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Consideration of variances in equilibrium reconstruction[1]

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Nova Photonics

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PPPL PRINCETON PLASMA PHYSICS LABORATORY

Abstract

The talk presents a theory of uncertainties in the reconstructions of the plasma current density and pressure profiles in the Grad-Shafranov equation. The associated technique was incorporated into the ESC code.

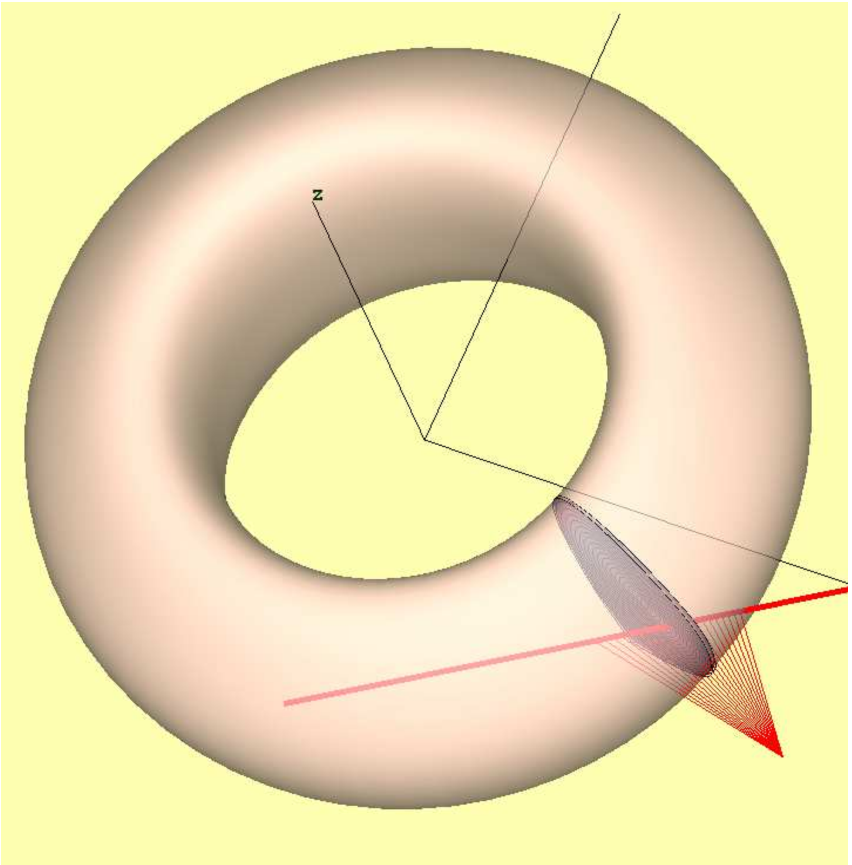
Potential variances in q- and p- profiles have been calculated for different sets of external and internal measurements envisioned for equilibrium reconstruction in ITER.

It was shown that complementing the external magnetic measurements with either Stark line polarization signals (MSE-LP) or with recently proposed for ITER by Nova Photonics line shift signals (MSE-LS) can significantly improve the reliability of the reconstructed plasma profiles and the magnetic configuration.

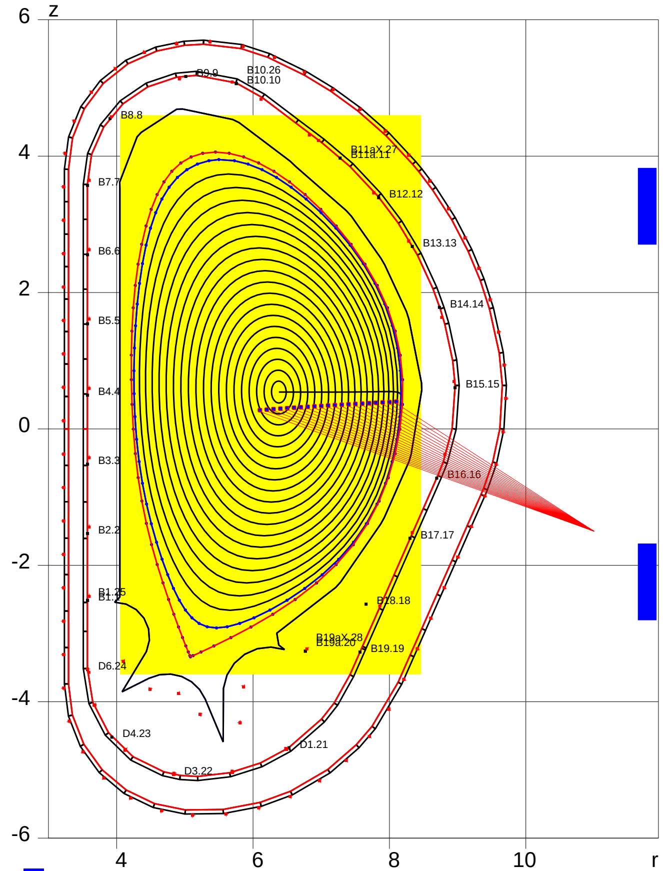
Capabilities of calculating variances, incorporated into the numerical code ESC, have completed the theory of reconstruction, which for a long time had a significant gap in ability to evaluate the quality of the presently widely used equilibrium reconstruction technique.

Contents

ITER B=5.3 T, I_{pl}=15 MA $\beta = 2.8$% equilibrium configuration



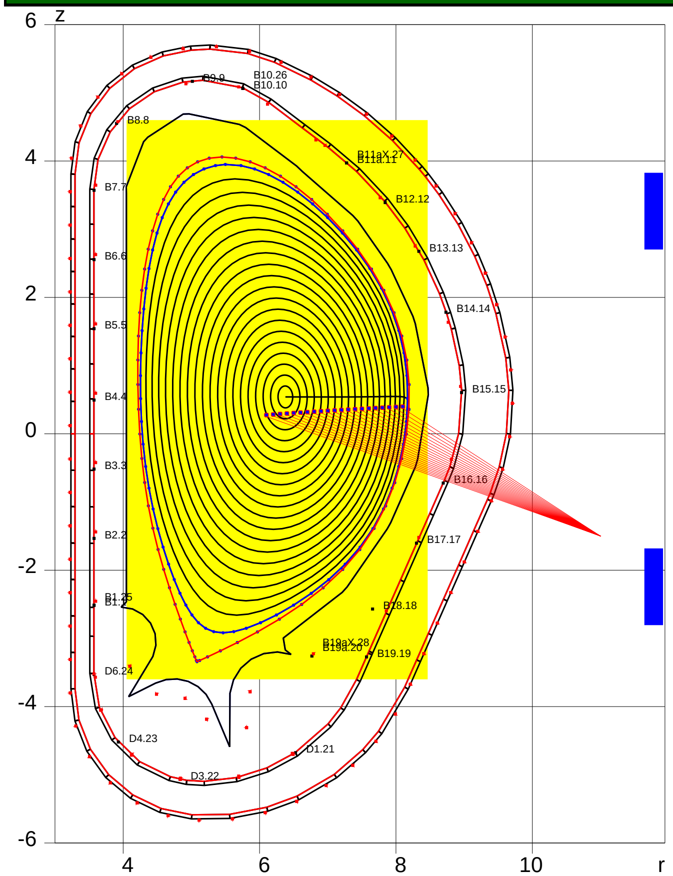
Center line of 1 MeV NBI in ITER



$\bar{\Psi}$-loops, B-coils, pickup points of MSE

One of unique features of ITER is its 1 MeV neutral beam injection

Measurements of the Line Shift due to MSE was proposed by Nova Photonics as a diagnostics of ITER configuration



Reference signal errors ϵ used here for calculating variances in equilibrium reconstruction in ITER:

Signal name	$\epsilon^{relative}$	$\epsilon^{absolute}$	*Comment*
B-coils	0.01	0.01 T	*local probes*
Ψ-loops	0.01	0.001 Vsec	
Φ-loop	0.01	0.001 Vsec	*diamagnetic loop*
MSE-LP	0.01	0.1^o	B_z/B_φ *from MSE line polarization*
MSE-LS	0.01	0.05 T	$\sqrt{|\mathbf{B}|^2 - (\mathbf{B}\cdot\mathbf{v})^2}$ *from MSE line shift*

MSE-LP and MSE-LS signals were assumed to be point-wise. This requires more realistic model from Nova Photonics.

The capabilities of equilibrium reconstruction with such a set of signal is the topic of the talk

The practice typically neglects making analysis of variances in reconstructed equilibiria

In tokamaks the Grad-Shafranov (GSh) equation describes the configuration

$$\Delta^*\bar\Psi = -T(\bar\Psi) - P(\bar\Psi)r^2, \quad T \equiv \bar F\frac{d\bar F}{d\bar\Psi}, \quad P \equiv \mu_0\frac{dp}{d\bar\Psi}, \tag{2.1}$$

Its solution can be perturbed by

1. perturbation of the plasma shape

$$\xi(a_{pl}, l), \quad \textit{and} \tag{2.2}$$

2. perturbation of two 1-D functions

$$\delta T(\bar\Psi), \quad \delta P(\bar\Psi). \tag{2.3}$$

The question, neglected by present practice, is what level of perturbations cannot be distinguished given the finite accuracy of measurements.

The level of variances $\xi, \delta T, \delta P$ determines the very value of reconstruction and of the entire diagnostics system

The theory of variances has been created in 2006 by L.Zakharov, J.Levandowski, V.Drozdov and D.McDonald

The problem is reduced to solving the linearized equilibrium problem

$$\bar{\Psi} = \bar{\Psi}_0 + \psi, \quad \Delta^* \psi + T'_{\bar{\Psi}} \psi + P'_{\bar{\Psi}} \psi = -\delta T(a) - \delta P(a) r^2 \qquad (2.4)$$

for N *possible perturbations*

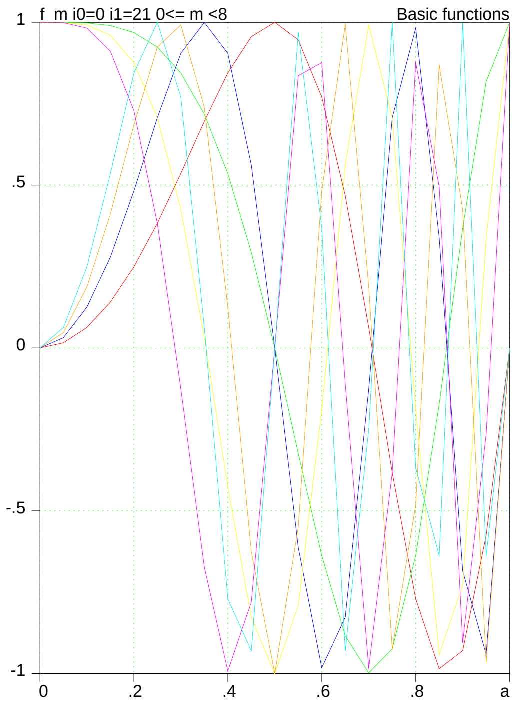
$$\xi = \sum_{n=0}^{n<N_\xi} A_n \xi^n(l), \quad \delta T = \sum_{n=0}^{n<N_J} T_n f^n, \quad \delta P = \sum_{n=0}^{n<N_P} P_n f^n,$$
$$N = N_\xi + N_J + N_P, \quad f^{2n} = \cos 2\pi n a^2, \quad f^{2n+1} = \sin 2\pi n a^2, \qquad (2.5)$$

where l is the poloidal coordinate at the plasma boundary, and $0 \le a \le 1$ is the square root from the normalized toroidal flux.

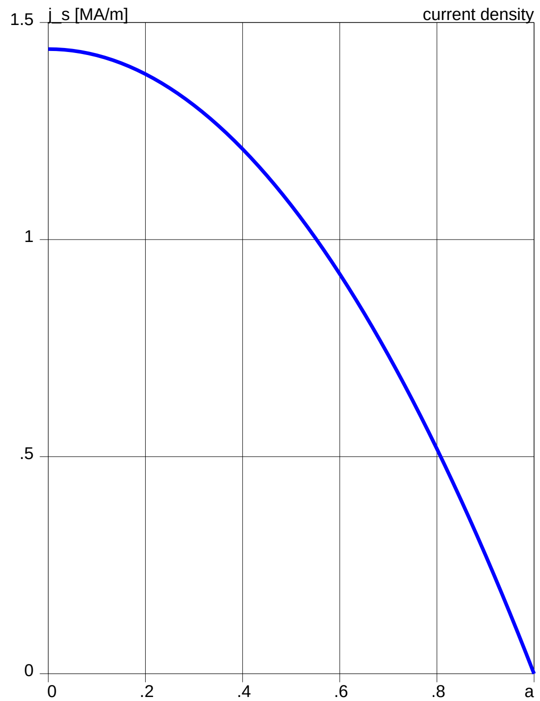
The response of the diagnostics to each of N *solutions ψ^n can be calculated in a straightforward way.*

ESC is based on linearization of the GSh equation. It was complemented with a routine for analysis of variances

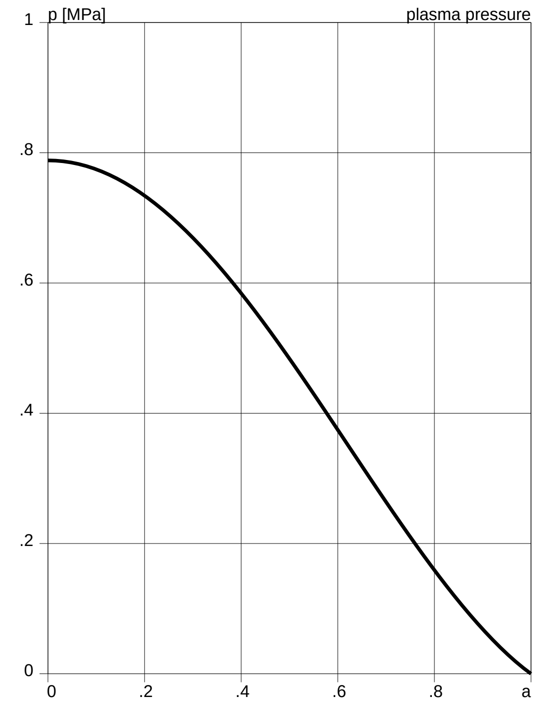
8 functions $f^n(a^2)$ has been used to perturb $P(\bar\Psi), T(\bar\Psi)$



Trigonometric expansion functions $f^n(a^2)$



background current density profiles $\bar j_s(a)$



background pressure profile $\bar p(a)$

ESC can use an extended set of basis functions

After solving the perturbed GSh equation, the problem is reduced to a matrix problem

Let vector $\vec{X}$ contains the amplitudes of perturbations

$$\vec{X} \equiv \left\{ \underbrace{A_0, A_1, \ldots, A_{N_b-1}}_{N_\xi \ of \ \xi}, \underbrace{T_0, \ldots, T_{N_T-1}}_{N_T \ of \ \delta T}, \underbrace{P_0, \ldots, P_{N_P-1}}_{N_P \ of \ \delta P}, \right\} \tag{3.1}$$

and vector $\delta\vec{S}$ represents the signals

$$\delta\vec{S} \equiv \left\{ \underbrace{\delta\Psi_0, \ldots, \delta\Psi_{M_\Psi-1}}_{M_\Psi \ of \ \delta\Psi}, \underbrace{\delta B_0, \ldots, \delta B_{M_B-1}}_{M_B \ of \ \delta B_{pol}}, \underbrace{\delta S_0, \ldots, \delta S_{M_S-1}}_{M_S \ of \ \delta \mathrm{others}} \right\}, \tag{3.2}$$
$$M \equiv M_\Psi + M_B + M_S, \quad M > N.$$

32 $\Psi-, 1\ \Phi_{diamagnetic}$-loops, 64 B-probes, 21 MSE-LP (line polarization) and 21 MSE-LS (line shift) signals (both pointwise) were used in the analysis.

ESC calculates the response matrix A relating $\delta\vec{S}$ and perturbations $\delta\vec{X}$

$$\delta\vec{S} = \mathsf{A}\vec{X}, \quad \mathsf{A} = \mathsf{A}_{M\times N}. \tag{3.3}$$

The working matrix $\bar{\mathsf{A}}$ weights δS_m based on their accuracy

$$(\bar{\mathsf{A}})^n_m = \frac{1}{\epsilon_m}(\mathsf{A})^n_m, \quad \delta\bar{S}_m = \frac{1}{\epsilon_m}\delta S_m, \quad \bar{\mathsf{A}}\vec{X} = \delta\vec{\bar{S}}, \tag{3.4}$$

where ϵ_m is the error in the signal S_m. SVD expresses the matrix $\bar{\mathsf{A}}$ as a product

$$\begin{aligned} &\bar{\mathsf{A}} = \mathsf{U}\cdot\mathsf{W}\cdot\mathsf{V}^T, \\ &\mathsf{U} = \mathsf{U}_{M\times N}, \quad \mathsf{U}^T\cdot\mathsf{U} = \mathsf{I}, \quad I^n_m = \delta^n_m, \\ &\mathsf{W} = \mathsf{W}_{N\times N}, \quad W^n_k = w^n\delta^n_k, \\ &\mathsf{V} = \mathsf{V}_{N\times N}, \quad \mathsf{V}^T\cdot\mathsf{V} = \mathsf{I}. \end{aligned} \tag{3.5}$$

Here, w^n are the eigenvalues of the matrix problem.

The resulting vector of variances can be represented as a linear combination of “eigenvectors”, which are the columns of matrix V

$$\vec{X}^k = \vec{V}^k, \quad \mathsf{A}\vec{X}^k = w^k\vec{U}^k, \quad \bar{\sigma}^k \equiv \sqrt{\frac{1}{M}\sum_{m=0}^{m<M}\left(\mathsf{A}\vec{X}^k\right)^2_m} = \frac{w^k}{\sqrt{M}}, \tag{3.6}$$

Eq.(3.6) gives variances and normalized RMS $\bar{\sigma}^k$ in an explicit form. The perturbations $\vec{X}^k$ with $\bar{\sigma}^k > 1$ are “invisible” for diagnostics

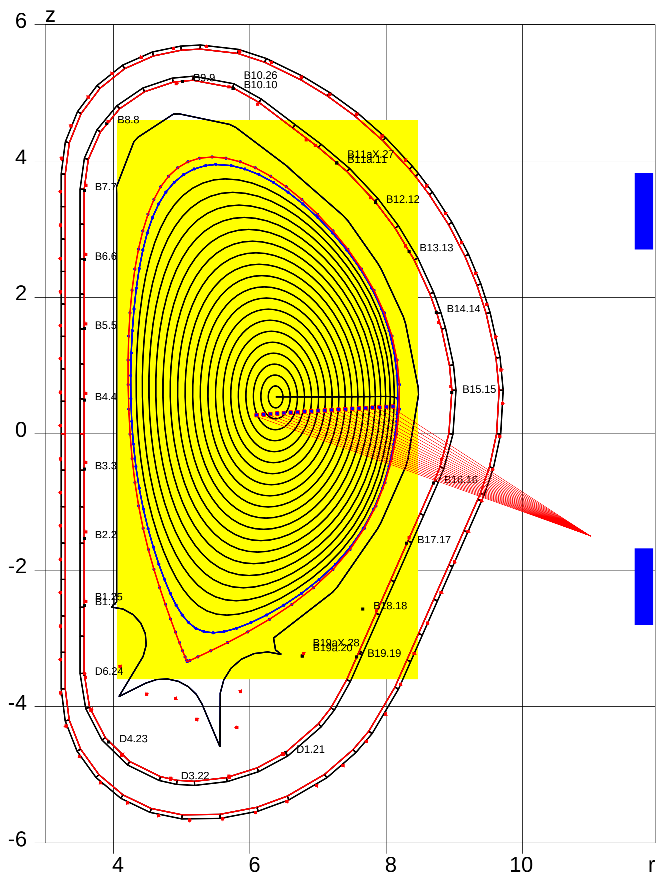
SVD of matrix A

$$\begin{array}{c|c} & (N) \\ \hline & \begin{pmatrix} \cdots & \cdots \\ \mathsf{A} & \\ \cdots & \cdots \end{pmatrix} \\ (M) & \end{array} = \begin{array}{c|c} & (N) \\ \hline & \begin{pmatrix} \cdots & \cdots \\ \mathsf{U} & \\ \cdots & \cdots \end{pmatrix} \\ (M) & \end{array} \times \begin{pmatrix} w_1 & & \\ \cdots & \cdots & \cdots \\ & w_k & \\ \cdots & \cdots & \cdots \\ & & w_N \end{pmatrix} \times \begin{array}{c|c} & (N) \\ \hline & \begin{pmatrix} \cdots & \cdots \\ \mathsf{V} & \\ \cdots & \cdots \end{pmatrix} \\ (N) & \end{array} \qquad (3.7)$$

Vector $\vec{X}$ in terms of eigen-vectors

$$\begin{pmatrix} X_1 \\ \cdots \\ \vec{X} \\ \cdots \\ X_N \end{pmatrix} = \begin{array}{c|c} & (N) \\ \hline & \begin{pmatrix} \cdots & \cdots \\ \mathsf{V} & \\ \cdots & \cdots \end{pmatrix} \\ (N) & \end{array} \times \begin{pmatrix} C_1 \\ \cdots \\ \vec{C} \\ \cdots \\ C_N \end{pmatrix} \qquad (3.8)$$

ITER configuration is used for illustrating the technique



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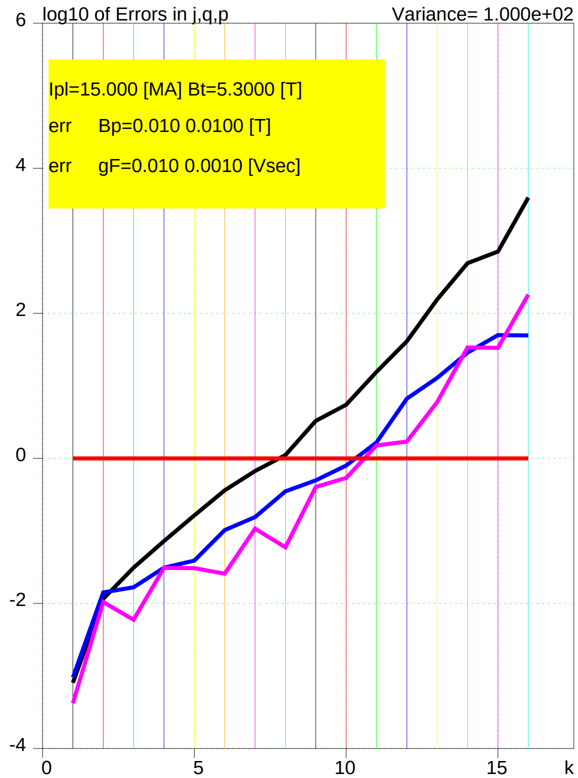
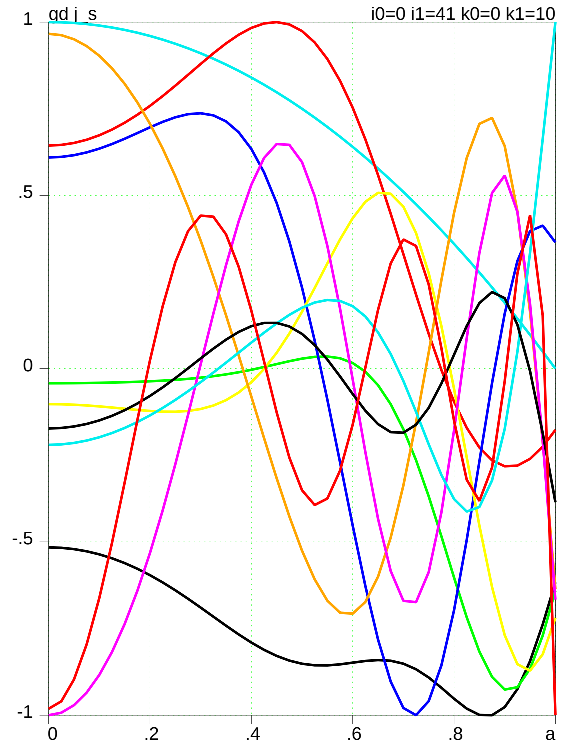
MSE-LP and MSE-LS signals were assumed to be point-wise. This requires more realistic model from Nova Photonics.

Different combinations of signal lead to different residual variances

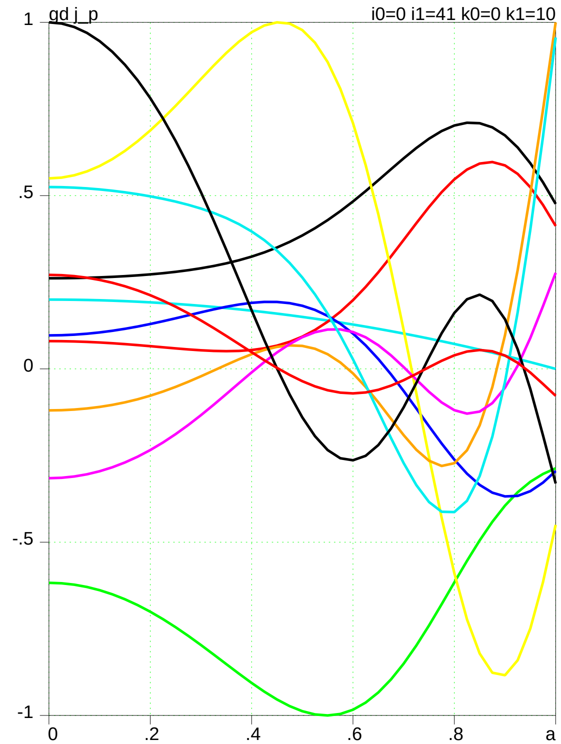
Plasma boundary is well specified, Φ-loop, B-coils are used



$\log_{10}\bar{\sigma}^k$, $\log_{10}\bar{\sigma}_q^k$, $\log_{10}\bar{\sigma}_p^k$ $(N_J = 8, N_P = 8)$



Eigen-variances $\delta j_s^k(a)$.

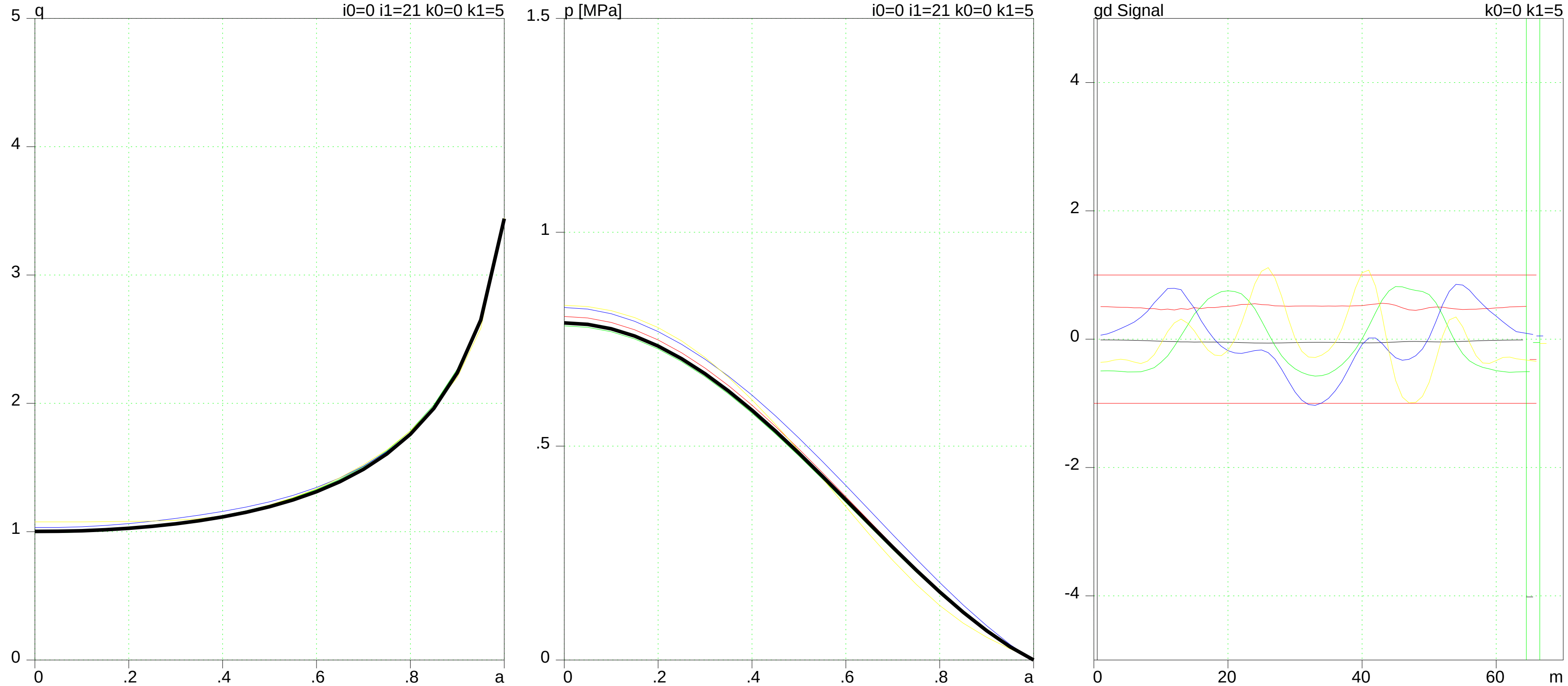


Eigen-variances $\delta j_p^k(a)$, $j_p \equiv P/R_0$

$\bar{\sigma}_q$ *and* $\bar{\sigma}_p^k$ *[MPa] on the left plot are RMS for* q*- and* p*-profiles*

Perturbations $j_s^{k>8}, j_p^{k>8}$ are invisible and cannot be reconstructed

Plasma boundary is well specified, Φ-loop, B-coils are used



$q-$ profile and variances for $k_J \leq 3$, $k_P \leq 2$

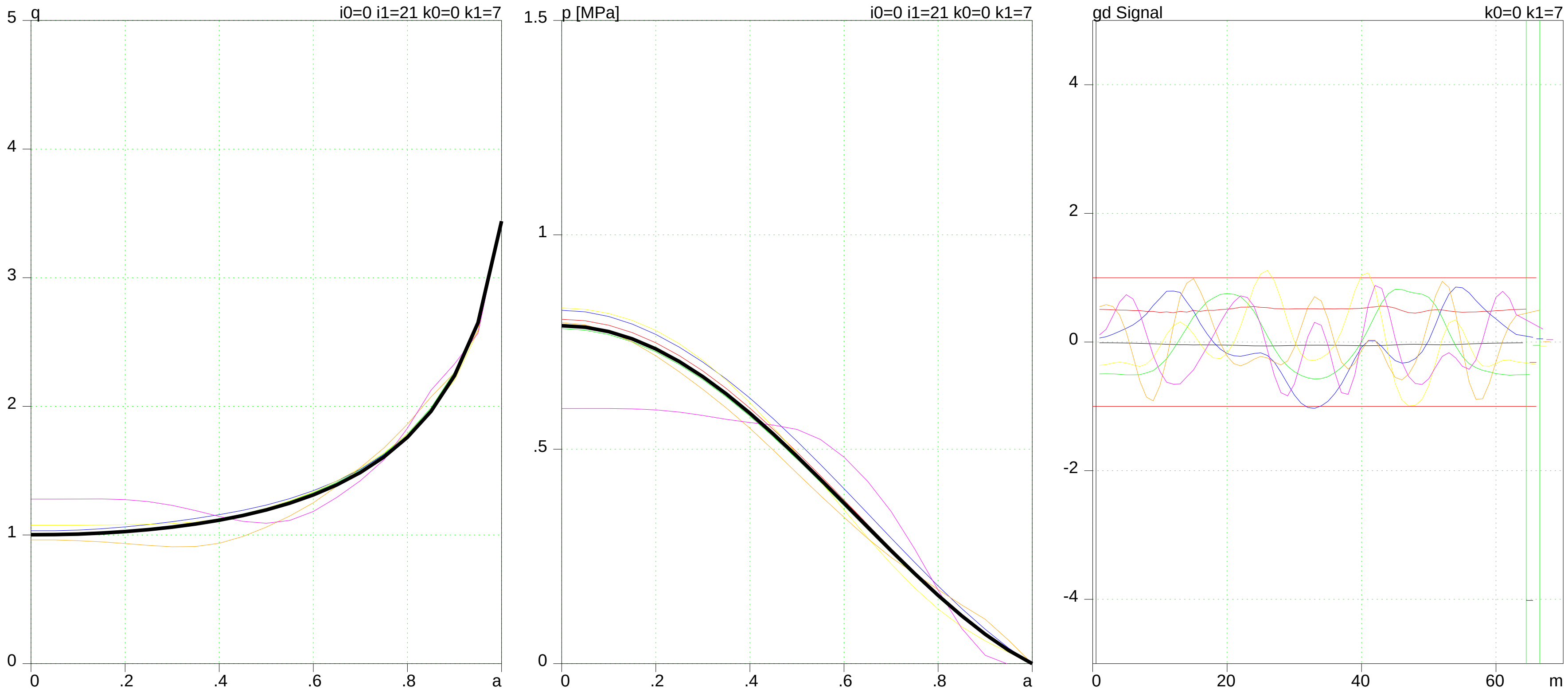
variances in $p-$profile as functions of a

Signals $\delta S_m/\epsilon_m$ generated by perturbations

For k_J+k_P=5, typically used, the reconstruction looks very good

KiloGb's of reconstructions "data" can be easily generated

Plasma boundary is well specified, Φ-loop, B-coils are used



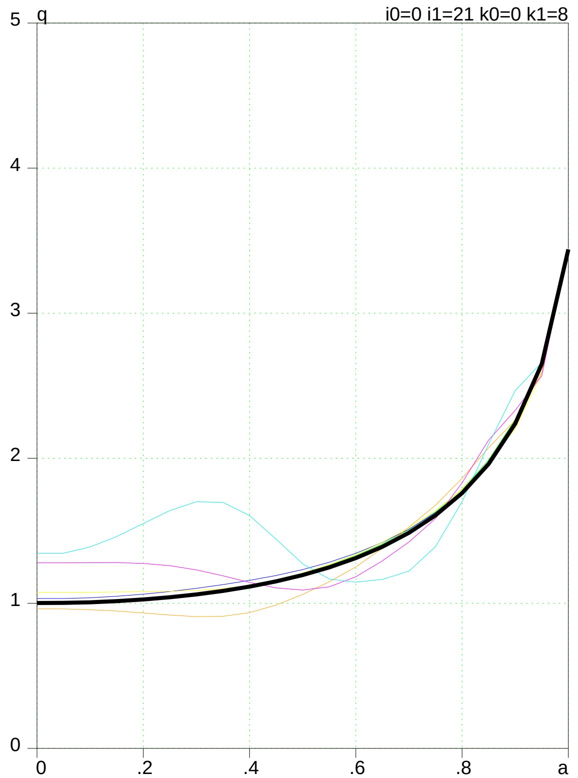
$q-$ profile and variances for $k_J \leq 4$, $k_P \leq 3$.

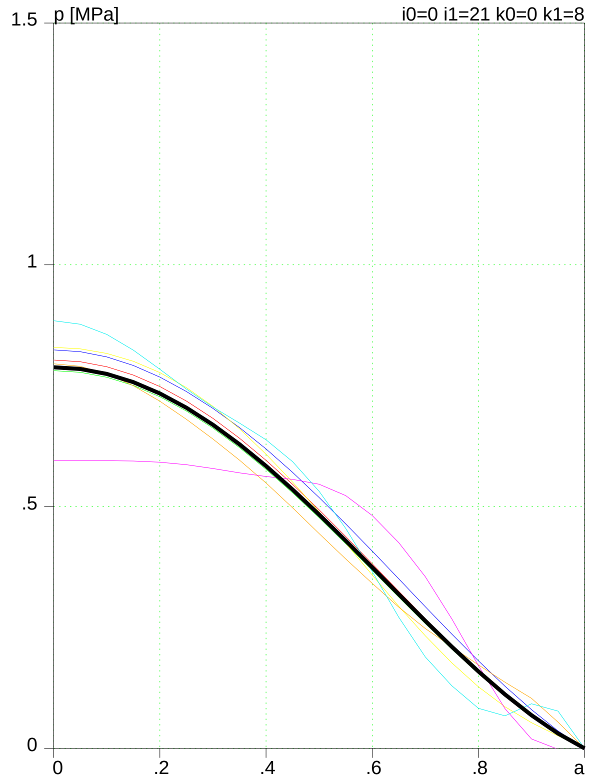
$p-$ profile and its variances as functions of a

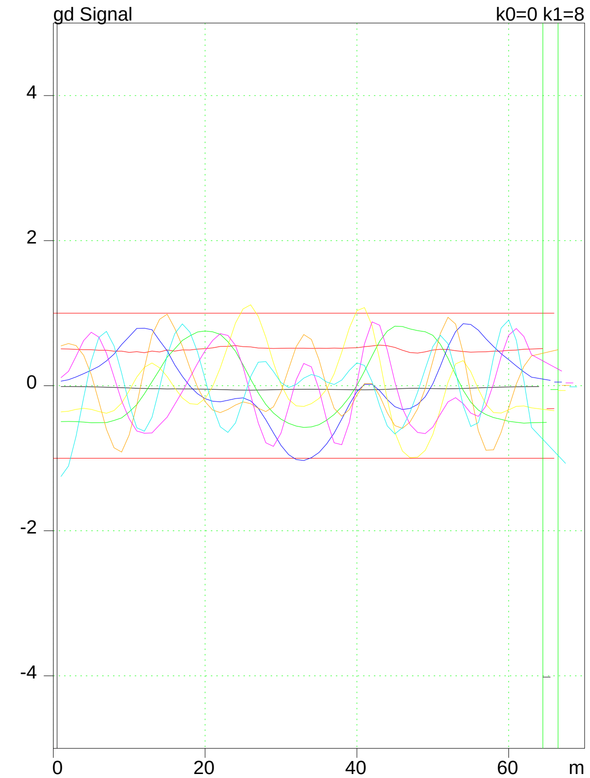
Signals $\delta S_m/\epsilon_m$ generated by perturbations

Testing k_J+k_P=7 shows that the reconstruction is, in fact, not so good

Plasma boundary is well specified, Φ-loop, B-coils are used







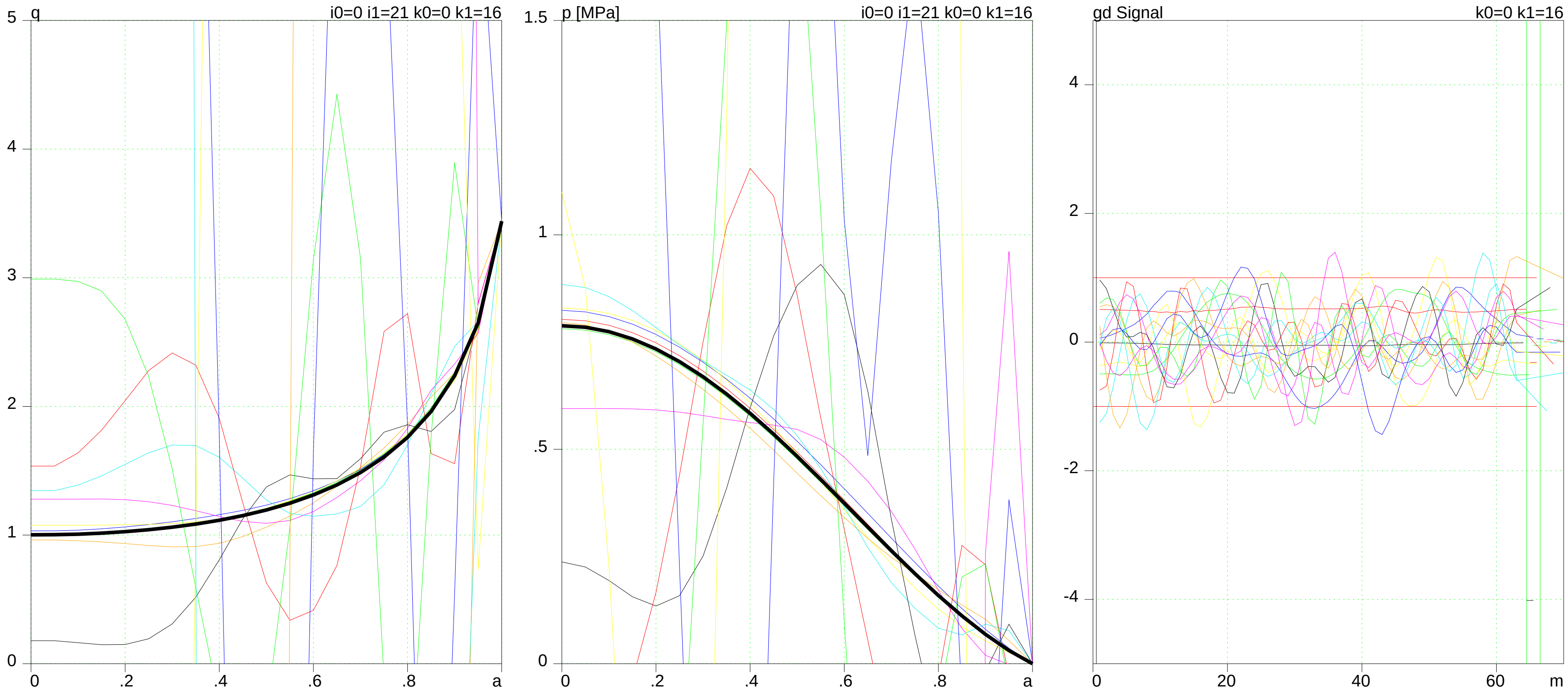
$q-$ profile and variances for k_J ≤4, k_P ≤4

$p-$ profile and its variances as functions of a

Signals $\delta S_m/\epsilon_m$ generated by perturbations

Testing k_J+k_P=8 shows that even the q reconstruction is doubtful

Plasma boundary is well specified, Φ-loop, B-coils are used



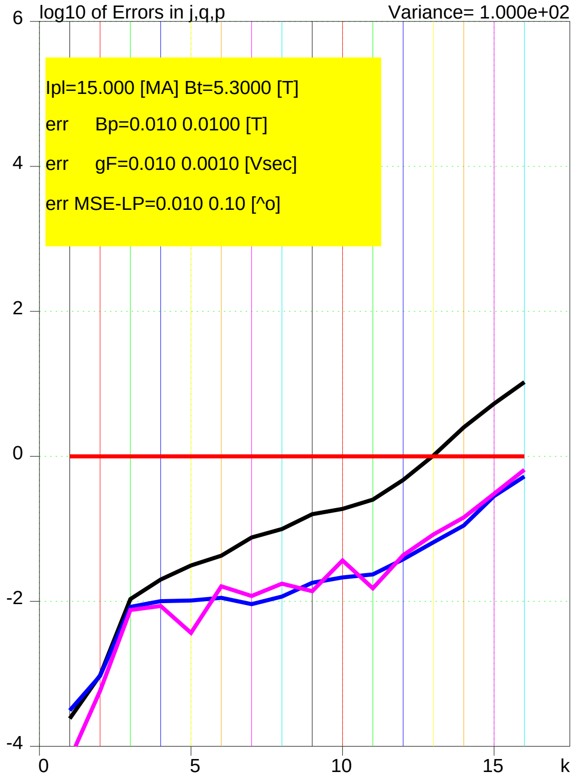
$q-$ profile and variances for $k_J \leq 8$, $k_P \leq 8$

$p-$ profile and its variances as functions of a

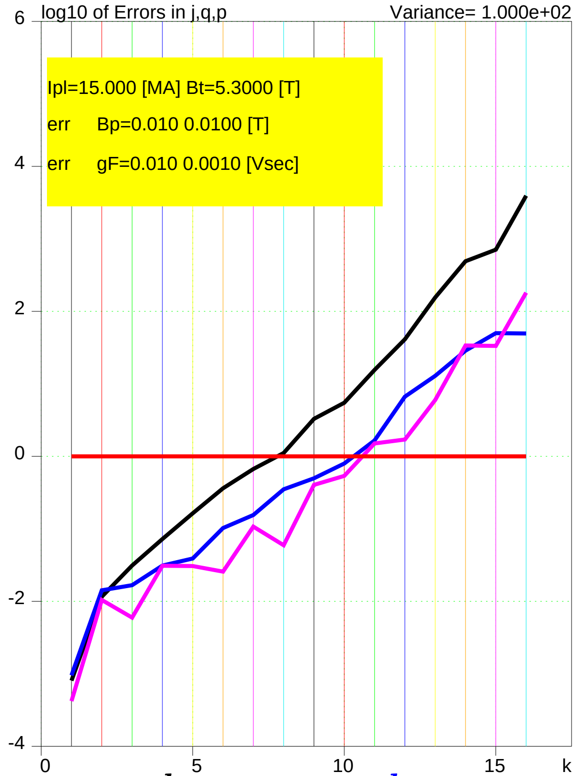
Signals $\delta S_m/\epsilon_m$ generated by perturbations

Test of $k_J + k_P$=16 shows that with no constrains the reconstruction has no scientific value and is a sort of “beliefs”

Fixed plasma boundary with (Φ & B & MSE-LP) signals



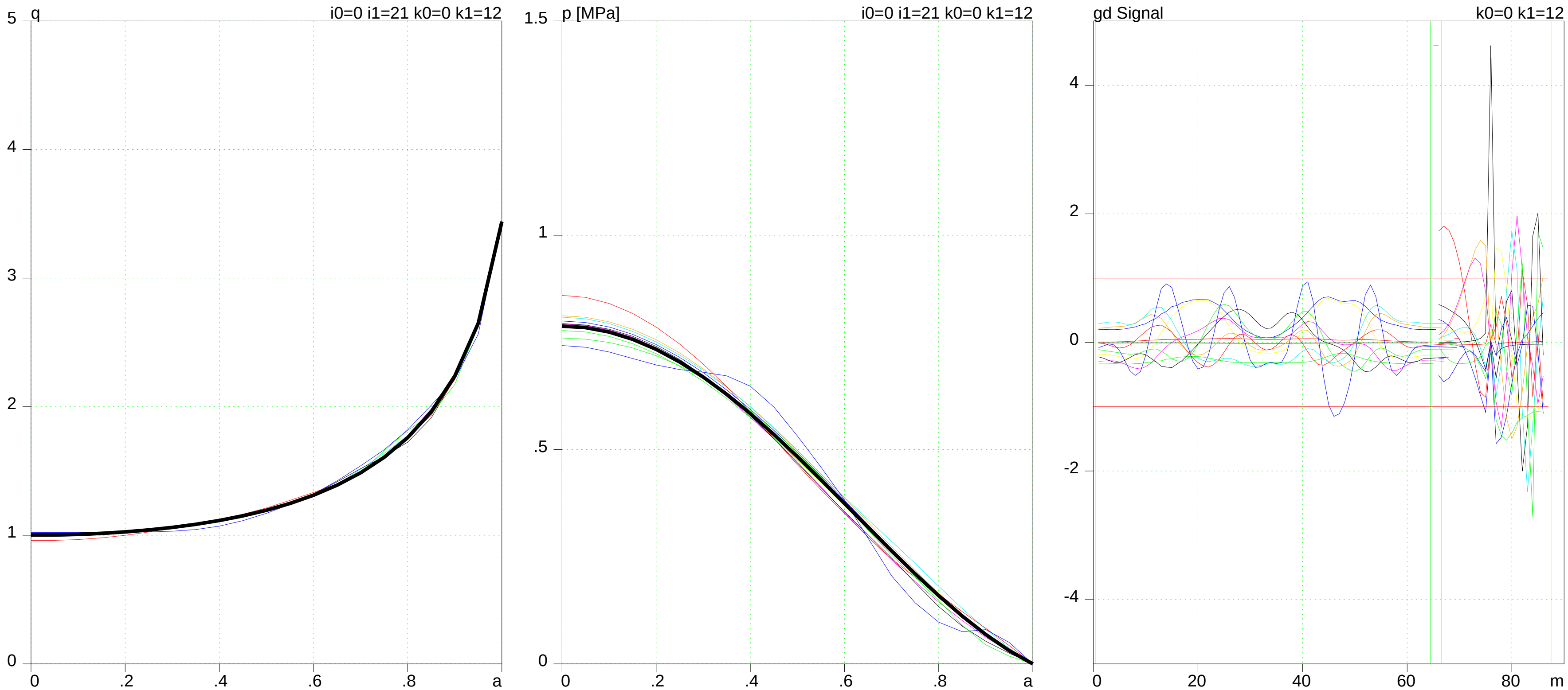
$\log_{10}\bar{\sigma}^k$, $\log_{10}\bar{\sigma}^k_q$, $\log_{10}\bar{\sigma}^k_p$
in case of Φ & B & *MSE-LP*



$\log_{10}\bar{\sigma}^k$, $\log_{10}\bar{\sigma}^k_q$, $\log_{10}\bar{\sigma}^k_p$
in case of Φ & B *only*

Use of MSE-LP drops largest RMS $\bar{\sigma}$, makes 12 perturbations visible, and dramatically improves reconstruction of q, p

Fixed plasma boundary with (Φ & B & MSE-LP) signals



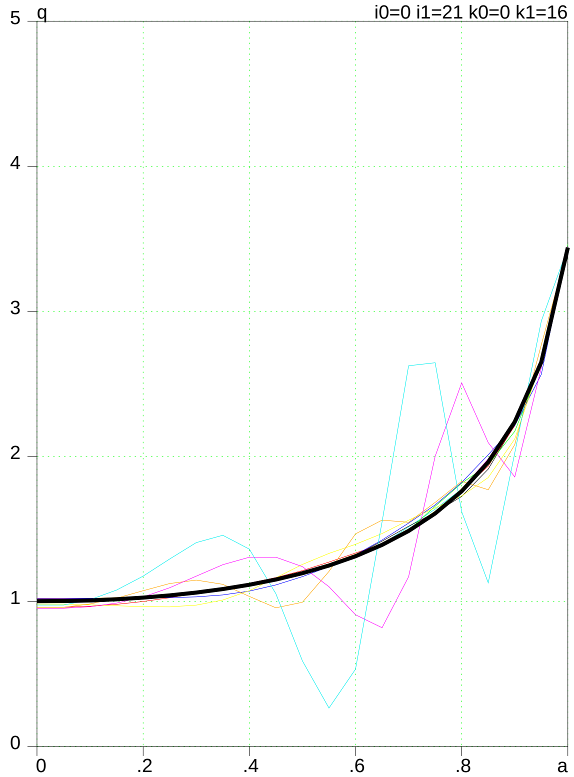
$\boldsymbol{q}-$ profile and variances for $\boldsymbol{k_J} \leq 6$, $\boldsymbol{k_P} \leq 6$

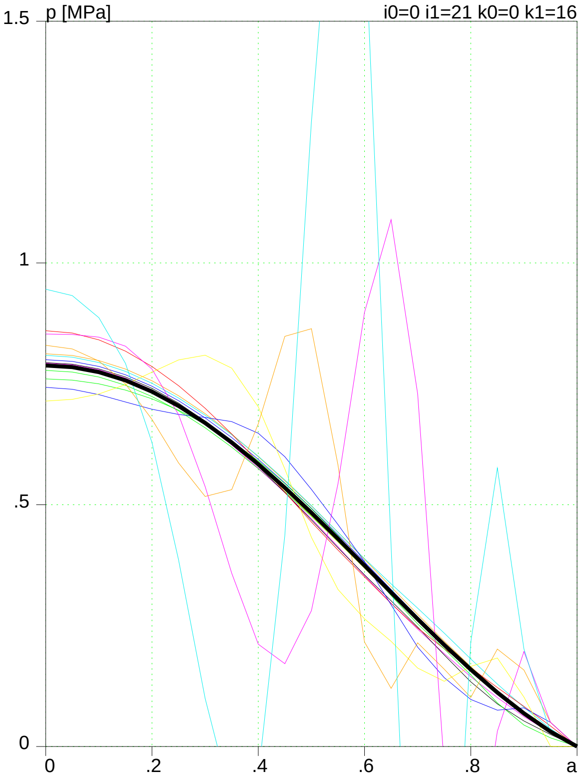
$\boldsymbol{p}-$ profile and its variances as functions of $\boldsymbol{a}$

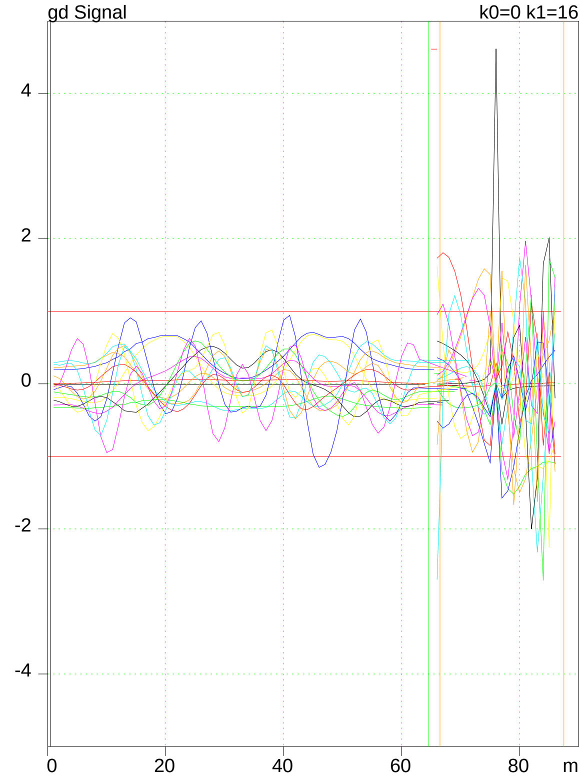
Signals $\boldsymbol{\delta S_m/\epsilon_m}$ generated by perturbations

Testing $N = 12$ shows that MSE-LP allows to reconstruct both q- and p-profiles

Fixed plasma boundary with (Φ & B & MSE-LP) signals







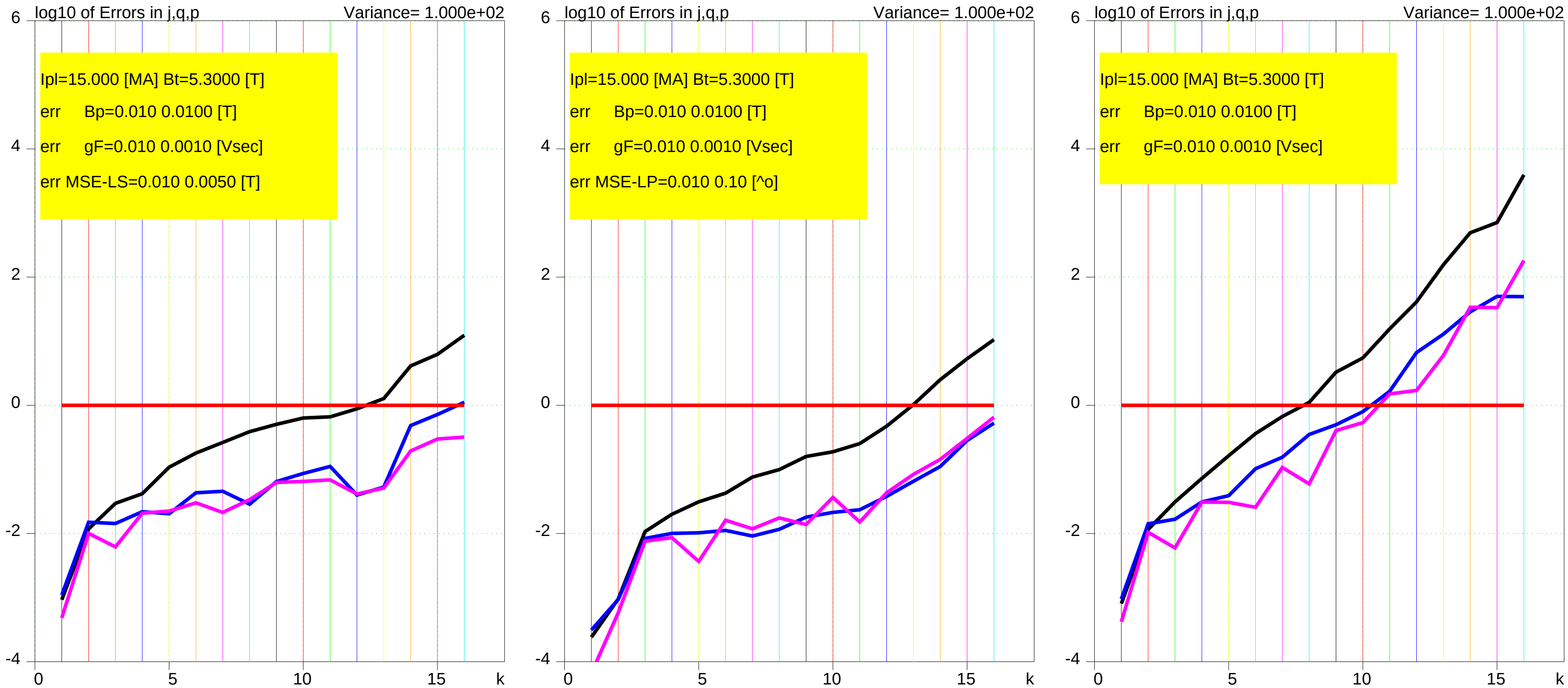
$q-$ profile and variances for all k

$p-$ profile and its variances as functions of a

Signals $\delta S_m/\epsilon_m$ generated by perturbations

Only perturbations with $k \geq 14$ might be potentially troublesome

Fixed plasma boundary with (Φ & B & MSE-LS) signals



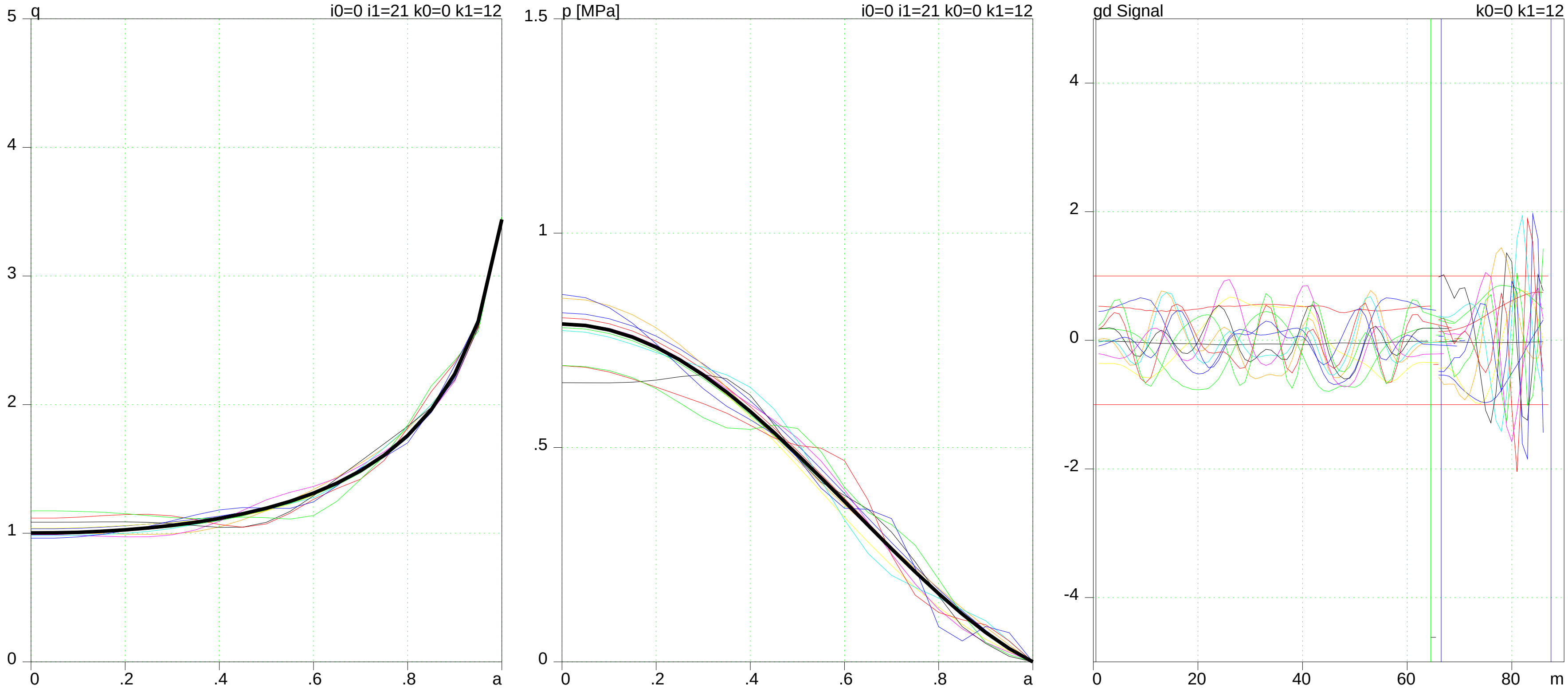
$\log_{10}\{\bar{\sigma}^k,\ \bar{\sigma}^k_q,\ \bar{\sigma}^k_p\}$ in case of (Φ & B & MSE-LS)

$\log_{10}\{\bar{\sigma}^k,\ \bar{\sigma}^k_q,\ \bar{\sigma}^k_p\}$ in case of (Φ & B & MSE-LP)

$\log_{10}\{\bar{\sigma}^k,\ \bar{\sigma}^k_q,\ \bar{\sigma}^k_p\}$ in case of (Φ & B) only

Use of MSE-LS can compete with MSE-LP in its value for reconstruction

Fixed plasma boundary with (Φ & B & MSE-LS) signals



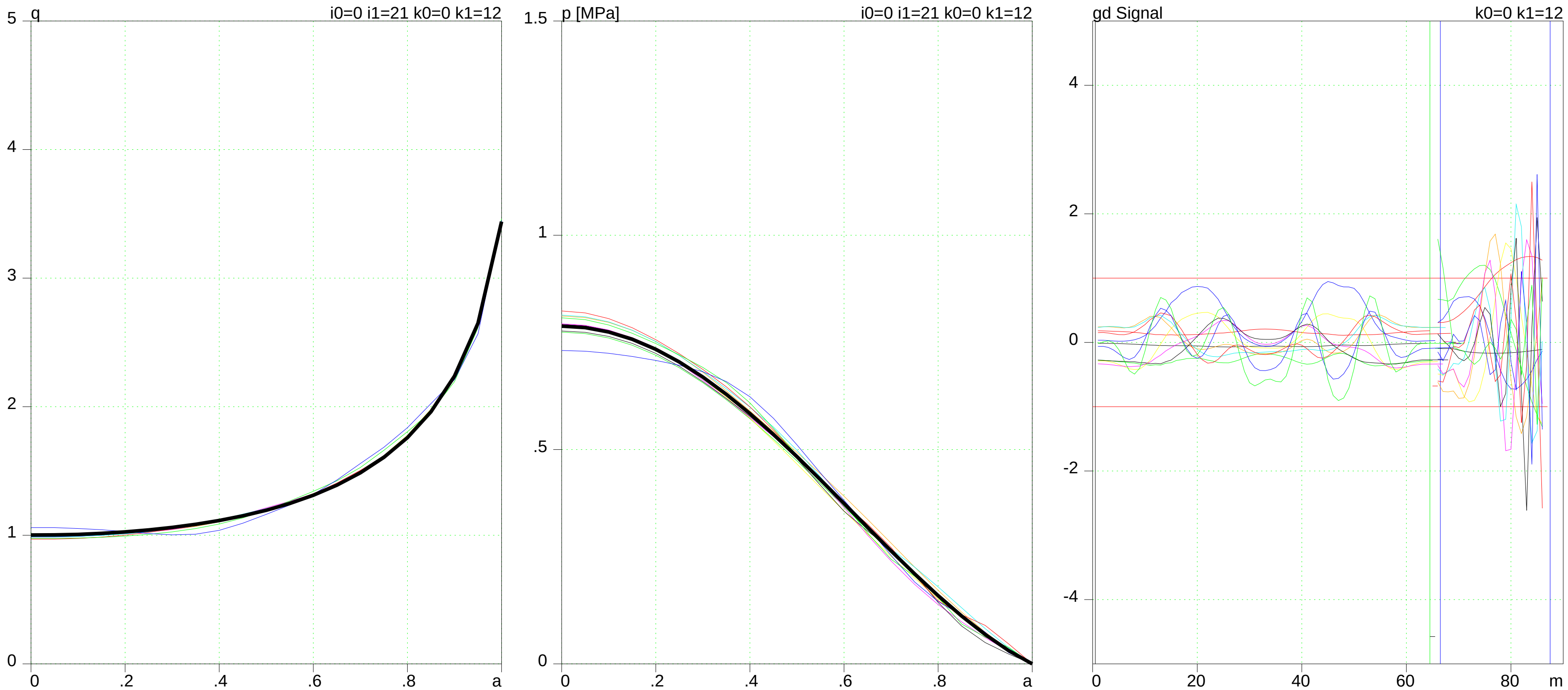
$q-$ profile and variances for all k

$p-$ profile and its variances as functions of a

Signals $\delta S_m/\epsilon_m$ generated by perturbations

Perturbations with $k \leq$12 can be reconstructed using MSE-LS

Same case with the improved relative accuracy of MSE-LS



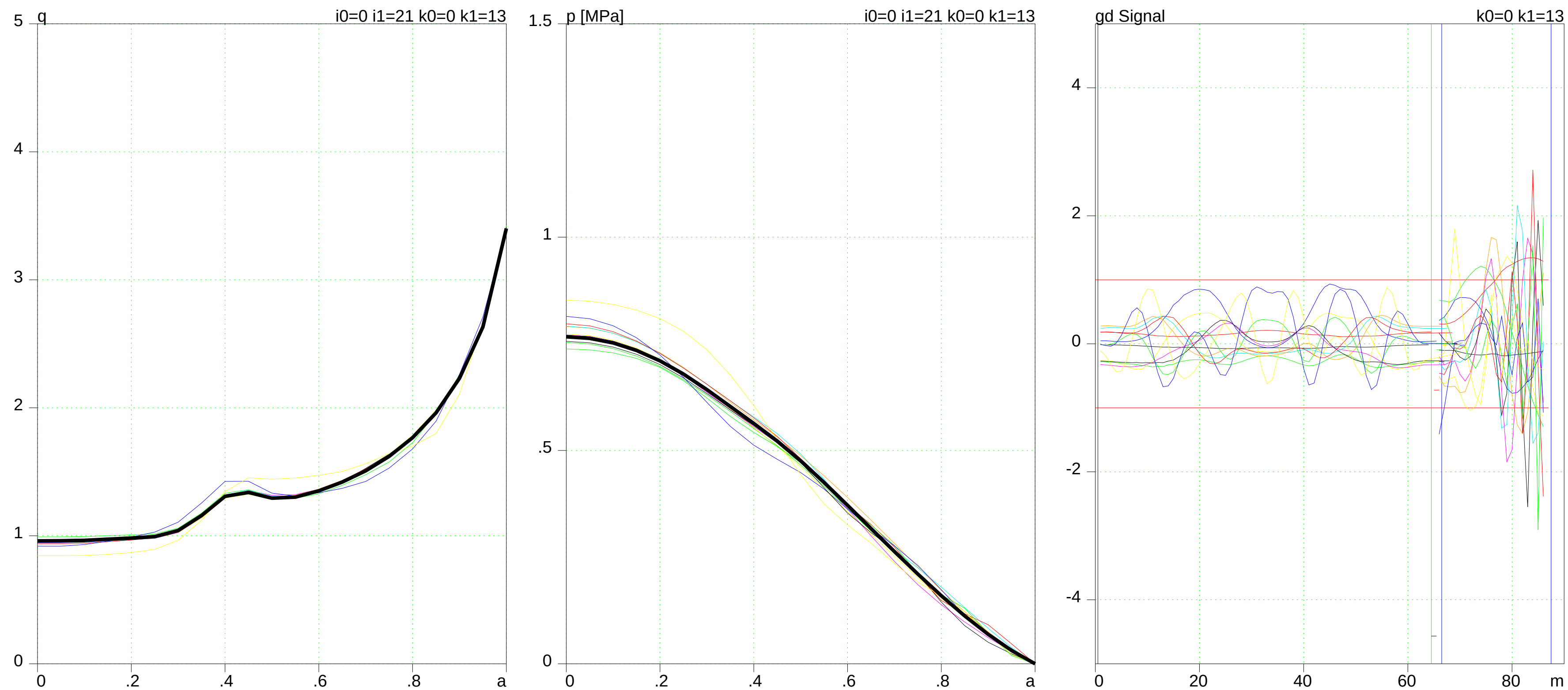
$\boldsymbol{q}-$ *profile and variances for all* $\boldsymbol{k}$

$\boldsymbol{p}-$ *profile and its variances as functions of* $\boldsymbol{a}$

Signals $\boldsymbol{\delta S_m/\epsilon_m}$ *generated by perturbations*

A realistic reduction of relative error $\epsilon^{relative}_{MSE-LS} \rightarrow 0.1\%$ improves the pressure profile reconstruction

Same case with $\epsilon^{relative}_{MSE-LS} \to 0.1\%$ and non-monotonic $\bar{j}_s$



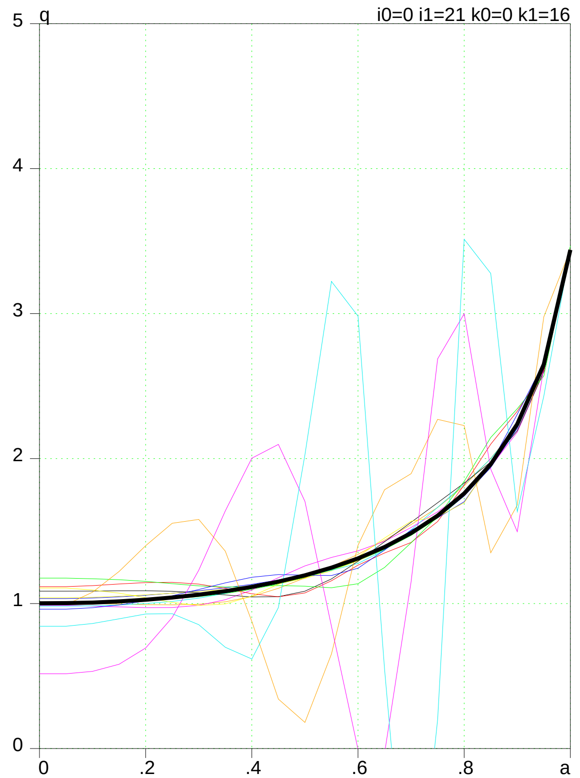
$q-$ profile and variances for all k

$p-$ profile and its variances as functions of a

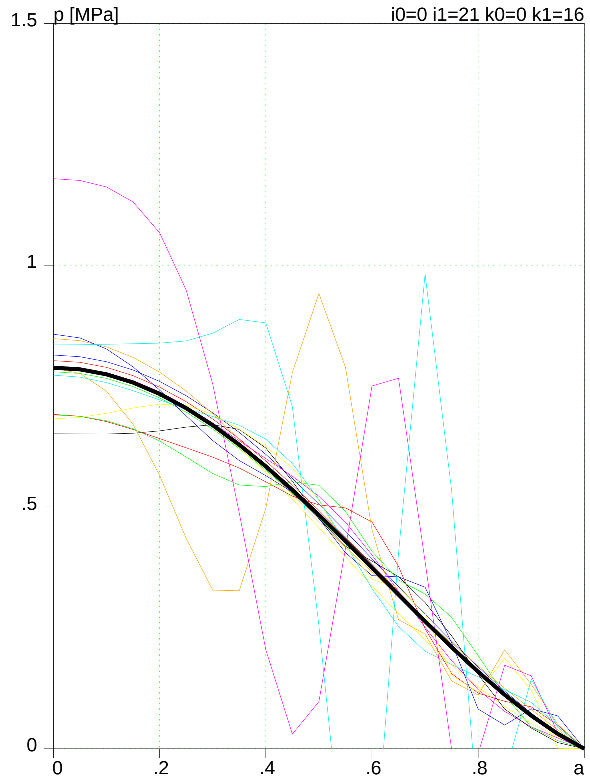
Signals $\delta S_m/\epsilon_m$ generated by perturbations

MSE-LS can pick up the details of the current drive

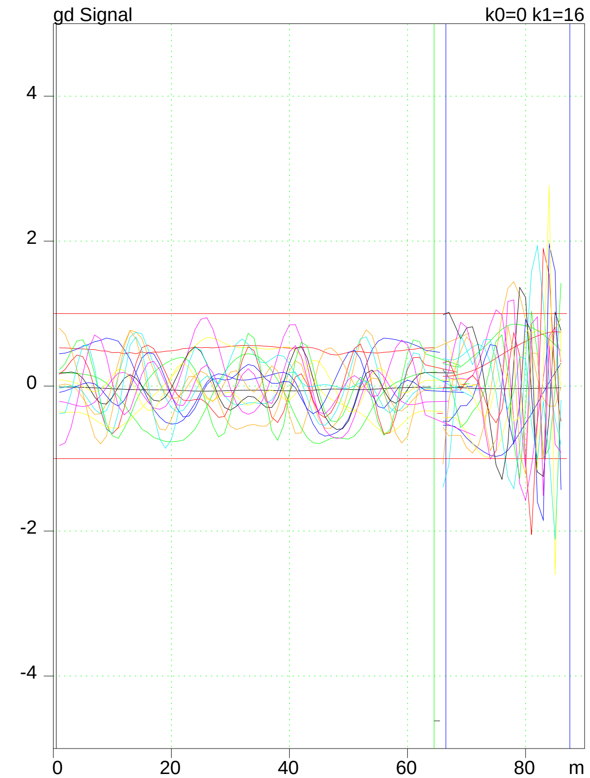
Back to reference fixed boundary and (Φ & B & MSE-LS)



$\boldsymbol{q}-$ profile and variances for all $\boldsymbol{k}$



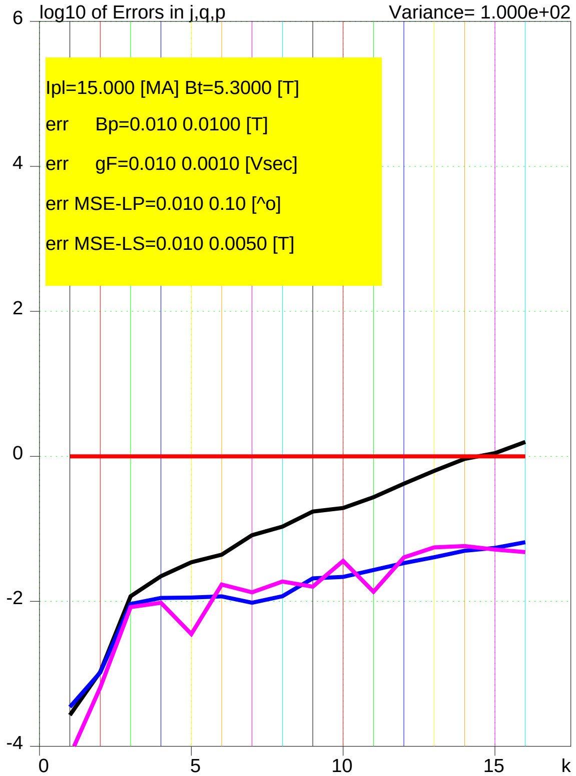
$\boldsymbol{p}-$ profile and its variances as functions of $\boldsymbol{a}$



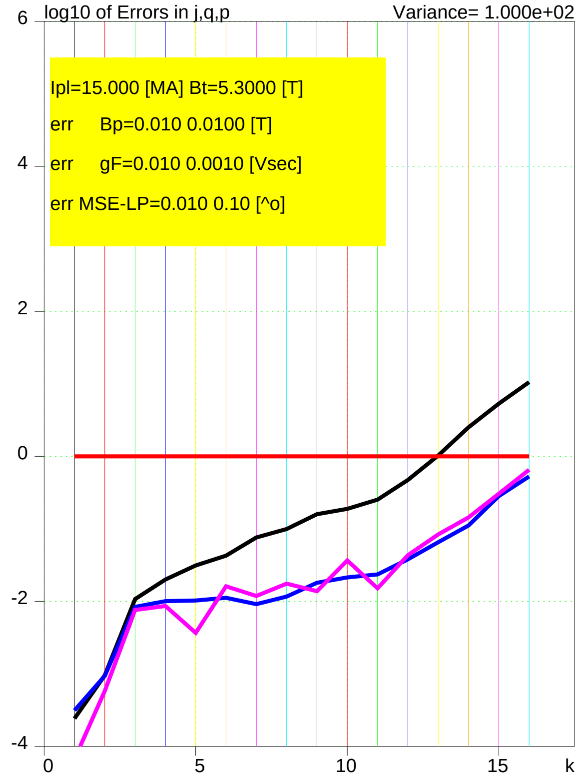
Signals $\boldsymbol{\delta S_m/\epsilon_m}$ generated by perturbations

With MSE-LS only perturbations with $k \geq$13 might be potentially troublesome

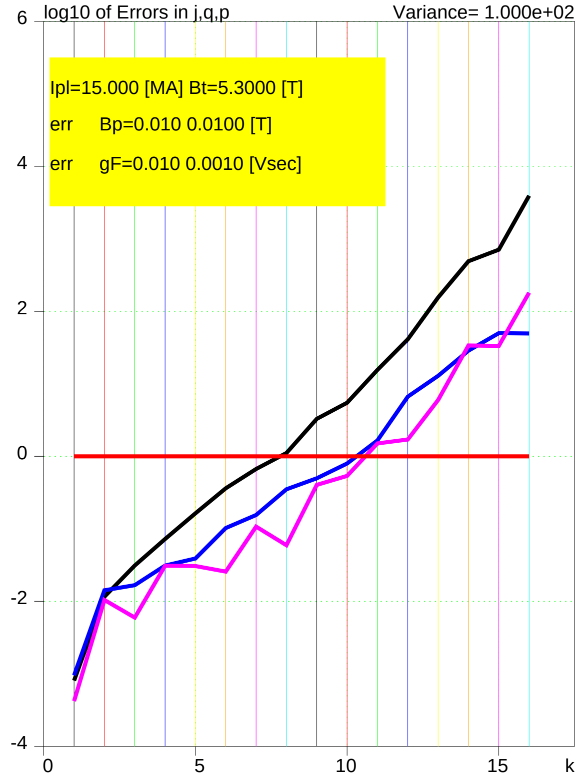
Fixed plasma boundary with (Φ & B & MSE-LP & -LS) signals



$\log_{10}\{\bar{\sigma}^k,\ \bar{\sigma}_q^k,\ \bar{\sigma}_p^k\}$ in case of (Φ & B & MSE-LP & MSE-LS)



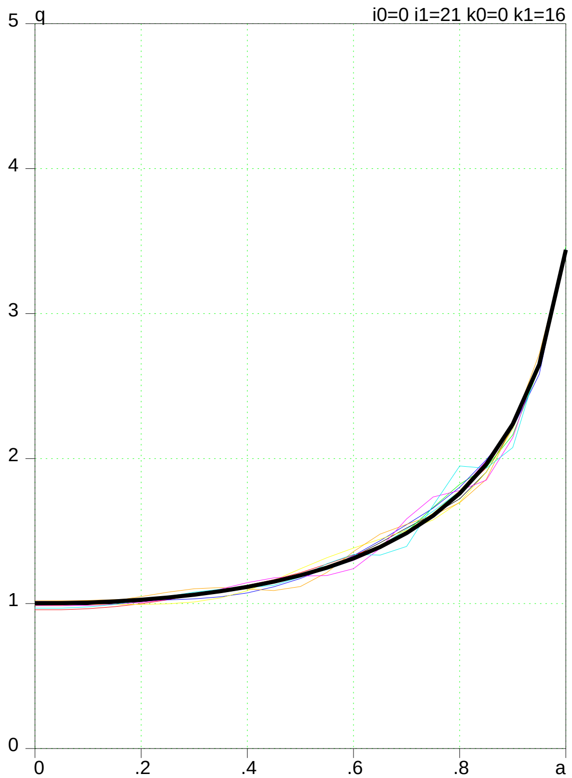
$\log_{10}\{\bar{\sigma}^k,\ \bar{\sigma}_q^k,\ \bar{\sigma}_p^k\}$ in case of (Φ & B & MSE-LP)



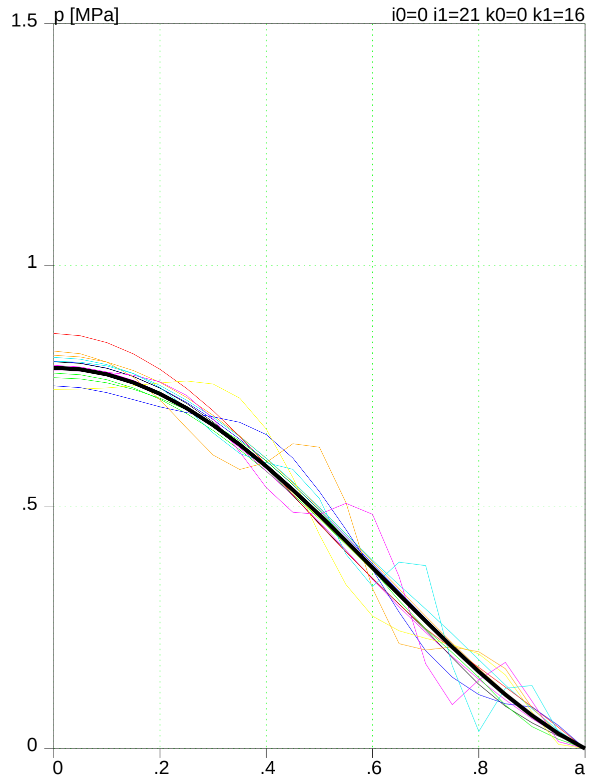
$\log_{10}\{\bar{\sigma}^k,\ \bar{\sigma}_q^k,\ \bar{\sigma}_p^k\}$ in case of (Φ & B) only

Both MSE-LP & LS allows for a reliable reconstruction of q- and p-profiles

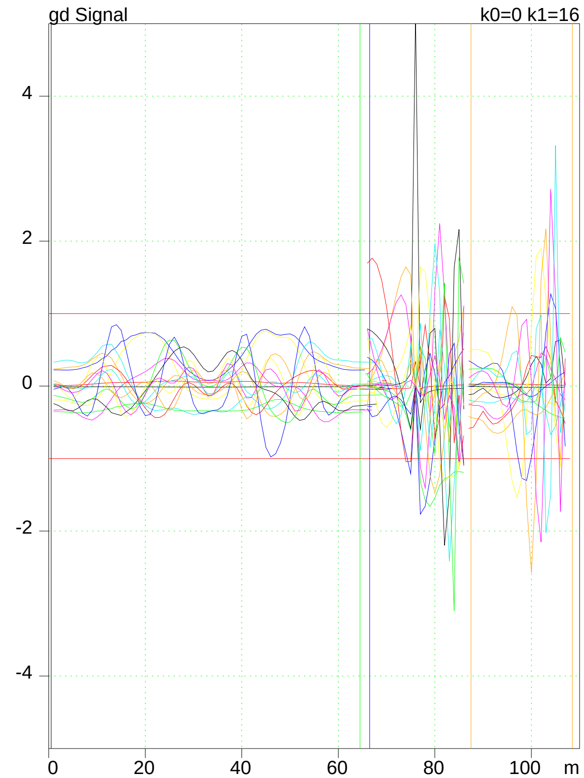
Fixed plasma boundary with (Φ & B & MSE-LP&LS) signals



$\boldsymbol{q}-$ profile and variances for all $\boldsymbol{k}$



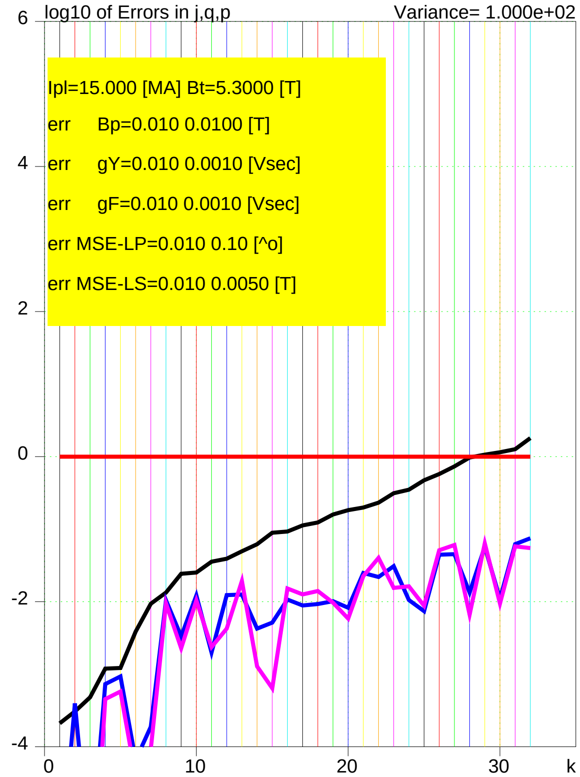
$\boldsymbol{p}$-profile and its variances as functions of $\boldsymbol{a}$



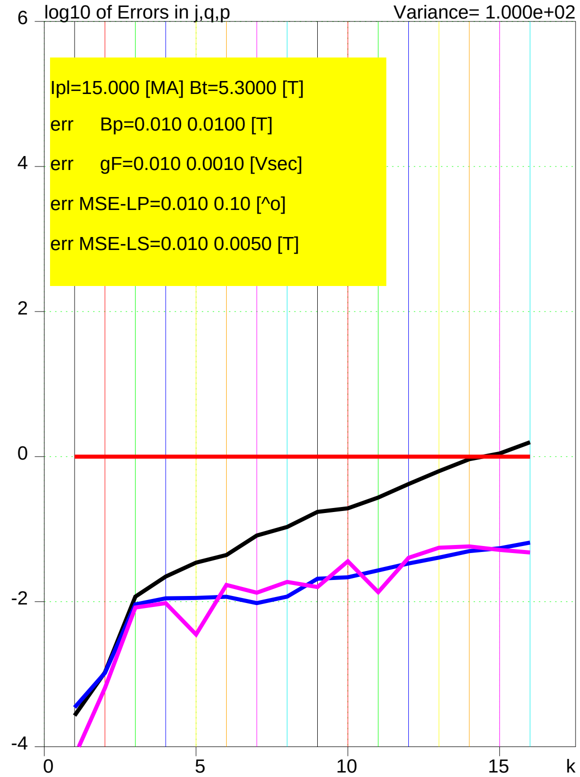
Signals $\boldsymbol{\delta S_m/\epsilon_m}$ generated by perturbations

q- and p-profiles can be reconstructed in all spectrum of k

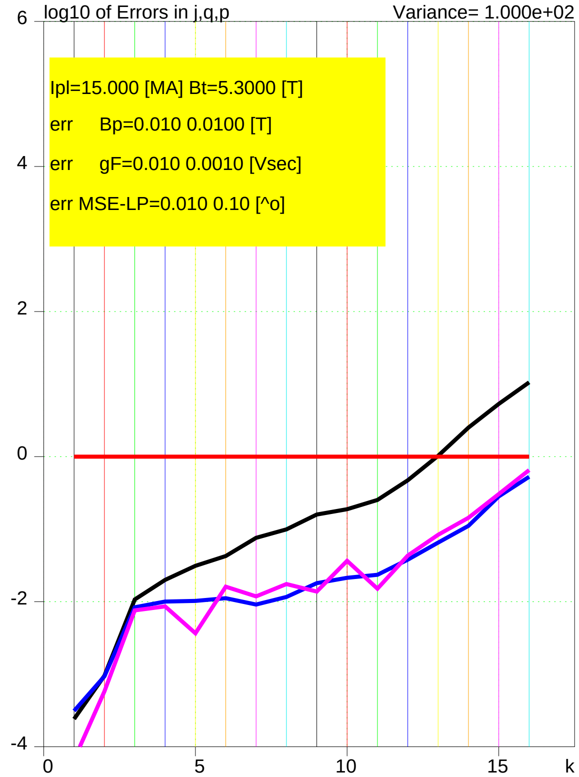
Free boundary plasma with (Φ & B & MSE-LP & -LS) signals



$\log_{10}\{\bar{\sigma}^k,\ \bar{\sigma}^k_q,\ \bar{\sigma}^k_p\}$ in case of (Φ & B & MSE-LP & MSE-LS), $\vec{\xi} \neq 0$



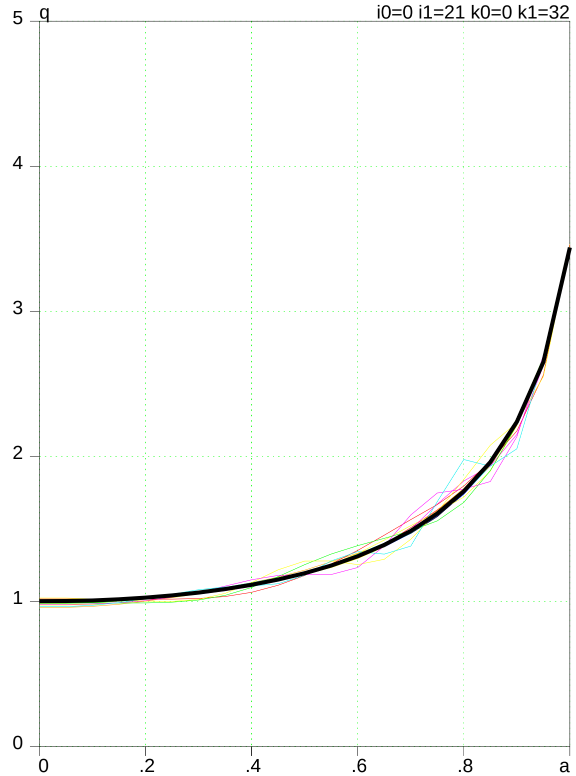
$\log_{10}\{\bar{\sigma}^k,\ \bar{\sigma}^k_q,\ \bar{\sigma}^k_p\}$ in case of (Φ & B & MSE-LP & MSE-LS), $\vec{\xi} = 0$



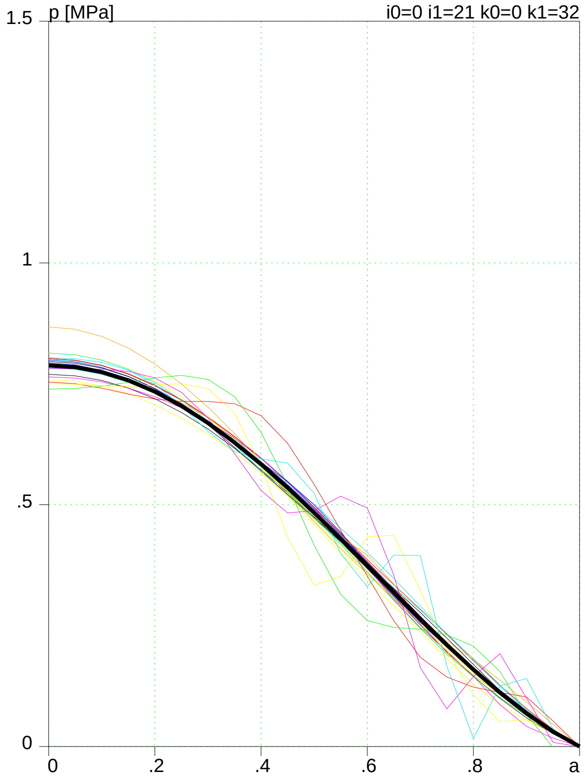
$\log_{10}\{\bar{\sigma}^k,\ \bar{\sigma}^k_q,\ \bar{\sigma}^k_p\}$ in case of (Φ & B & MSE-LP), $\vec{\xi} = 0$

Free boundary expands the k range but does not affect the reconstruction

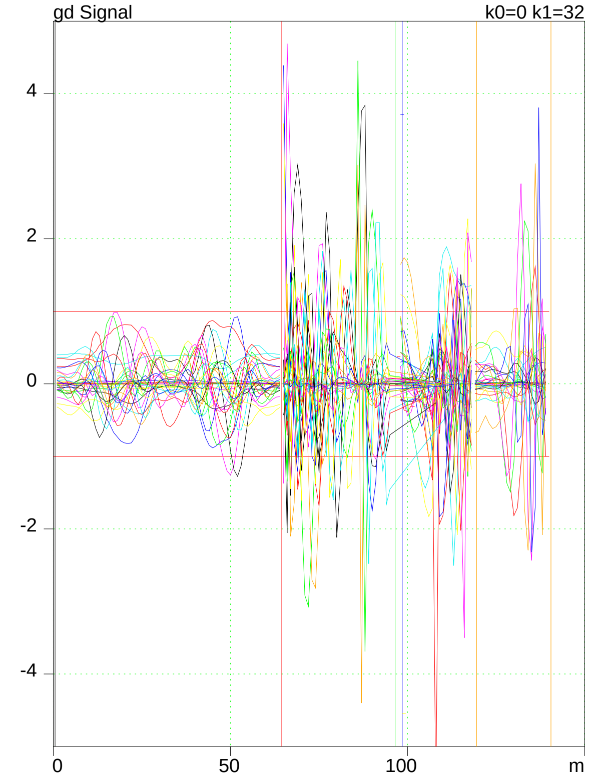
Free boundary plasma with (Φ & B & MSE-LP & -LS) signals



$q-$ profile and variances for all extended k



p-profile and its variances as functions of a

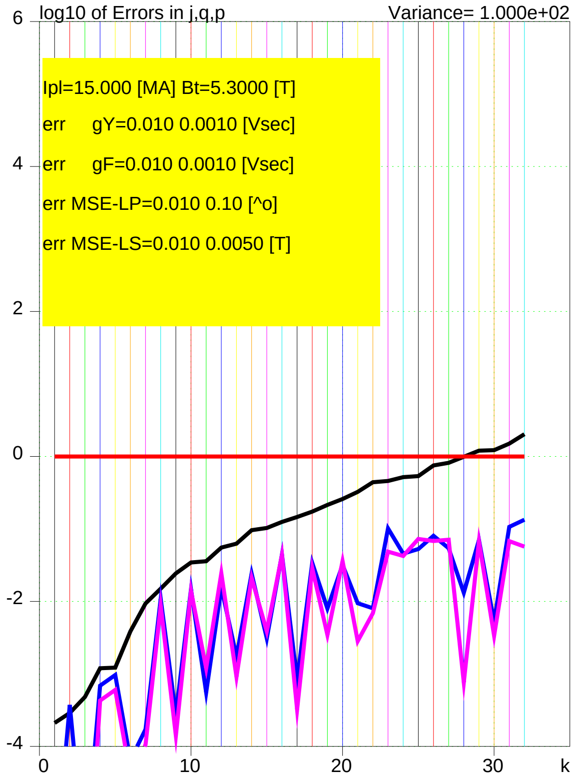


Signals $\delta S_m/\epsilon_m$ generated by perturbations

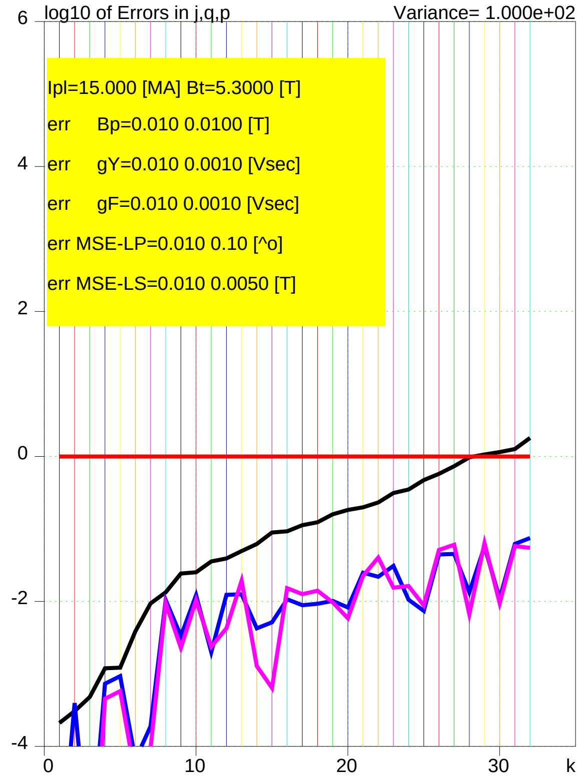
q- and p-profiles can be reconstructed in all extended spectrum of k

4.6 Curious case, NO B-signals, $\xi \neq 0$, Φ & both MSE-LP & MSE-LS

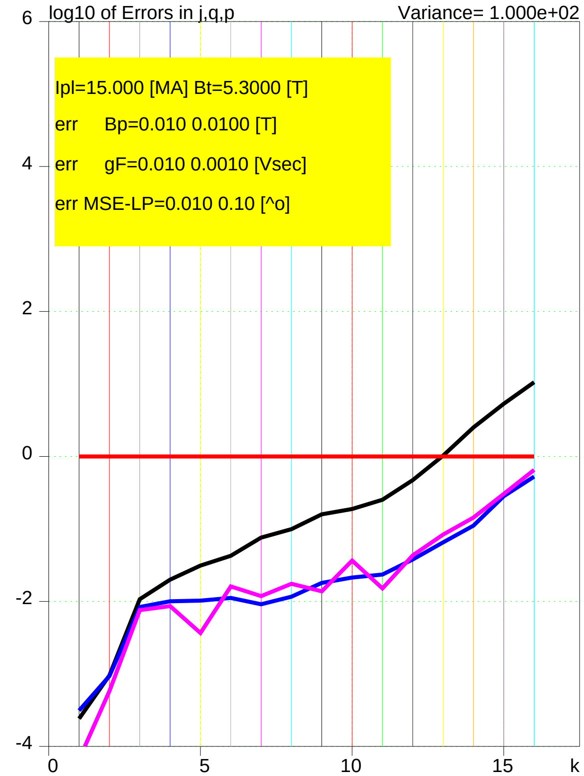
Free boundary, (Φ & MSE-LP & -LS) signals, NO B-signals



$\log_{10}\{\bar{\sigma}^k,\ \bar{\sigma}_q^k,\ \bar{\sigma}_p^k\}$ in case of (Φ & B & MSE-LP & MSE-LS), $\vec{\xi} \neq 0$



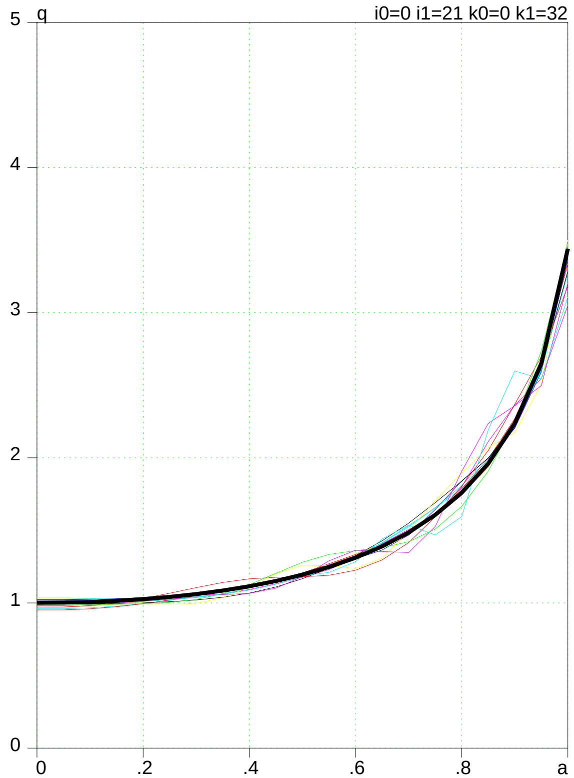
$\log_{10}\{\bar{\sigma}^k,\ \bar{\sigma}_q^k,\ \bar{\sigma}_p^k\}$ in case of (Φ & B & MSE-LP & MSE-LS), $\vec{\xi} \neq 0$



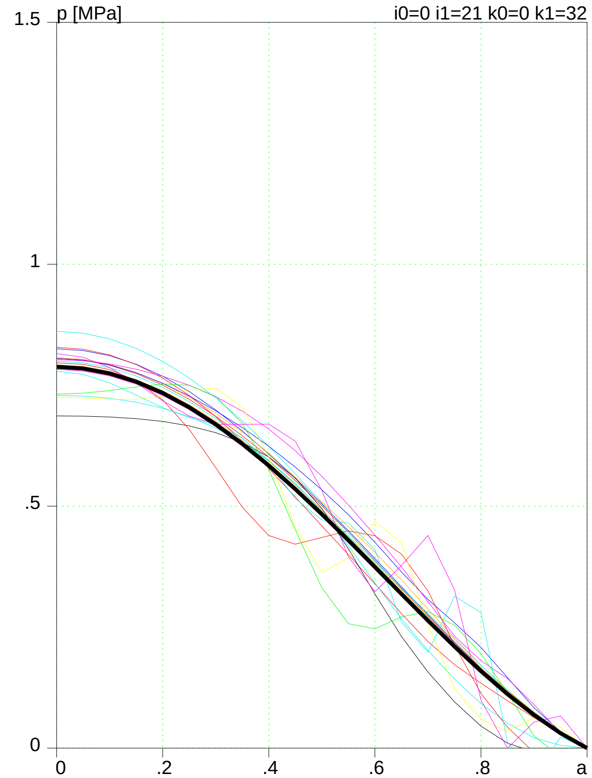
$\log_{10}\{\bar{\sigma}^k,\ \bar{\sigma}_q^k,\ \bar{\sigma}_p^k\}$ in case of (Φ & B & MSE-LP), and $\vec{\xi} = 0$

(MSE-LP & MSE-LS) together can do the job for external B-coils

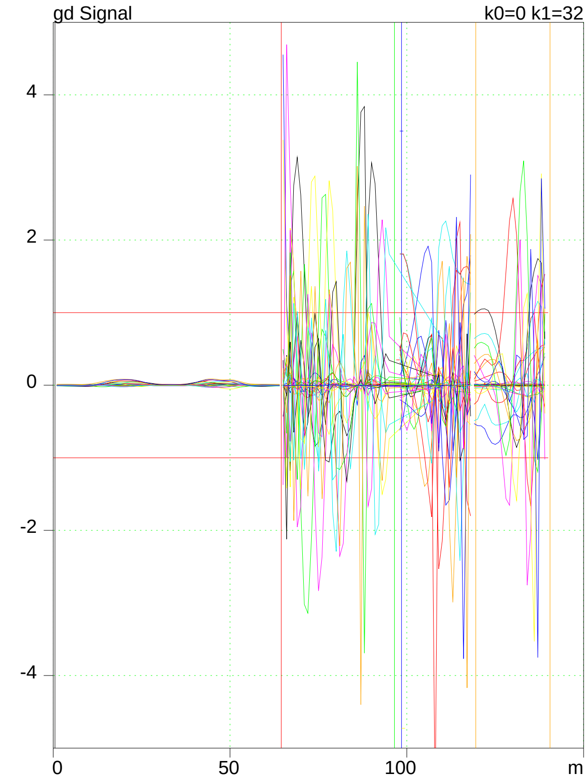
Free boundary, (Φ & MSE-LP & -LS) signals, NO B-signals



$\boldsymbol{q}-$ profile and variances for all extended $\boldsymbol{k}$



$\boldsymbol{p}$-profile and its variances as functions of $\boldsymbol{a}$



Signals $\boldsymbol{\delta S_m/\epsilon_m}$ generated by perturbations

q- and p-profiles can be reconstructed over extended spectrum of k even with NO B-coil signals

The capability of calculating variances, now developed, has completed the theory of equilibrium reconstruction

1. *The quantitative evaluation of the quality of diagnostics systems on existing and future machines can be done based on spectrum of "visible" perturbations*
2. *It was confirmed that the internal measurements of the magnetic field are crucial for reconstruction.*
3. *Either* MSE-LP *(line polarization) or* MSE-LS *(line shift) signals from the plasma in addition to external measurements allow for a complete reconstruction (of both* q*- and* p*-profiles).*
4. *The presented technique can be used to optimize the diagnostic set on any tokamaks. Contribution of any signal can be evaluated.*
5. *The proposal by* Nova Photonics *to utilize* MSE-LS *signals would significantly enhance the equilibrium reconstruction capability in* ITER.

The extension of the theory should be focused on realistic simulation of signals used in reconstructions