IX. Boundary Conditions

The implementation of a sophisticated numerical algorithm and the development of a flow code usually are trivial tasks when compared with the work of actually solving a particular fluid dynamics problem. We can always assess the applicability of a numerical algorithm based on stability and accuracy considerations. The writing of specialized input and output routines, while not unimportant, is usually mechanical. The real stumbling block comes with the selection, implementation and assessment of boundary conditions (BC).

There is a hierarchy of decisions which are made when the boundary condition problem is attacked. The important aspects of boundary condition development are:

1. The physical definition of the flow problem must be satisfied. For example, inviscid flow requires tangency at solid surfaces, or we may want to specify pressure at some boundary.
2. The physical conditions must be posed in terms of the mathematics of the problem. Characteristic theory suggests the number of conditions required at a boundary. The condition of no slip for viscous flow is imposed by setting the flow velocities to zero at solid surfaces.
3. The mathematical conditions are numerically approximated.
4. The numerical interior scheme may require more boundary information than the physics provides. For example, standard central differencing as an interior scheme requires all flow quantities at boundaries, but this may not be consistent with mathematical theory. Additional numerical boundary conditions may be adjoined.
5. The combination of interior numerical scheme and boundary scheme should be checked for stability and accuracy.
6. Finally, we must assess the efficiency and generality of a flow code in terms of its ability to handle a wide variety of problems and flow conditions.

The physical definition of the flow problem is the first and foremost consideration. Once a geometry and topology have been chosen, then physics dictates the constraints on the boundaries.

9.1 Characteristic Approach

The concept of characteristic theory is best demonstrated with the one-dimensional Euler equations, where

$$\partial_t Q + \partial_x (AQ) = 0 \tag{9.1}$$

represents the model equation. Assuming that A is a constant coefficient matrix we can diagonalize Eq. (9.1) using the eigenvector matrix X, so that

$$\partial_t \left(X^{-1}Q\right) + \Lambda_A \partial_x \left(X^{-1}Q\right) = 0. \tag{9.2}$$

Defining $X^{-1}Q = W$, we now have a diagonal system, with

$$\Lambda_A = \begin{bmatrix} u & 0 & 0 \\ 0 & u+a & 0 \\ 0 & 0 & u-a \end{bmatrix} \tag{9.3}$$

At the left boundary of a closed physical domain, see Fig. 9, where say $0 < u < a$, (for example, subsonic inflow for a channel flow) the two characteristic speeds $u, u+a$ are positive, while $u-a$ is negative. At inflow then, only two pieces of information enter the domain along the two incoming characteristics and one piece leaves along the outgoing characteristic. At the outflow boundary one piece of information enters and two leave. We would like to specify the first two components of W, which are the two incoming characteristic variables and then handle the third characteristic variable such that its value is not constrained, i.e., is determined by the interior flow.

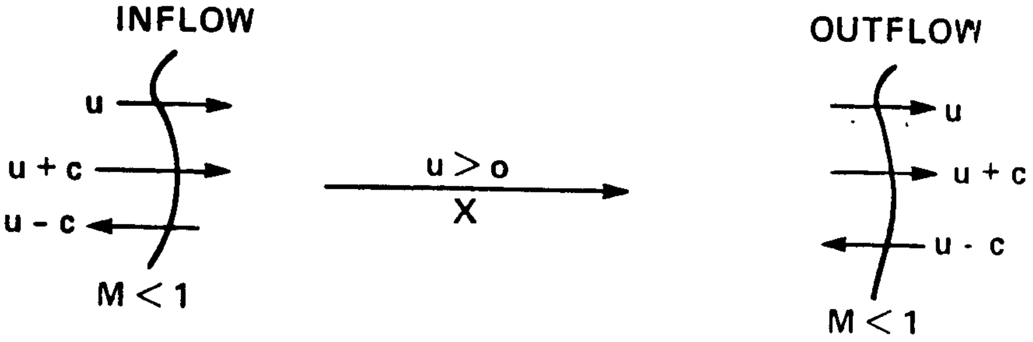

FIGURE 9. Characteristics at Subsonic Inflow and Outflow Boundaries of a Closed Domain.

It is not necessary to fix values only in terms of the characteristic variables, other flow quantities could be employed, as long as they lead to well posed conditions (that is, conditions which guarantee the stability of the mathematical problem). Yee [34] provides an excellent survey and development of boundary conditions within the framework of an implicit algorithm. The major constraints which occur are that the correct number of boundary values corresponding to incoming characteristics are specified and that the actual implementation is stable and well posed. Chakravarthy [35] presents an implicit characteristic boundary procedure. In this the eigenvectors of the system are coupled with the chosen fixed boundary values and one sided finite differences to develop an equation which is solved for the flow variables at the boundaries.

9.2 Well Posedness

A simple check on the well posedness of boundary conditions is obtained as part of Chakravarthy's development. As an example, let us consider one-dimensional flow with subsonic inflow and subsonic outflow. Then two variables can be specified at inflow, associated with the first two eigenvalues, and one variable can be specified at outflow, associated with the third eigenvalue. As specified values take, $\rho = \rho_{in}$, $\rho u = (\rho u)_{in}$ and $p = p_{out}$. These can be written as

$$B_{in}(Q) = \begin{bmatrix} q_1 \\ q_2 \\ 0 \end{bmatrix} = B_{in}(Q_{in}) \tag{9.4a}$$

$$B_{out}(Q) = \begin{bmatrix} 0 \\ 0 \\ (\gamma-1)(q_3 - \frac{1}{2}q_2^2/q_1) \end{bmatrix} = B_{out}(Q_{out}) \tag{9.4b}$$

with $q_1 = \rho,\ q_2 = \rho u,\ q_3 = e.$

Forming the Jacobians $C_{in} = \partial B_{in}/\partial Q$, and $C_{out} = \partial B_{out}/\partial Q$ we have

$$C_{in} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C_{out} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ [(\gamma-1)/2]\,u^2 & -(\gamma-1)u & \gamma-1 \end{bmatrix} \tag{9.5}$$

The eigenvector matrix X^{-1} is

$$\begin{bmatrix} 1-\frac{u^2}{2}(\gamma-1)a^{-2} & (\gamma-1)ua^{-2} & -(\gamma-1)a^{-2} \\ \beta[(\gamma-1)\frac{u^2}{2} - ua] & \beta[a-(\gamma-1)u] & \beta(\gamma-1) \\ \beta[(\gamma-1)\frac{u^2}{2} + ua] & -\beta[a+(\gamma-1)u] & \beta(\gamma-1) \end{bmatrix} \tag{9.6}$$

with $\beta = 1/(\sqrt{2}\rho a)$.

The condition for well posedness of these example boundary conditions is that $\overline{C}_{in}^{-1}$ and $\overline{C}_{out}^{-1}$ exist where

$$\overline{C}_{in} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta[(\gamma-1)\frac{u^2}{2} + ua] & -\beta[a+(\gamma-1)u] & \beta(\gamma-1) \end{bmatrix} \tag{9.7a}$$

and

$$\overline{C}_{out} = \begin{bmatrix} 1-\frac{u^2}{2}(\gamma-1)a^{-2} & (\gamma-1)ua^{-2} & -(\gamma-1)a^{-2} \\ \beta[(\gamma-1)\frac{u^2}{2} - ua] & \beta[a-(\gamma-1)u] & \beta(\gamma-1) \\ (\gamma-1)\frac{u^2}{2} & -(\gamma-1)u & \gamma-1 \end{bmatrix} \tag{9.7b}$$

These matrices are formed by adjoining the eigenvector associated with the outgoing characteristic to the Jacobian matrices of the boundary conditions. The inverses of the above matrices will exist if $det(\overline{C})$ is nonzero. For the two boundaries we have $det(\overline{C}_{in}) = \beta(\gamma - 1) \neq 0$ and $det(\overline{C}_{out}) = \beta(\gamma - 1)a \neq 0$ and so the boundary conditions are well posed. Other choices for specified boundary values can be similarly checked.

Rather than go into any more detail on boundary condition theory we refer the reader to Refs. [34] and [35]. For the remainder of this section, we shall discuss some of the current types of physical conditions which are used in ARC2D.

9.3 Computational Mapping of Boundaries

Usually in a flow field computation, we are faced with a variety of boundary surfaces and conditions. Our experience has been mostly in external flows and so we shall outline below some of the more commonly used boundary conditions. The curvilinear coordinate transformations are made such that physical boundaries are mapped to boundaries in the computational domain. This makes the formulation and implementation of boundary conditions easier.

At this stage in the development of ARC2D we have kept the boundary conditions explicit. This gives us a more flexible clean code since all BC are handled in just one subroutine. We realize that implicit boundary treatment will enhance the stability and convergence rates of the codes. Our experience has been that the basic code in its present form is fairly robust and can be implemented for a wide variety of cases. The user is then responsible for the implementation of boundary conditions.

A particular set of BC for an airfoil calculation is used below for demonstration purposes. The geometry is mapped onto the computational rectangle such that all the boundary surfaces are edges of the rectangle, for example see Fig. 10 .

In “O” mesh topologies the wake cut boundary is periodic and can be handled as such where periodic solvers are used in the implicit inversions.

A. Body Surfaces

At a rigid body surface, tangency must be satisfied for inviscid flow and the no slip condition for viscous flow. In two-dimensions body surfaces are usually mapped to $\eta =$ constant coordinates. The normal component of velocity in terms of the curvilinear metrics is given by

$$V_n = \frac{\eta_x u + \eta_y v}{\sqrt{(\eta_x^2 + \eta_y^2)}} \tag{9.8a}$$

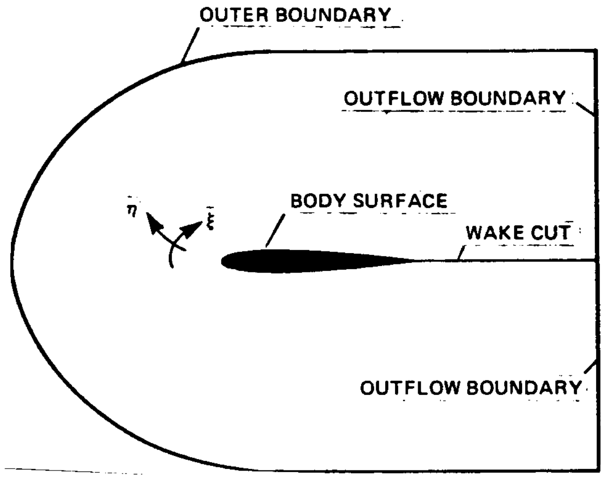

FIGURE 10. Topological Mapping of an Airfoil onto a "C" Mesh.

and the tangential component is

$$V_t = \frac{\eta_y u - \eta_x v}{\sqrt{(\eta_x^2 + \eta_y^2)}} \tag{9.8b}$$

Therefore, tangency is satisfied by $V_n = 0$ (no flow through the body). The tangential velocity V_t is obtained at the body surface through linear extrapolation for inviscid cases and is set to zero for viscous cases. The Cartesian velocities are then formed from the inverse relation between them and Eq. (9.8) where

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1.}{\sqrt{(\eta_x^2 + \eta_y^2)}} \begin{bmatrix} \eta_y & \eta_x \\ -\eta_x & \eta_y \end{bmatrix} \begin{pmatrix} V_t \\ V_n \end{pmatrix} \tag{9.9}$$

The extrapolation of V_t produces less error if the mesh lines are clustered to the body surface. The velocities of Eqs. (9.8a), and (9.8b) are scaled such that the metric variations are removed which decreases the errors in the extrapolations, especially for nonorthogonal meshes.

The pressure on the body surface is obtained from the normal momentum equation

$$\begin{aligned} &\rho[\partial_\tau \eta_t + u\partial_\tau \eta_x + v\partial_\tau \eta_y] - \rho U(\eta_x u_\xi + \eta_y v_\xi) = \\ &(\eta_x \xi_x + \xi_y \eta_y) p_\xi + ({\eta_x}^2 + {\eta_y}^2) p_\eta = p_n \sqrt{{\eta_x}^2 + {\eta_y}^2} \end{aligned} \tag{9.10}$$

where n is the local normal to the body surface. Equation (9.10) is solved at the surface using central second-order accurate differences in ξ and one-sided first- or second-order accurate differences in η. For steady uniform incoming flow free-stream stagnation enthalpy H_0 is held constant along the body in inviscid flow. Using the equation for enthalpy $H_0 = (e+p)/\rho$ and the computed velocities and pressure, a value of density is obtained at the body. Adiabatic or constant temperature walls are used for viscous and unsteady flows to obtain density at the surface. In either case, total energy e is decoded from the equation of state.

B. Free Surfaces Stretched grids are usually used to place far field boundaries far away from body surfaces. When bow shocks and attached shocks are generated at a body surface care is taken to ensure that the shocks are sufficiently weak when they reach far field boundaries so that they are not reflected or at least they reflect outside the flow domain. A nonreflective characteristic like boundary procedure is used at far field boundaries.

For subsonic free stream locally one-dimensional Riemann invariants are used at the outer far field boundaries. The locally one-dimensional Riemann invariants are given in terms of the normal velocity component as

$$R_1 = V_n - 2a/(\gamma - 1) \quad \text{and} \quad R_2 = V_n + 2a/(\gamma - 1) \tag{9.11}$$

The Riemann invariants R_1, R_2 are associated with the two characteristic velocities (locally one-dimensional) $\lambda_1 = V_n - a$ and $\lambda_2 = V_n + a$ respectively. Two other equations are needed so that four unknowns (the four flow variables) can be calculated. We choose V_t and $S = \ln(p/\rho^\gamma)$ where S is entropy. At the far field boundaries shown in Fig. 10, the normal n is directed away from the boundary. For subsonic inflow $V_n < 0$ and the characteristic velocity $\lambda_2 > 0$, therefore the characteristic variable R_2 can be specified along with two other conditions. The Riemann invariant R_2, V_t and S are all set to free stream values. The other characteristic velocity $\lambda_1 < 0$ and R_1 is extrapolated from the interior flow variables. On subsonic outflow $V_n > 0$ and $\lambda_2 > 0$ while $\lambda_1 < 0$ so only R_1 is fixed to free stream and R_2, V_t and $\ln(S)$ are extrapolated. Once these four variables are available at the boundary the four flow variables Q can be obtained. For supersonic inflow boundaries all flow variables are specified and for supersonic outflow all variables are extrapolated.

Along singularities or cuts in the geometry (such as the wake cut in a "C" mesh), averaging is used to provide continuous flow variables. As mentioned above periodic conditions are used for "O" meshes.

C. Far Field Circulation Correction

For lifting airfoils in subsonic free stream, circulation at the far field boundary is accounted for to first-order (following Salas, et. al. [36]) by imposing a compressible potential vortex solution which is added as a perturbation to the free