want to be sure, and to do this they will have to figure things out for themselves. Have the members of the class plan how they will help their audience understand the play. After performing the play and helping their audience understand it, have them think about the other students' experience and compare it with their own.

   - Did the other students experience some of the same frustrations that they did?
   - What strategies did the other students find most useful for understanding the play?
   - Did all of their plans for helping the other students understand work? What would they do differently another time?

A Usual Day at Unusual School

A three-scene play for up to 22 actors

Scene I: At the secretary's desk outside the principal's office.

The secretary is seated at her desk. On her shirt is a big sign that reads 'Ms. Smart'. Terry's name is on his shirt. The Principal's name, Mr. Bumbleton, is on the front of his shirt, too. Terry and the principal shake hands in the doorway to his office. Terry leaves. Mr. Bumbleton stands in the doorway and listens to the conversation between Terry and Ms. Smart.

Terry: Boy! Was that ever confusing. Are you going to explain the rest?

Ms. Smart: Certainly. I think this is what he wants me to explain to you: You probably already know that here at Unusual School each person either tells the truth or lies. All the time. We call the people who always tell the truth Brights, and we call the ones who lie all of the time Braves. Since you want to come to this school...

Terry: Well, I just moved here. I don't think I *want* to come to this school. I just moved into the district. I *have* to.

Ms. Smart: Well, sure, okay. Since you are here, the first thing that you have to do when you come to this school is figure out who all the Brights are and who all the Braves are. Then it's not confusing at all to be here.

Terry: That's simple enough. I'll just ask them.

Ms. Smart: Of course. Just remember that whatever they tell you, if it is a Bright talking, the statement is true, and if it is a Brave, whatever it is they say is false.

Mr. Bumbleton: If you believe everything she says, you will make mistakes. You can't get reliable information from just anyone in unusual school–especially from this person. She lies!

Ms. Smart: Well, er, yes. Of course, Mr. Bumbleton, of course you would remind him of that. Good luck, Terry.

End of Scene I

Scene II. In the classroom.

Students are doing various things all around the classroom, standing together, talking in groups, drawing on the chalk board, looking at books, etc. Students turn and look at Terry. Some of them point to him, but no one comes over and talks to him. All of the students wear signs on their shirts with their names on them. The teacher comes to the front of the room. The sign on her shirt says, 'Ms. Boss'.

Ms. Boss: Students, please. I don't want to see anyone sitting in their seats right now. And I want everyone to all talk at once.

Students move quickly to their seats and sit down quietly. The desks are arranged so that the students sit in pairs.

Ms. Boss: As most of you haven't noticed, we do not have any new students in our class. His name is not Terry, and he does not want to get acquainted with you. I don't want you to answer any of the questions that he asks you. Would everyone please close their books and stop reading.

Terry: *(to audience)* Oh brother. I thought just the kids were Brights and Braves. But clearly this teacher must be a Brave. What about Mr. Bumbleton and Ms. Smart? I have a lot to figure out. I better get started.

(Approaches the desk where Jan and Mike are sitting.

Terry: Boy, this sure is a hard project I have to do here. Can you help me out at all?

Jan: Braves always say that they are Brights.

Terry: That makes sense. I'll remember that.

Mike: Well, if I was a Brave, I wouldn't say I was a Bright.

Terry: *(scribbles something in a notebook)* Gee, you two, thanks a lot. *(Moves over to the desk that Pat and Denise share.)* Say, I was wondering if you could help me with ...

Pat: She says she's a Brave. Don't believe her.

Denise: Oh you lie all the time!

Terry: *(writes in the notebook)* Hey, thanks a lot! Talk to you later. *(Continues on to the desk where John and Emily are sitting.)* Well, who do we have here?

John: One Bright, one Brave.

Emily: Don't believe it. He always lies.

Terry *(writes in his notebook)* Thanks for the help. Catch you after while. *(Moves on to Billy and Susan's desk.)* So, who sits here?

Billy: Exactly one of us is a Brave.

Terry: And which one is that?

(Billy and Susan both speak at the same time.)

Billy: *(pointing to Susan)* Her.

Susan: *(pointing to Billy)* Him.

Terry: I see. *(writes in his notebook)* Thanks a lot. See you later. *(Terry goes over to the desk where Sandra and Pete are sitting. Neither one of them look up from their books.)* Ummm, I wonder if I could ask you two a couple of questions.

Sandra: We're busy. Go away.

Terry: Oh, no, you don't understand. You see, I'm new here, and didn't you hear the teacher when she said that....

Pete: Didn't you hear what Sandra said. She said she doesn't want to talk to you. Go away!

Terry: Oh, please, it won't take but an instant. I'm new here, and I really have to find out whether the two of you are Brights or Braves.

Pete: OK, OK, then, we're both Braves. Now leave us alone!

Terry: *(writes in notebook)* Whew! That was a hard one. It's been, uh, nice, uh, I mean, maybe we can talk again sometime. Later. *(Terry leaves and goes to the desk of Dan and Alice.)* Hi there, uh...

Alice: We're both the same kind.

Terry: I wonder what I am supposed to make of that.

Alice: If I am a Brave, then so is he.

Terry: Oh, I see. Thanks. *(Writes in the notebook.)* Now, whose next? *(Goes over to Mark and Pam.)* Hi. Which of you two is the type that would claim to be a Bright?

Mark and Pam: *(They both raise their hands.)*

Terry: Oh, I see. *(Writes in notebook.)* Thanks, you two. I guess there's only a few people left to check with. *(Goes over to the table with Tony and Becky.)*

Terry: Good morning! Can you tell me anything about yourselves?

Tony: She's the only Brave at this desk.

Becky: No, he is.

Terry: *(writes in notebook)* Thanks an awful lot, you two. Only one more desk to visit. *(Walks over to Tom and Connie.)*

(Tom and Connie answer exactly together to each of Terry's questions.)

Terry: Are you Brights?

Tom and Connie: Yes!

Terry: Are you Braves?

Tom and Connie: No!

Terry: Are you lying?

Tom and Connie: No!

Terry: Is your partner lying?

Tom and Connie: Yes!

Terry: Is somebody lying?

Tom: No!

Connie: Yes!

Terry: *(writes in notebook)* Well, I think that does it! It wasn't as hard as I thought it would be. This school is pretty fun. I think I want to stay here for sure. I wonder what I am supposed to do next.

Ms. Boss: Terry, I don't want you to come over here, and I don't want to see that list either.

Terry: Ok, sure. *(He closes up the notebook and starts to sit down, then pauses...* Oops! I mean, I'm sorry. *(Hurries over to Ms. Boss and shows her the list.)*

Ms. Boss: *(Looking at the paper.)* This is completely wrong. You shouldn't go back to the office with it. I don't want you to listen to what I say, because it is not important. You cannot influence whether you will become a Bright or a Brave. If the next thing that you say is true, from that moment on you will be a Brave. If the next thing that you say is not true, you will become a Bright. Now, you must stay here and not go back to the office.

(Terry pauses and thinks. Then he leaves the classroom.)

End of Scene 2.

Scene 3: In the school office.

Terry enters and hands a piece of paper proudly to Mr. Bumbleton.

Terry: In this building, there can only be 11 Brights and 10 Braves.

Mr. Bumbleton: And I see you have not yet decided which one you will be.

(Terry and Mr. Bumbleton laugh and shake hands triumphantly.)

End of Scene 3.

The End

Chapter 7

Algorithms and Ice Cream for All

7.1 Overview

Where shall we locate ice cream stands in our town so that no one has to travel too far to buy a treat? The problem-solving strategies for this problem apply to many other situations that require planning for facilities. Students will have a chance to grapple with the notion of proof and decide what makes a solution satisfactory. They are also exposed to an important unsolved problem in mathematics.

Big Ideas and Key Concepts

• Dominating set • Algorithm • Computatonal Complexity • Minimality • Proof • One-way function • Infinity

NCTM Curriculum Strands

• Mathematics as Problem Solving • Mathematics as Communication • Mathematics as Reasoning • Mathematical Connections • Number Sense, Numeration and Number Relationships • Whole Number Computation • Spatial Sense • Representation

Vocabulary

• vertex • edge • degree • graph • dominating set • one-way function • algorithm • greedy algorithm • brute force algorithm

Materials and Preparation

- Copies of blackline masters for each student.
- Overhead transparencies of each blackline master

Evaluation

In these activities students are faced with a problem that has no straightforward, reliable method of solution. Finding the right answer is not as important as the experience of looking for answers and evaluating the quality of the answers that they do find. Some questions that may prove useful in evaluation student performance are: How frustrating is it for the student not to find a quick solution to the problem? What strategies does the student use to solve the problem? What makes the student abandon one strategy and try another? Can the student describe the strategies that were used and comment on their effectiveness?

The purpose of discussing proof and mathematical truth is to expose students to these notions, and to make them familiar with the roles that certainty and uncertainty play in mathematics. Although students should be learning to articulate their reasoning processes, they should not be expected to produce flawless proofs. They should recognize that the question, "Do we know this for sure?" is one that mathematicians continuously ask.

Mathematics in Maps and Planning

"What is the minimum?" is a very important question that mathematicians ask in many different contexts. The Ice Cream Stands problem presented in this section is one where, when you have found a solution, you must then ask, "Is this the *best* solution that I can find?" The notion of **minimality** is used as the basis for evaluating the quality of solutions.

The map of the town that is used in the activities is a collection of dots and lines that mathematicians call a **graph**. Graphs are described, along with some of the terminology that is used for talking about them in the chapter **Games on Graphs**. You may want to refer to that information if you are not already familiar with graphs such as these—they are made up of dots and lines, and unrelated to graphs that chart information with which you are undoubtedly familiar.

The technical term for an optimal solution to the Ice Cream Stands problem is a **minimum dominating set**. The problem itself is probably best understood in the form of the ice cream stands story itself. If the dots are houses and the lines are roads, then what is the smallest set of locations to put the ice cream stands so no one has to walk down more than one street to get to one? A set of dots or **vertices** where the ice cream stands go in a solution to this problem is called the **dominating set**.

A dominating set for a graph is easy enough to find. But ice cream stands are expensive to build—as are warehouses, "hub" airports, electrical switching stations, etc. So it is that many situations which involve planning the location of important facilities can be modeled by a graph for which we ask the question, "What is a *minimum* dominating set?" We want the one with the *smallest* possible number of members.

It turns out that for an arbitrary graph or map, there is no reliable step-by-step procedure that you can use to find a minimum dominating set, other than checking all the possibilities. When you find a dominating set that might be minimum, there is not always a way to be sure it is without checking all the possibilities for a better solution.

Is checking all the possibilities so bad? Couldn't you get a computer to do it? A computer can certainly check *one* possibility extremely quickly. This does not mean, however, that the computer can check all the possibilities in a reasonable amount of time. To know if that can be done, you have to know how many possibilities there are. The number of possibilities will vary with the size of the graph: the more vertices it has, the more possibilities there will be. But how many more? This question belongs to a branch of mathematics concerned with **computatonal complexity**.

The best way to try and figure out how many possibilities there might be is to look at some examples, beginning with some very small ones, and try to find some kind of trend or pattern in the way the possibilities grow with the size of the graph.

A graph with only one vertex is trivial: there is one possibility. For a graph with two vertices, you could place ice cream stands on both vertices, or on one or the other, so there are three possibilities. For a graph with three vertices, you have to count the number of possibilities for each quantity of ice cream stands that is possible to use. There is only one way to arrange three ice cream stands, of course. How many ways to arrange two? One?

For a graph with four vertices, you continue in the same way, only there will be even more possibilities. It takes a lot of careful counting to begin to get a feel for how the numbers grow. Generally speaking, the number of possibilities to check grows at a much faster rate than the number of vertices in the graph. This kind of growth is called **exponential**.

There are some problems that mathematicians and scientists would like to solve whose solutions are minimum dominating sets, but the graphs are large enough that it would take a fast computer years and years to produce a solution—longer, perhaps than the estimated age of the universe!

The fact that the Ice Cream Stands puzzle appears to be so difficult to solve allows us to use it to play with the idea of a **one-way function**. In the first activity, you can explore this idea by seeing how a person can create puzzles like these with the solution in mind first, but completely disguise the solution by the time the puzzle is finished. If you go from the solution to the puzzle, a good solution is always known to the maker of the puzzle. But you can't go the other way with such ease. (Another example of a one-way function is finding the prime factors of a large number. You can look up two large prime numbers from a table in a book, multiply them together and ask someone else to find the factors and they will most likely have a very hard time indeed!)

Of course this one-way trick isn't going to help someone who is trying to figure out where to locate the big and little computers in a large interconnected computing system, because no one created that problem with the solution already in mind! One-way functions are important, however, in **cryptography**, the mathematics of secret codes and secure passwords. If you had a clubhouse with a huge graph on the door, and only members knew what the minimum dominating set for the graph was, the minimum dominating set could be the password because the chances of someone else figuring it out would be slim.

7.2 The Ice Cream Stands Problem

Description

In this activity, a graph is used to represent the map of a city. (See **Games on Graphs**.) The problem that is explained in the story is simple enough, but, surprisingly, there is no simple, straightforward way to solve it. Students will find themselves experimenting with a variety of approaches.

Materials

- Copies of Figure 7.1 for each student.
- Overhead transparencies of Figures 7.1 and 7.2.

Instructions

1. Pass out copies of Figure 7.1 and present the following to the students:

   What you have in your hands is a map of a town. It's a somewhat unusual way to draw a map. The lines on this map represent streets and the dots are street corners. The map doesn't have any houses on it, but we do know that there is at least one house at each corner.

   This would be a nice town to live in except for one problem: you can't get ice cream anywhere in the town. The Icicle & Iceberg Ice Cream Company has decided to do something about that. Ivan and Ivana Icicle, the owners, want to do something good for their town, but they also want to make money. They are going to build ice cream stands all over town where people can go to get ice cream, and they want it to be easy for the people to them.

   At first, Ivan and Ivana had hoped to put an ice cream stand on every corner, knowing how, in the summertime, they would just rake in the money. But ice cream stands are expensive to build: you have to buy all that lumber, and nails, and windows, etc. Then you have to put big freezers inside them, and pay people to work in them all day, and so forth. There was no way they could afford to build that many ice cream stands.

   So they decided instead that people would still eat lots of ice cream if they only had to walk down the street to get it. Their second plan was to build the ice cream stands so that people could get ice cream either right there on the corner where they live, or at the very most, have to walk down only one street to find a corner where there was an ice cream stand parked.

   Now, all they have to do is figure out where to put the ice cream stands. Where should they go? How many do they have to build?

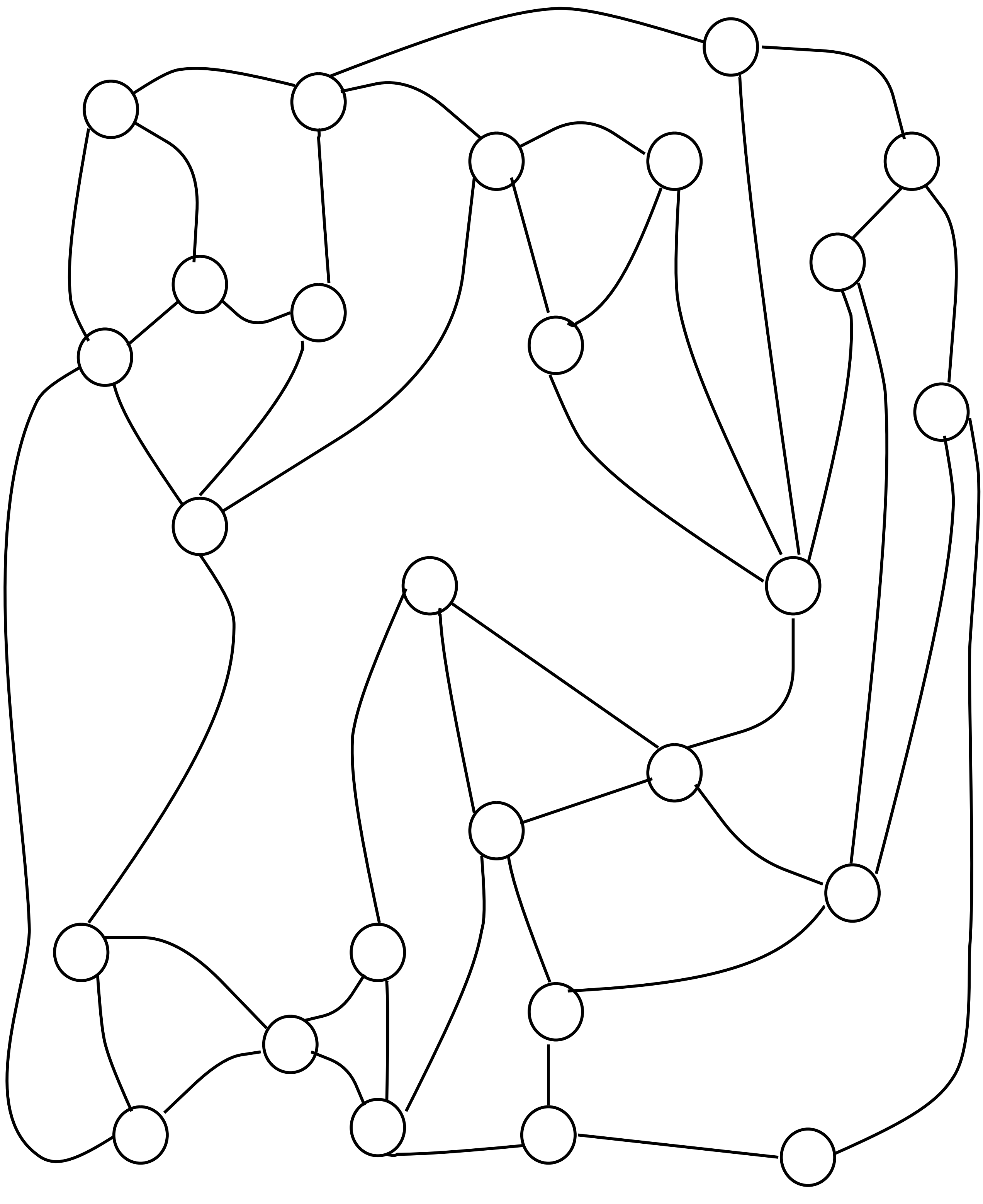


Figure 7.1: The map for the Ice Cream Stands puzzle

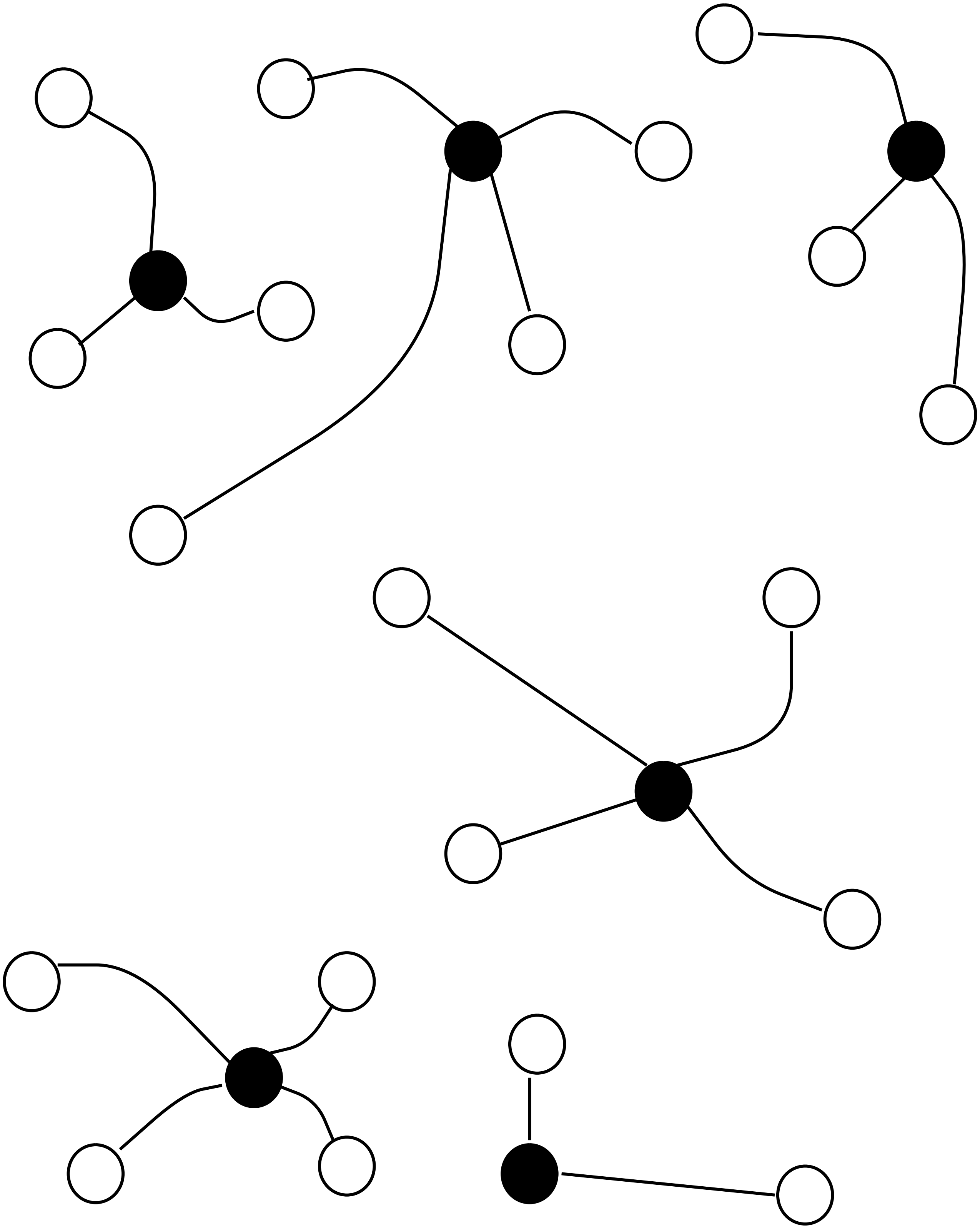


Figure 7.2: The solution to the Ice Cream Stands puzzle. First the solution is made, then the puzzle, not vice-versa.

2. Show the students how to use the unifix cubes to mark the places where they think that the ice cream stands should go. Place a marker of one color on a corner where you will put an ice cream stand. Then put markers of a second color on all the corners that are one street away from that ice cream stand. All the people who live in those houses will walk one street to get to the ice cream stand on the next corner.

3. Have the students experiment with their maps and decide where they think the ice cream stands should go. As students find configurations of ice cream stands that will serve all the houses, remind them that the ice cream stands are expensive to build, and that Ivan and Ivana want to build as few as possible.

4. You can tell students that it is possible to build only 6 ice creams stands and serve all of the houses in this town, or you can let them try to discover that this is the minimum on their own.

5. After students have had a chance to work on the puzzle and solve it, show them how it was made. Put Figure 7.2 on the overhead. Explain that you make a puzzle like this by drawing the solution first. It is easy to see what the solution to the puzzle is when looking at Figure 7.2. Then place Figure 7.1 on top of Figure 7.2 on the overhead. Almost by magic, the easy-to-see solution disappears. After drawing the solution to the puzzle, the puzzle-maker draws just enough disguising lines to make the solution hard to find.

6. Invite students to make a similar puzzle of their own.

7. It is not necessary to make puzzles with ice cream stands only. Perhaps the students would like to make a puzzle about something that is more interesting to them than ice cream. One way to begin is by asking, "What kinds of things do you think a town should have many of so that there is one close to everyone's house?" Students' ideas may include: convenience stores, video rentals, swimming pools, shopping malls, parks etc.

Discussion

1. What strategies did you use to try to figure out where to put the ice cream stands? How did you check to make sure that no house was too far away from an ice cream stand? Do you think it is possible to arrange the ice cream stands in a different way so that Ivan and Ivana won't need to build so many of them?

2. If you didn't use ice cream stands in the puzzle that you made, how did you decide what to use instead? How did you make your own puzzle? What happened when you showed your puzzle to other people? Were they able to solve it?

3. Do you think it would be possible to make a puzzle like this, and then come up with a solution that is even better (i.e., takes fewer ice cream stands) than the one you built into the puzzle in the first place? Try to make a puzzle where that happens.

4. Explain to students that this puzzle is an example of what mathematicians call a **one-way function**. If you start with the solution and create the puzzle it's easy. If you start with the puzzle and have to find the solution it's not. One way it's easy, one way it's hard. A place that one-way functions are useful is for inventing secret codes. Invite students to find other examples of one-way functions.

5. Ask students to think of other situations in real life that might present themselves as a puzzle like this. Some of their ideas might include: finding locations for warehouses, main and branch offices, electrical switching stations, telecommunication centers, airports, hamburger joints, hospitals, firestations, and restrooms. Remind students that when people are faced with problems like this in real life, someone hasn't made the puzzle ahead of time. Trying to find strategies for solving puzzles like this is important when you can't just say, "I give up. What's the solution?"

7.3 Brute Force and Other Algorithms

Description

An **algorithm** is a step-by-step method for solving a problem. This activity takes a closer look at two of the algorithms that can be used to solve the Ice Cream Stands problem. Students have a chance to think about ways to evaluate the effectiveness of an algorithm.

Instructions

1. Explain to the students that the reason why finding a solution to the Ice Creams Stands problem was so hard is that problems like this are very hard in general. A set of step-by-step directions for finding a solution to a problem is called an algorithm. No one has found a good (fast) algorithm for solving problems like the Ice Cream Stands problem.

2. Tell students that you are going to take a look at the best algorithm known for solving this problem. It's pretty simple: check every single possible way to put the ice cream stands. When you've done that, pick the best one. This is called a **Brute Force Algorithm**.

3. Ask students whether they think that this plan for solving the problem will work out well. Of course the answer depends on how many possibilities there are to check. Have students discuss how they could find that out. You will need an algorithm — a step-by-step recipe — for checking all of the possibilities and making sure that none are left out.

4. Have students design an algorithm for checking all the possibilities for locating ice cream stands in the town in Figure 7.1. How many possibilities will they have to check?

Discussion

1. Why do you suppose the Brute Force Algorithm was given that name?

2. Students are likely to suggest that the Brute Force Algorithm is acceptable because all the possibilities could be checked by a computer. Have the students imagine that they have a computer that can check one possibility in one second. With what size of graph or map does checking all the possibilities become impossible in a reasonable amount of time?

3. Another kind of algorithm that mathematicians often look for when trying to find one that will work for a problem is called a **greedy algorithm**. A greedy algorithm for the Ice Cream Stands Problem would be to first find the intersection or intersections that had the most streets coming into them, and put ice cream stands there. Then put ice

cream stands at the intersections with the next highest number of streets coming in to them, and so forth. How does this algorithm work for the Ice Cream Stands problem? Will this algorithm produce a *minimum* dominating set for Figure 7.1?

4. Have students review the strategies that they used when solving the Ice Cream Stands problem. Do their strategies suggest other algorithms that might work? For problems like this one, where there is no reliable "best way" to solve it, mathematicians usually try several kinds of algorithm just to see what kinds of solutions can be produced. Sometimes those solutions will give them ideas for other things to try. Sometimes they have to be content with the best solution they can find, never knowing if there is a better one.

5. If mathematicians know more than one algorithm that might work, another thing that they try sometimes is to switch back and forth from one algorithm to the other. Would an approach like that work for the Ice Cream Stands problem?

6. Using Brute Force to solve a problem involves checking all the possibilities. Is it possible to draw a map for the Ice Cream Stands problem where the number of possibilities you would have to check is infinite?

7.4 Proving that You Have Found the Minimum

Description

Sometimes it is possible to be certain that there is no better solution to the Ice Cream Stands problem than the one you have can be found. This activity explores some of the ways that students can find this out.

Materials

- Copies of Figure 7.3 for each student or small group of students.

Instructions

1. Have students look at the map of Twotown in Figure 7.3. If Ivan and Ivana Icicle were tyring to figure out where to put the ice cream stands in Twotown, they would probably not have too much trouble.

2. Have students figure out where the ice cream stands should go. Make sure the whole class agrees on a solution. (Students will see that Twotown is aptly named!)

3. Have the students explain why they are sure there is no solution to the Twotown puzzle that will only use one ice cream stand.

4. With contributions from all members of the class, write a statement on the board that explains why the solution for Twotown must have more than one ice cream stand. Will this prove that two ice cream stand is the best solution?

5. Have the students look at the map of Puzzzletown and solve the Ice Cream Stands Puzzle for this map.

6. Tell the students that, just as with Twotown, there will come a point when they are trying to reduce the number of Ice Cream Stands that they use, where they will see it is impossible to solve the problem with fewer Ice Cream Stands than what they have already used.

7. Have the class again collaborate to compose a clear explanation of why the solution they have found for Puzzletown is the minimum. The chain of reasoning to do this is more complicated than what was required for Twotown. Some students are likely to grasp the reason before others do. Remind students as they try to convince one another that they are trying to base their explanations on logical, true statements—even if they feel more inclined to impassioned persuasion.

8. Invite the students to try other dominating set puzzles to see if they can always be so sure their solution is the best one.

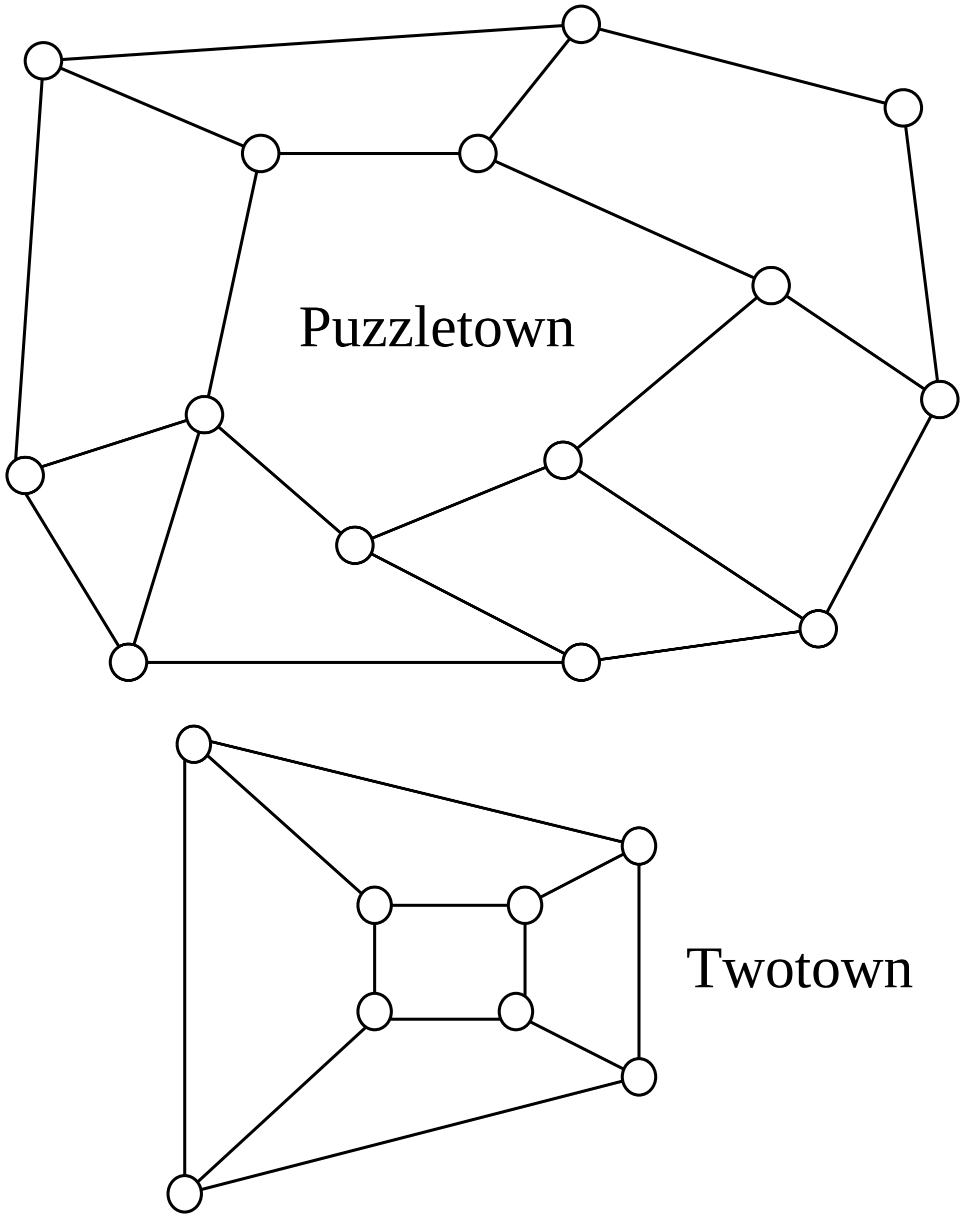


Figure 7.3: Some simpler maps for the Ice Cream Stands puzzle. It is easier to articulate the strategies for solving the puzzle by analyzing simpler versions of it.

Discussion

1. What was it that made you realize that two ice cream stands would be enough for the Twotown puzzle? Did you find it difficult to put your understanding into words? Did you need to invent any special words to make everything more clear? Were there phrases that you used that ended up being useful?

2. Most people find the Puzzletown problem harder than the Twotown one. Was that your experience? What could be the reason for this?

3. What was necessary to make it clear that you had found the Puzzletown solution that was the best one? How was this different from what you needed to show that your Twotown solution was the best?

4. To verify the Puzzletown solution, you have to hold a lot of information in your head all at once. Do you find that hard to do? What tricks did you use to help yourself do that?

5. Talk about how difficult it is to explain something to someone who hasn't understood it in the same way as you do, and how difficult it is to try to understand something that someone else is explaining to you. Make a list of ideas that would be good advice for everyone who is either struggling to explain or struggling to understand.

6. The original Ice Cream Stands Puzzle in Figure 7.1 can be solved using no more than 6 ice cream stands. You can demonstrate that this is true. It takes some experimentation and careful thinking, but it can be done. Discovering it will make you feel triumphant!

7. It would be nice if we could always be sure that our solutions to these problems were the best, but this is not the case. Do you have any ideas about what details in the problem will make it so that you can verify that your solution is the best one? What details in the map might make this difficult to do?

Chapter 8

Welcome to the Hotel Infinity!

8.1 Introduction

Overview

When turn-of-the-century mathematician Georg Cantor approached the subject of infinity with mathematical rigor, he encountered many paradoxes. At the Hotel Infinity, these paradoxes come to life.

Big Ideas and Key Concepts

• Infinity • Transfinite arithmetic • One-to-one-correspondence

NCTM Curriculum Strands

• Mathematics as Communication • Mathematics as Reasoning • Mathematical Connections • Number Sense and Numeration • Concept of Whole Number Operations • Whole Number Computation • Number Systems and Number Theory

Vocabulary

• infinite • finite • countable • uncountable • one-to-one correspondence • set • transfinite arithmetic • continuum

Materials and Preparation

This activity works well in combination with an art lesson. Skillful guidance on conceiving and producing illustrations, as well as for using the medium to be employed for the illustration will make the activity more satisfying and exciting.

Evaluation

Transfinite arithmetic is subtle and paradoxical. Students should be expected to enjoy the story and appreciate the paradoxes, but not resolve them. Appropriate responses to the paradoxes include bewilderment, arguing, and laughter.

Infinity is for Children—And Mathematicians!

Most everyone is familiar with the infinity symbol–the one that looks like the number eight tipped over on its side. The infinite crops up in everyday speech as a superlative form of the word *many*. How many is that? How big is infinity?

You can't count to infinity. Yet we are comfortable with the idea that there are infinitely many numbers to count with: no matter how big a number you might come up with, someone else can come up with a bigger one: that number plus one–or plus two, or times two. Or times itself. There simply is no biggest number. Is there?

Is infinity a number? Is there anything bigger than infinity? How about infinity plus one? What's infinity plus infinity? Infinity times infinity? Children to whom the concept of infinity is brand new, pose questions like this and don't usually get very satisfactory answers. As adults, we know that these questions don't seem to have very much bearing on our daily lives, so their unsatisfactory answers don't concern us.

At the turn of the century, in Germany, the Russian-born mathematician Georg Cantor applied the tools of mathematical rigor and logical deduction to these questions in search of satisfactory answers. His conclusions are paradoxical to our everyday experience, yet they are mathematically sound. The world of our everyday experience is finite. We can't exactly say where the boundary line is, but beyond the finite, in the realm of the **transfinite**, things are different.

Cantor is the founder of the branch of mathematics called Set Theory, which is at the foundation of much of 20th century mathematics. At the heart of Set Theory is a hall of mirrors—the paradoxical infinity. Georg Cantor was known to have said, "I see it, but I do not believe it."

The **set** is the mathematical object which Cantor scrutinized. He defined a set as any collection of well-distinguished and well-defined objects considered as a single whole. A collection of matching dishes is a set, as well as a collection of numbers. Even a collection of seemingly unrelated things like, {toothbrush, elephant, toaster, 6} is a set. They are well-defined and can be distinguished from one another.

Sets can be large and small. They can also be finite and infinite. A finite set has a finite number of members. No matter how many there are, given enough time, you can count them all. Cantor's surprising results came when he considered sets that had an infinite number of members. Sets such as all of the counting numbers, or all of the even numbers are infinite sets.

In order to study infinite sets, Cantor first formalized many of the things that are intuitive and obvious about finite sets. At first, it seems like these formalizations are just a whole lot of trouble, a way of making simple things complicated. Because the formalisms are clearly correct, however, they provide a powerful tool for examining things that are not so simple, intuitive or obvious.

Cantor needed a way to compare the sizes of sets, some method for determining whether sets had the same number of members. If two sets didn't have the same number of members, he needed a method for telling which one was larger. Of course this is simple for finite sets. You count the members in both sets. If the number is the same, they are the same size. If the number of members in one set is greater than the number of members in the other, then that set is larger.

You can't count the members in an infinite set, though, so this method won't work for comparing their sizes. If you have two infinite sets, you need some other way to tell if one is larger.

The formal notion that Cantor used for comparing sizes of sets is the idea of a one-to-one correspondence. A one-to-one correspondence pairs up the members of one set with the members of another. We could pair up the elements of the ridiculous set {toothbrush, elephant, toaster, 6} with the numbers {1,2,3,4}. It is possible to do this so that one member of each set is paired up with one member of the other, no member is left out, and no member has more than one partner.

This may seem like a lot of work for small finite sets, but it provides a way to think about the sizes of infinite sets, so that children and mathematicians have a tool to look for answers to questions like: What's bigger, the set of all counting numbers, or the set of all counting numbers with a shoe thrown in?

The set of all counting numbers is $\{1, 2, 3, ...\}$. Is there a way to put its members in a one-to-one correspondence with the set { shoe, 1, 2, 3, ...} ? The answer is *yes*.

$$\begin{array}{cccccccc} \{ & 1, & 2, & 3, & 4, & \ldots & \} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\ \{ & \text{shoe}, & 1, & 2, & 3, & \ldots & \} \end{array}$$

Our intuition and experience with finite things in the world makes us want to protest that when we pair up the members of these sets, the first one, the set without the shoe, will run out of members before the other one does. But these are infinite sets. We will not get to the end. Neither set will run out of members. The pairing goes on and on. Ad infinitum.

In the story *The Hotel Infinity* the pairing of hotel guests with hotel rooms puts the members of a set of hotel guests in one-to-one correspondence with a set of hotel rooms.

Infinity plus one is infinity. Once you slip into the realm of the infinite, the rules of arithmetic are not the same anymore. The rules of **transfinite arithmetic** take over.

There are many other infinite sets whose sizes are interesting and surprising to compare. Consider the set of all even numbers {2, 4, 6, ...}. Is it half as big as the set of all counting numbers? We can try putting their members in one-to-one correspondence with each other to try and find out:

$$\begin{array}{cccccc} \{ & 1, & 2, & 3, & 4, & \ldots & \} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\ \{ & 2, & 4, & 6, & 8, & \ldots & \} \end{array}$$

We won't run out of even numbers before we run out of counting numbers or vice versa, so the members of the two sets can be placed in one-to-one correspondence with each other. This means that they are the same size. There are as many even numbers as there are counting numbers. Infinity times two is infinity. Infinity divided by two is infinity.

What about the set of all rational numbers–the numbers that we can express as fractions: {1/2, 3/4, 5/6,...} Part of the set of rational numbers consists of {1/1, 1/2, 1/3, 1/4,...}. These are all the fractions whose numerator is one. There are an infinite number of those. Then there are {2/1, 2/2, 2/3, 2/4...} These are all the fractions whose numerators are 2—another infinite set. We could make infinitely many such lists, each one with infinitely many rational numbers on it. So to figure out how many rational numbers there are, all we would have to do would be multiply infinity times infinity. Certainly that must be more than infinity. If it is, the size of the set of rational numbers will be bigger than the size of the set of counting numbers and we won't be able to place them into a one-to-one correspondence with one another.

When the all of the busses pulled up to the Hotel Infinity, fully loaded, with every one needing a room, it was as though all of the rational numbers showed up to a dance where all of the counting numbers were, and everyone started looking for a partner. If you imagine each person on the bus as having a bus number and a seat number, you can list the people on the first bus as {1/1, 1/2, 1/3,...} the people on the second bus as {2/1, 2/2, 2/3...} So if you can pair up each person on the busses with a room, you can pair up every rational number with a counting number. Could it be true that infinity times infinity is infinity?

Perhaps, thought Cantor, once you start dealing with infinities, everything is the same size. Oddly enough, this did not turn out to be the case. Cantor developed an entire theory of **transfinite arithmetic**, the arithmetic of numbers beyond infinity. The results are surprising. Although the sizes of the infinite sets of counting numbers, even numbers, odd numbers, square numbers, etc., are the same, there are other sets, the set of numbers that can be expressed as decimals, for instance, that are larger. Cantor's work revealed that there are hierarchies of ever-larger infinities. The largest one is called the **Continuum**.

Some mathematicians who lived at the time of Cantor did not want to accept his work at all. The fact that his results were so paradoxical was not the problem so much as the fact that he considered infinite sets at all. Some mathematicians at the time of Cantor held that mathematics could only consider objects that could be constructed directly from the counting numbers. You can't list all the elements in an infinite set, they said, so anything that you say about infinite sets is not mathematics. The most powerful of these mathematicians was Leopold Kronecker who even developed a theory of numbers that did not include any negative numbers.

Although Kronecker did not persuade very many of his contemporaries to abandon all conclusions that relied on the existence of negative numbers, Cantor's work was so revolutionary that Kronecker's argument that it "went too far" seemed plausible. Kronecker was a member of the editorial boards of the important mathematical journals of his day, and he used his influence to prevent much of Cantor's work from being published in his lifetime. Cantor did not know at the time of his death, that not only would his ideas prevail, but that they would shape the course of 20th century mathematics.

8.2 Illustrating the Hotel Infinity

Description

As students listen to the story *The Hotel Infinity*, they will try to picture how it looks and what happens there. Then they will make illustrations that can be displayed with the text of the story.

Materials

- Art materials for making illustrations.
- Optional: extra copies of the story.

Instructions

1. Explain to the students that you are going to read them a story, and that they should listen and try to make pictures in their minds of the things they hear about. Tell the students that they will be drawing the things that they see in their minds.

2. Read the story aloud.

   NOTE: Another way to present the story is for the character George, the narrator of the story to visit the class and tell the story. The advantage to doing it this way, is that George can interact with the students, invite interruptions, answer their questions, and ask them what they think as the story progresses.

3. Make the text of the story available to the students to refer to and reread.

4. Ask the students how they imagine the Hotel Infinity looks. If you visited there, what would you like to do? A few of the rooms are described in the story. What other things do you suppose the clever and imaginative Sam put in some of the other rooms? Encourage the students to conjure up details that are not in the story: what did the lobby look like? how were the hallways shaped? what did the busses look like? describe Kronecker's clothes and his truck.

5. Introduce the students to the medium you have chosen for them to use for their illustrations. Add to your presentation whatever you feel is necessary for them to make illustrations that they will enjoy and find satisfying.

6. Give the students plenty of time to work on their illustrations and complete them.

7. On a large and spacious wall, preferably one in a more public area than the classroom, arrange the illustrations interspersed with the text of the story.

8. Encourage the students to discuss and ask questions about the details of how the hotel works.

Discussion

1. Once the children have heard the story, no doubt they will have ideas and questions that relate to the mathematics behind the Hotel Infinity. Students may claim that no such place could possibly really exist. The narrator in the story, George, however reminds us that although he no longer remembers how the hotel or the busses that held infinitely many people were made, they did indeed exist, because he helped to build them and he worked at the hotel for many years.

2. Encourage the children to be creative and brainstorm about ways that the hotel and the busses could be put together. Since the Hotel Infinity is a place full of paradoxes, the following other paradoxes that involve the idea of infinity may help give students ideas for conceiving of how things worked:

   Zeno's paradox, or the Tortoise and the Hare. The Hare challenges a tortoise to a race. Since the hare can run twice as fast as the tortoise can, the tortoise isn't very interested. So the hare agrees to give the tortoise a head start that is exactly half the distance to the finish line. The tortoise agrees and they have the race. The tortoise plods along. When he gets exactly half the distance from where he started to the finish line, the hare, who runs twice as fast, has gone twice as far, and is just at the point where the tortoise began. The tortoise is still plodding. When he again covers half the distance to the finish line, the hare has reached the point where the tortoise was. This happens again and again. And again. And again. And the hare never does catch up with the tortoise? Or does he?

   The Great Galumphing Gnome. The Great Galumphing Gnome has the most unusual way of walking. He can start out with a step of any size. Sometimes that step is very little and sometimes that step is very big. Always, though, his second step is twice the size of the first, his third step twice the size of the second, the fourth step twice as far as the third, and so forth. (It's not known what happens when he walks backwards.)

   The Great Galumphing Gnome's Sister, Gert. Her steps are all the same size. Sometimes she takes her first step very quickly, and sometimes her first step is very slow. When she takes her second step, it is twice as fast as the first one. Her third step is twice as fast as the second one, and so forth. Can brother and sister ever take a nice stroll down the street together side by side?

   The Incredible Moving Sidewalk. This sidewalk is a conveyor belt of possibly infinite length. When you turn it on it goes very slowly, but every minute, it's speed doubles. How long does it take to get from here to there?

3. Another idea for imagining how the Hotel and the busses could have been made is to look at prints by M.C. Escher.

4. Students may want to experiment with ways that the Hotel Infinity was laid out. Early in the story George tells us that "it was all very organized, the way that the Hotel was set up, so even though the numbers got very large, the rooms were easy to find." Later in the story we learn that Hallway Number Eight leads to all the rooms whose numbers begin with 8.

5. George tells us that it was no problem to fit in all the people on the Continue 'Em Tours busses, even when 150 busses all showed up at once. Do you think George had to make the announcement that emptied all the odd numbered rooms 150 times?

6. Do you think that George's scheme for unloading the busses was going to work? You can test it out by using rows of some kind of markers to represent the first 10 or so people on the first ten or so busses. Remove the markers according to the scheme that George describes in the story. Does it seem that all of the people will "move off of the busses in an orderly way so that no one will get skipped, and no one will have infinitely many people in front of them"? Will that be sufficient to file them all into the odd numbered rooms?

7. Georg Cantor spent a lot of time thinking about sets of numbers, particularly sets that had infinitely many numbers in them, such as the counting numbers, the odd numbers, the rational numbers (which we express as fractions), the real numbers (which we express as decimals) and so forth. He was trying to understand the notion of sizes of infinities. See if you can get to the bottom of this paradox: Consider the set of all sets. Now consider the set of all sets who do not have themselves as members.

8.3 The Story of the Hotel Infinity

So! You want to know about Sam Cantori, his hotel and those busses. Let me guess! You want to know everything—all the exactly's and the how-so's. You only want the truth, you say. The whole truth—complete and exact. I can tell it to you.

A warning to you, though: if you only want to hear things that are easy to believe, why don't you just tell me now and save me the trouble of wasting my breath. And if you are the type that likes to say words like "impossible", or if you think you are going to interrupt and try to tell me that things I saw with these two eyes couldn't have happened, or things that I made with these two hands couldn't have been built—well, why don't you just get up right now and go swap stories with Mr. Kronecker. And when you get there, don't let him say a word until he tells you how he learned to talk. Ha!

But if you want to listen, I'm glad to tell you all the stories I know, and if you help me, I'll try to remember the rest. If we can get all the remembering done, maybe you could build it all back up again. Then you could have some of the fun—because, oh, it was fun we had at the Hotel Infinity.

It was a long time ago that Sam hired me. I think he picked me because I had already lived and worked a long time and could do many things. He didn't have to teach me how to build a hallway, or a room, or how to put in pipes for water, or wires for electricity. I could build with metal just as easily as with wood, and had worked on motors and engines since I was 9, so there was nothing in the putting together of the building or the busses that I couldn't do.

Being on the job with Sam was different than with any other boss I've had. I'd get a staircase or a hallway all framed in the way he said, then he'd check the blueprints he kept locked in the safe, and have me make tiny changes—a little twist for the steps, a slant on the ceiling, a bit of lean on the wall. None of his changes made any sense for anything except making a good building bad. But I did what he said. I did what he told me. None of this "Why do I have to do it this way, boss?" or "I think I know a better way, boss," or "This doesn't make any sense to me, boss." None of that. It was all "Yes, sir," and "Right away, sir," and "What next, sir?"

Sam Cantori paid me to do, he didn't pay me to think. Once I started thinking about what it was I had done, it came to me that I had done some things that were downright impossible, things I'd stand here today and tell you couldn't be done, except that I did 'em myself.

There didn't seem to be anything orderly about the way we worked either. One day it was floors, doorknobs, and fan belts. Next day he'd put me to working on transmissions, sinks, and wallpaper. If he tilted a mirror a little crooked, or had me raise one end of a step, he might then say, "Now go do that to all the rest of the busses and in all the rooms while I study the plans and decide what to do next."

All the time I worked, the questions I never asked built up in me somehow, because I remember the morning when I started to be different. I showed up for work, right on time like I always do, carrying my tools. And there was Sam in his office. All the blueprints were gone, all the plans, all the lists, all the sawdust, the nails, the screws, the paint, the grease, the sawhorses, the gears. The floor was swept clean as could be, and what a sight was Sam! His slick and shiny hair showed the tracks that the comb had left. His chin was clean-shaven, pink and smooth. The suit he had on was made of the finest cloth I'd ever seen or touched, and a cane dangled from his wrist. In one hand was a silk top hat, in the other a gleaming gold watch. He snapped the watch case shut, tucked it into his vest pocket and looked up at me as I came in.

"Sam!" I cried, "What's—"

He held up his white-gloved hand for silence, and motioned me over to his desk. He shoved two papers in my direction. They were drawings.

"Make these two signs, George. When you've got them hung up, put your tools on the pile with the other things out back. I'm going to call Kronecker to haul all that junk away. Then go to the tailor, the cobbler, and the barber. They're expecting you. No more of this backbreaking sweat and sawdust. You're the one who will greet all the guests at the Hotel Infinity!"

The moment I saw those drawings, something changed in me, and I turned into a questioning, attention-paying person. Oh, I didn't speak any questions out loud, but it was the first time an idea came into my head about not doing something exactly the same as Sam Cantori told me to do it. "Make these two signs and we're done," he'd said. Impossible!

One sign was the big one to go out front: *Hotel Infinity*, it said. *All rooms One Dollar*. That's a strange enough name, and a price sure to lose money, but that wasn't the problem. Sam was rich and only doing this for fun anyway, and he could name his hotel anything he wanted. It was the other sign, the one that said *Vacancy* that bothered me. Not too strange, just a little bit strange, like those crooked walls and bending hallways. But this was a different morning. For some reason I was thinking about things.

Now there's nothing wrong with a *Vacancy* sign that hangs under the name of your hotel to let people know there's room for them. But everybody knows that you have to have a little sign that says, *No* to hang up in front of the *Vacancy* sign so it says *No Vacancy* when the hotel gets full. You get in trouble if you advertise a vacancy, then tell someone you're all full up and they can't come in. I started to bring it up with Sam, but he was already on the phone with the junk man Kronecker negotiating a price for our big pile of leftover materials and tools. So I made that little sign, too, and hid it under the pot of the avocado tree. Here it is. As you can see, we never hung it up.

When I came back from getting myself spruced up like Sam, Kronecker was leaning against the handle bars of that propeller-driven solar-powered truck he made out of hot water tanks and bicycle parts, staring at the big sign in the parking lot. I knew he was going to be glad to get that junk pile. He's a funny man, that Kronecker. He won't use anything he doesn't make himself out of bits and pieces of everyday things, so all that junk was a pile of treasure for him. That's why I was surprised when he whirled and shook a bony finger at me.

"Cantori told me what you two were up to," he growled. "Hotel Infinity, indeed. No such place can be made, yet you pretend that it can. Such a thing is incorrect, evil, and wrong. I will see to it that this is stopped. And yes, I *will* take your tools, because you two should never be allowed to claim to build another thing!"

Sure, Kronecker's threat was strange, but Kronecker was strange. There was no time to argue or worry about what he said. The first guests were already arriving. It's no surprise that the place was already beginning to fill up. Where else can you get a hotel room for a dollar?

Sam stood up from his big leather chair and greeted the members of the first family in line. "What is it you like? What do you like the best?" he boomed, pumping the hand of the youngest child.

"Purple!" came the answer—along with a grin that had one tooth missing.

"And you?' He reached for the hand of the older child and gave it a big shake. "What do you want in your room?"

"Polka dots!"

"Purple with polka dots. Hmmm. Will those be orange polka dots or crimson ones?" he asked the parents.

Theh parents turned to each other and shrugged, "Orange will be just fine," they said.

"Wonderful!" clapped Sam. "Room 91 is the one for you. I'll take your dollar. Then all you need to do is read this very important paper here. Grownups, please make sure the kids understand. And when you've signed it, my assistant George will show you how to find your room."

Now, that paper was a funny thing. Sam always took charge of that. The paper had all the usual things on it, you know, clean up your mess, fix what you break, check-out time is 11AM. But there was this one extra thing at the bottom that Sam always acted like it was so important. It was written in letters twice as big as the rest, and Sam always made sure that they saw it. It said, "AND I PROMISE TO OBEY EVERY REQUEST THAT IS MADE OVER THE LOUDSPEAKER, NO MATTER HOW SILLY OR INCONVENIENT IT MIGHT SEEM." Well, the silly part was that we didn't ever use the loudspeaker, even though I knew full well that it went everywhere in the whole place. I put it there myself, you know.

Well, my job was to show the people to their rooms. I didn't have to go all the way there. The place was too huge for that. Sam would hand me a piece of paper with their room number on it. I would tell them the number and start them on their way. When we were just new and people were just starting to find out about us, I told them numbers like 6, and 28, and 42. In a while, as the hotel got more popular, I was giving out more interesting numbers like 523, or 954, and even 1,255. After a long time, I got used to handing out numbers like, 3,244, and 8,999,223. It was all very organized the way the hotel was set up, so even though the numbers got very large, the rooms were easy to find. I liked the big numbers the best.

One day, when I had just sent a family off to room 4,693,203,355,226 I asked Sam the first question ever. "How many rooms do we have in this place, anyway?" He looked at me kind of funny. "Just curious, sir," I said quickly.

"You built it, George, you should know," he answered. "Infinitely many." I didn't ask any more questions for a long time. But I did watch, and I did think.

One day, Sam said he would be out for a while. He had an appointment with the head of a tour company that was probably going to buy the busses for sure. He said that I was to be in charge while he was gone. I would be the one who sits at the big desk and calls a big welcome to everyone who comes in the door: the families, the little children and their pets, the salespeople, the convention-goers, the bicycle travelers, the bus loads of tourists. I couldn't wait.

"Make sure they understand that paper," Sam cautioned. "Don't let anyone in without signing it. And oh, by the way–there's only one room empty right now. It's 26." He gave a wave and was jauntily off, swinging his cane.

Only one empty room! Thank goodness I had made that sign. It was going to have to say *No Vacancy* out there before this day was over for sure! I kind of wondered how many people actually were in the hotel just now. That would be interesting to know. But the office was filling up with folks like it always did.

At the same time that cars were pulling up in the lot outside, a few people were dragging in suitcases to leave. A line had already formed at the right-hand side of the desk where the sign said, *Check In*. I had to get that sign up fast. When I tried to tip the pot of the avocado tree to pull the little *No* sign out, it wouldn't budge. The tree had grown too huge and heavy since I slipped it under there. I was going to need a lever to get the sign out, and a hammer to hang it up. But Kronecker had taken away all the tools years ago. It was useless. What was I to do? I had to think.

As quickly as I began to think, I had an idea. It wouldn't work for sure, but it would buy me some time. "Good morning," I said to the first person in the check-in line. He wore a suit and carried a briefcase. I gave him the paper, made sure he agreed to all of the rules, including the one about the loudspeaker, took his dollar, and sent him off to room 26. Next in line were two young kids wearing backpacks.

"We're traveling by bicycle." they said. "Usually we camp out, but we heard about this place. The Hotel Infinity. History making. We could use a shower and a soft bed, so we decided to stay here tonight. Here's our dollar."

"Just a minute," I said, and swung myself around in Sam's big leather chair to face the first person in line in front of the left-hand side of the desk where the sign said *Check Out* . The woman handed me the key to Room 4,326.

"Was everything okay," I asked her.

"Oh, it was just fine," she said. "I've never slept so well in a bed shaped like a turtle before." She picked up her suitcase and walked toward the door. The man behind her started to hand me a key, but I held up my hand, smiled and said, "Be with you in just a minute, Sir." Quickly, I swung the chair back around to face the bike trip kids and checked them into room 4,326. The man who I'd left waiting checked out of room 4, which is where I checked the next person into. That's the way it went all morning long. I kept checking people into whichever room it was that someone just checked out of. I was lucky. After the last person checked out, there was exactly one person left to check in. What a relief!

The lobby was empty. I leaned back into the soft leather of Sam's chair, propped my feet on the desk and looked at my reflection in the polished shine of my fancy shoes. I was proud of myself. Just a little bit of thinking, and I had figured out what to do. Outside, I could hear splashing in the pool, the clop-clop of the ponies feet, and the music from the carousel. People were playing tennis, strolling on the sidewalks, pitching horseshoes, shooting arrows at the archery targets, and sipping cool drinks. A perfect day at the Hotel Infinity. Then, right there before my eyes, a car pulled up. Oh no! Nobody would be checking out until tomorrow, and I had forgotten to put up the sign.

Every door of the car creaked slowly open, and a whole tired-looking family got out, along with two cats, a Saint Bernard, and a poodle. The girl was carrying a bird cage, and her brother had a fish bowl. The man pushed the office door open and tossed a crumpled dollar bill on the desk. He didn't look friendly.

"We need a room," he gruffed. "A big one. Now. Been driving since yesterday morning." He held his hand out for a key. "Which one is it?"

I started to explain that I didn't have a room, but he interrupted me. He threatened me and scared me. He told me I was going to go to jail for running a hotel that advertised a vacancy, and refusing to give him a room. He said I was discriminating against families, against tired people, against loud people, against rude people, against people who had been driving since yesterday morning. He grabbed the phone and was calling his lawyer, when in burst Sam.

"George! I've done it!" he shouted. "I sold them! I've sold all the busses to Continue 'Em Tours. All of them, George! Sold! It's a wonderful day, George!"

"Oh, Sam, Sam, no. It's a horrible day," I groaned. "Forget the busses, we're ruined. Something terrible has happened." Sam's face fell, and I started to explain. "The hotel is full, Sam, and there's no place to put this man. He's mad, he's going to sue. We're going to jail! Oh Sam, it's so terrible, Sam, I should have told you sooner, we need to change the sign..."

Sam wasn't even listening. He strolled over to the man and said, "Excuse me, Sir, my untrained assistant has made a mistake. He meant no harm, he just doesn't know how to find an empty room when the hotel is full. It is true that we already have an infinite number of guests but this is the Hotel Infinity. We have an infinite number of rooms, so of course there is space for you. Please, have a seat, this will only take a moment." He bustled over to me, fishing into his vest pocket for a piece of paper which he pressed into my hand.

"At last, George," he whispered, grinning. "We get to use the loudspeaker. Go announce this."

I went over to the table in the corner with the microphone on it, angry and muttering all the while. "Untrained assistant! Haven't I worked for Sam more years than the two of us put together can remember? Finding an empty room when the hotel is full. An infinite number of guests. An infinite number of rooms. An infinite number of too-crazy things is what we've got here!" I flipped the switch on the loudspeaker system, and much to my surprise and relief, it worked after all these years. I leaned over the microphone and read what was on the paper.

"Ladies and gentlemen, families and friends. Please recall that you have agreed to do exactly as we ask when we make a request on the loudspeaker, even if what we ask seems silly or inconvenient. This won't be hard and it won't take long. All we ask is this: would everyone please gather up your belongings and move to the room whose number is exactly one greater than the number you are in. Got that? Everyone move quickly now, yes that's right, to the room whose number is one greater than the one you are in."

The people had signed. The people had agreed. So, although the people were somewhat confused, they began to move. The trapdoor in the ceiling of room 34 creaked open, and that family climbed out and moved on to room 35. The businessman in room 578 appeared in the hallway and asked a boy where room 579 was.

"Oh, my mom and I just left there," he said, "it's back that way just around the bend in the hallway. We're looking for room 580 now."

The third graders on a class trip quickly gathered their things and moved out rooms 222, 223, and 224 and into to rooms 223, 224, and 225. The grandparents in room 3,465,892 moved to room 3,465,893. And the lonely old man who had been in room 1 did not look nearly so glum as he stood on the ladder, pushed up on the round door over his head, then climbed up and popped through the floor into to room 2.

"Well!" said Sam, as he turned to the man, who still looked rather angry. "We are most sorry for the inconvenience, sir. So sorry. As you can imagine, it has taken many years for this hotel to get full. This is the first time it has happened, and my assistant here wasn't aware of the procedures. Please accept my apologies, and let me return your dollar so that you can stay in room number 1 for free. But first, of course, you must sign the paper, here."

That Sam! He was such a charmer! Oh, those were the days—before all that fuss with Kronecker heated up.

From then on, more people came than left every day, so the hotel was always full. We used the loudspeaker to get everyone to move down whenever we needed an extra room. It got to be known far and wide that this would happen, and people began coming to the hotel just to see how many different rooms they could get to stay in. They might start out in an upstairs room with dinosaurs on the wall paper, and then the loudspeaker call would come, and they would have to move to the room with the checkerboard bedspreads. If it was a particularly busy day, a lucky family might, all in one day, get to stay first in the room with the elephant-shaped bathtub, then in the one with jungle plants and trapezes. After dinner they might get sent to the room that has the bathtub slide, and end up going to sleep in the one with the fold-down beds. People didn't mind moving. That turned out to be half the fun.

Because I was paying more attention, I was learning to manage the hotel. Sam could leave me in charge, and I could handle everything just as smoothly as he could. Good thing, too, because Sam had to be away more and more due to all the trouble Kronecker was causing.

I think it was Kronecker's peculiarity about having to know how things were built up from simple parts that set him off. And Sam is pretty stubborn, too. He said that if Kronecker wanted to be a fool and not believe the hotel existed just because he didn't know how it was made, that was up to him. Then Kronecker bought up all the newspapers and wouldn't run Sam's ads because he said they violated truth-in-advertising laws. That's why they ended up in court. Lots of people from the hotel came to see, but of course they didn't all fit in the courtroom.

Sam stood up in front of the judge and declared, "At this moment there are infinitely many people inside the building that Kronecker claims everyone should ignore." He pounded his fist on the witness stand. "He has never set foot in the door! If he did come inside, see the people come and go, try and count them if he likes, and watch our operation, he would see how it works. It is indeed infinite. Why, I'm sure we could find room for any number of new guests right now."

"Preposterous!" shouted Kronecker "He tells me to come visit the infinite as though it is something waiting to be discovered or explored, skipping over entirely the fact that ordinary people can not figure out a way for it to get there. If he cannot tell me how that place is constructed, it is neither real nor useful and people should stay away! And furthermore—"

"Listen to this ridiculous man!" burst in Sam. "I am sure he cannot tell me how to construct a stomach, yet this has never prevented him from staying away from his lunch!"

There was a lot of shouting back and forth like that, and the judge kept having to pound his gavel and make them calm down. In the end, though, he pretty much sided with Sam when he said, "He claims that there are infinitely many people inside, and that there is still room for more. I agree, Mr. Kronecker, that this is quite puzzling, but just because something is puzzling, we can't be telling people it is dangerous and they should stay away. There are ways, Mr. Kronecker, that you could prove that what Mr. Cantori claims about his hotel is not true. If there aren't infinitely many people there, as you claim, just count them and tell us how many there are really. And it also seems to me that if someone is turned away because the hotel is full, you would have proved your point about Mr. Cantori's advertising being false. If you can present the court with this kind of information, we will reopen this case, but for now I rule that you must allow Mr. Cantori to advertise. Case Dismissed!"

Sam laughed and looked at Kronecker, "Come on over and count. Or come on over and wait for us to turn a guest away. You will be at either task for infinitely many days!" He paused, and scratched his head. His voice got quieter. "Or maybe a little longer. I'm not sure. I'll have to think about that."

Kronecker shook his bony finger and threatened revenge, but things settled down. A good thing, too, because I don't know what I'd have done if I'd been there alone the day that one of Sam's busses pulled up—fully loaded and needing rooms for everyone.

I told you I made those busses just like I made the hotel, back when all I did was follow orders, before I started asking questions. Did I tell you that those busses were made just like the hotel? That the busses held infinitely many people? Sam saw it pull up before I did. He rushed right out and began talking to the driver. I knew it would be time for my sign for sure. I hoped the driver wouldn't be too mad—or the people on that bus either. Where else is a bus loaded with infinitely many people ever going to be able to let all the passengers off so they can take a shower and sleep in a nice warm bed? Oh, I hoped Kronecker wouldn't get wind of this.

Well, wasn't I surprised when Sam Cantori shook the driver's hand and started waving the people off the bus. As he walked inside he was fishing into his vest pocket with his eyes all bright. He handed me a note, and flashed me a grin. "Loudspeaker time, George. By Zeno! I'm glad you made those busses. I don't see how we'd have ever gotten to this any other way!"

The lobby was filling up with people, and more and more of them were pouring out of the bus. No time to study the note—I just went to the loudspeaker and read it off.

"Ladies and gentlemen, families and friends. Recall that you agreed that you would follow our requests on the loudspeaker, no matter how silly or inconvenient they seem. Here is what we must ask you to do. Please look on the outside of your door and copy your room number down on a piece of paper. Everybody got that? Now, everyone, please multiply that number by 2. Yes, double the number, and write down your answer. Check your work please, make sure that it is correct. That's your new room number, folks. If you would please, gather up your things as quickly as you can, and move into the room that has your new number."

And they did. The person in room 1 moved to room 2. The family in room 2 moved to room 4, the kids in room 3 moved to room six, and so on in all the rooms. The room 4,886 was now occupied by the people that had been in room 2,443. The driver was motioning people off the bus, there were people everywhere, infinitely many people! Where would we put them all?

Sam pulled me aside and whispered, "It's OK, George, I told them they could take any odd-numbered room that they wanted. There are infinitely many odd numbers, you know, so it should come out just right, don't you think?"

Well, it did come out just right. And people kept coming. For years and years they came to see and visit and change rooms at the Hotel Infinity. Sometimes it was a carload of people that would come, sometimes just one person on roller skates or on a unicycle, sometimes a regular bus, and sometimes one of Sam's busses for infinitely many people. It always worked. Once five of the Continue 'Em Tour busses came at once. Another time there were 10. And one time over a hundred and fifty of those bus loads pulled up at once. No problem! We could always fit them in. Until the day when—well, you've probably all heard plenty about that.

Some say that Kronecker had a hand in the disaster. Maybe he did, but you want the truth, so I can only tell you what I know. He was the one who rented all of the Continue 'Em Tour Busses that day, filled them up and sent them over. I know that because the Continue 'Em Tour director called and told us. She knew all about the hotel, of course, but even she was a little concerned about what could happen if all of the busses showed up at once. I told her not to worry, and passed the message along to Sam just to let him know, and to my surprise, he looked alarmed!

He hurried to the safe and unlocked it for the first time in oh-so-many years. The crinkled papers he pulled out were covered with mathematical-looking writing.

"When I discovered this trick, I was amazed that such a thing could be possible," he said. "Now, how does it go again?"

I could hear rumbling in the parking lot. The busses were arriving. He began to shuffle faster and faster through the brittle pages, and as he did, they crumbled into hundreds, thousands, maybe infinitely many flakes of old yellowed paper.

"It's lost!" he cried out. "The secret is lost! If I cannot remember, Kronecker will have beaten us," and he buried his head in his hands.

"Sam," I said, "It cannot be so bad. We have accommodated many busses before. We will just do the usual trick. The people like the loudspeaker. They don't mind moving. So what if it is 200 or 500 or even a thousand busses. How many of those busses did we build anyway?"

He looked up at me in a way he never had before. He spoke to me, not like I was his assistant, his builder, or his employee. Sam spoke to me like the old friend he had become.

He said my name quietly. "George, you don't know, do you?"

I shook my head. I had stopped trying to count things so long ago. Of course I didn't know.

"George, I have maybe done a terrible thing keeping it all so secret, secret even from you. I left it up to you to notice, George, but I didn't really think you would. You didn't notice, did you, that every time we finished a room in the hotel we finished a bus."

I stared at him. "A bus for every room?" No, I hadn't noticed.

"Yes," he said. "As many busses as there are rooms."

"Then that means..."

"Yes," he said. "Infinitely many busses. And they are beginning to line up outside now. I cannot remember what to do to fit all those people in. We are finished. Let Kronecker say he has beaten us." He stood up and hobbled over to the avocado tree. "I need your help," he said, slipping the brass tip of his cane under the lip of the pot. We leaned hard together and the heavy tree tipped a bit. With a deft kick of his once-shiny shoe, he sent that little sign I had made skittering out across the floor.

"I'd always been amused by your little secret," he smiled sadly. "Go ahead. Put it up. Let the Hotel Infinity say *No Vacancy.* Kronecker can take a picture for all his newspapers."

Next to that potted tree, he stood, not straight and tall in the manner of the Sam I will always remember, but hunched over, his fine clothes hanging loosely from the bony hump between his shoulders. He turned and went through door number 8. The draught from the hallway that leads to all the rooms whose numbers begin with '8' caught the blueprints and they floated like giant feathers to the floor. The fragments of crinkled paper whirled and fluttered like snow. The last I saw before the door closed was a slow-moving figure that didn't carry his cane over his wrist in that old jaunty way. He used it to support himself as he hobbled along.

"Sam is old," I said to myself. The door clicked shut behind him, and the nail that fastened the wooden '8' to the door must have come loose, because the number tipped over on its side. "This hotel is old, too," I said. "So am I. But I'm *not* too old to think."

There was honking and shouting in the parking lot. The drivers who were awaiting their instructions for unloading the busses were getting impatient, and more busses were pulling up. I had disobeyed Sam when I made that little sign, and I was ready to disobey him again. First I made the usual announcement that emptied out all the odd-numbered rooms. That gave me some time to think.

"Infinitely many odd numbers," I said to myself. "Infinitely many rooms empty, I know that much for sure. How can there be more than infinitely many people on those busses?"

Then, I started out slowly toward that parking lot, and with every step I thought harder and deeper. When I got to the first bus, I had an idea that was worth a try.

I told the first couple of drivers to pass the word down that all the people would file off the busses and fill up the odd-numbered rooms. Now all I had to do was get all the folks to move off the busses in an orderly way so no one would get skipped, and no one would have infinitely many people in line in front of them. Then they could fill the odd numbered rooms in order.

"OK," I said. "Now, we'll take the first person off the first bus and the first person off the second bus." A lady with a briefcase. A man and his dog. No problem. "Now, we get the next two people off the first bus, one person from the second bus, one person from the third bus, and one person from the fourth bus."

You know, people who come to the Hotel Infinity are all so wonderful. Other folks might have minded the wait, but these people were cheerful, like it was a game. People who came to the Hotel Infinity counted on doing something unusual.

After the first person got off the fourth bus, we worked our way backwards, taking one person off the third bus, the second bus, and the first bus. Then one more off the first bus, and we worked our way back up, taking one person from each bus that was already in the game, and adding a person from the fifth bus and a person from the sixth bus this time. Back and forth we went, always letting two people off the first bus, and starting to empty two more busses each round.

It looked like it would work. More and more people were moving. At each round, more busses came into the game and the people began to file out. The odd numbered rooms in the hotel were slowly filling up. Word spread down from driver to driver about how the busses were being unloaded. You could hear whispering in the busses as the people passed the word to one another about how it was all working.

At least I think it was all working. I think it would have worked. After they didn't need my help on the busses anymore I went inside and sat in the big leather chair. The rooms in the seven hundreds were about half full when people started rushing out saying they smelled smoke. I didn't wait to sound the alarm. A big wooden building that old, and all those nice people—I couldn't put them in danger. I got on the loudspeaker and told everyone to get back on the busses–including the people in the even-numbered rooms. The guests were all out and the busses were pulling away for safety when the big blaze started to roar.

Maybe the building was just too old and it was hopeless. Maybe the fire fighters could have saved it if it had been made in a different way. You know there were all those twists and slants I told you about. Ever so slight, but they mattered. They only thought they could head down Hallway Eight to get around the other side of Hallway Seven where the fire seemed to have started. Of course it seems like it ought to go together that way, but nothing fit together inside that building the way you'd think it would fit. I guess that's the reason why it all fit.

Anyway, it's good they went down Hallway Eight instead of Hallway Two like they should have. Because they found Sam, hunched and hobbling, rounding the curve towards the eight millions. Oh, he gave him a time, he did, holding them at bay with his cane and hollering at them to get out, that there was no vacancy here, and that the Hotel Infinity couldn't be on fire because there *was* no Hotel Infinity anymore. They took him to the hospital figuring he was in shock about the fire.

Well before long it was true. There was no Hotel Infinity anymore, just hot ashes, coals, and thin plumes of smoke. I've been thinking since then that if we could have saved just one hallway we'd still be in business. But we didn't. All that's left is the sign that stood up outside, and this little sign here that I never put down after Sam handed it to me. I'm going down to the hospital to try and see Sam. I go every day about this time. So far, though, he refuses to see anyone. If he'll talk to me today I'm going to ask him if maybe it would be OK if I put the little sign up now.

Hotel Infinity, No Vacancy. People will see it and wonder if that's possible, wonder how such a thing could be true.